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Czechoslovak Mathematical Journal, Vol. 53 (2003), No. 2, 225–240

Persistent URL: `http://dml.cz/dmlcz/127795`

Czechoslovak Mathematical Journal, 53 (128) (2003), 225–240

THE BASIS NUMBER OF SOME SPECIAL NON-PLANAR GRAPHS

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(Received January 17, 1995)

Abstract. The basis number of a graph G was defined by Schmeichel to be the least integer h such that G has an h-fold basis for its cycle space. He proved that for $m, n \geqslant 5$, the basis number $b(K_{m,n})$ of the complete bipartite graph $K_{m,n}$ is equal to 4 except for $K_{6,10}$, $K_{5,n}$ and $K_{6,n}$ with $n = 5, 6, 7, 8$. We determine the basis number of some particular non-planar graphs such as $K_{5,n}$ and $K_{6,n}$, $n = 5, 6, 7, 8$, and r-cages for $r = 5, 6, 7, 8$, and the Robertson graph.

Keywords: graphs, basis number, cycle space, basis

MSC 2000: 05C35, 05C38

1. Introduction

Throughout this paper, we assume that graphs are finite, undirected, and simple. Our terminology and notation will be standard except as indicated. For undefined terms, see [3] and [4].

Let G be a graph, and let $e_1, e_2, \ldots, e_q$ be an ordering of its edges. Then any subset S of $E(G)$ corresponds to a $(0,1)$-vector $(a_1, a_2, \ldots, a_q)$ in the usual way, with $a_i = 1$ if $e_i \in S$ and $a_i = 0$ if $e_i \notin S$. These vectors form a q-dimensional vector space over $\mathbb{Z}_2$ denoted by $(\mathbb{Z}_2)^q$.

Let $\mathcal{C}(G)$, called the *cycle space* of G, be the subspace of $(\mathbb{Z}_2)^q$ generated by the vectors corresponding to the cycles in G. We shall say, however, that the cycles themselves, rather than the vectors corresponding to the cycles, generate $\mathcal{C}(G)$. It is well known that if G is a (p,q) connected graph, then the dimension of $\mathcal{C}(G)$ is

$$\dim(\mathcal{C}(G)) = \gamma(G) = q - p + k,$$

where p is the number of vertices, q is the number of edges, k is the number of connected components and $\gamma(G)$ is the cyclomatic number of G. In fact, given any spanning tree T in G, every graph $T + e$, $e \notin T$, contains exactly one cycle C_e, and the collection of cycles $\{C_e\colon\ e \notin T\}$ forms a basis of $\mathcal{C}(G)$, called the *fundamental basis corresponding to* T. While each edge outside of T occurs in exactly one cycle of this basis, an edge of T itself may occur in many cycles of the basis. This observation suggests the following definition.

Definition. Let h be a positive integer. A basis of $\mathcal{C}(G)$ is called *h-fold* if each edge of G occurs in at most h of the cycles in the basis. The *basis number* of G (denoted by $b(G)$) is the smallest integer h such that $\mathcal{C}(G)$ has an h-fold basis, and such a basis is called the *required basis* of G and denoted by $B_r(G)$. If B is a basis for $\mathcal{C}(G)$ and e is an edge of G then the *the fold of e in B* (denoted by $f_B(e)$) is defined to be the number of cycles in B containing e.

The *ring sum* of two graphs (subgraphs) G_1 and G_2 (written $G_1 \oplus G_2$) is the graph consisting of the vertex-set $V(G_1) \cup V(G_2)$ and of the edges which are either in G_1 or G_2 but not in both.

The *girth* of a graph is the length of its shortest cycle. An *r-cage*, $r \geqslant 3$, is a cubic graph of girth r with the minimum possible number of vertices. Tutte [8] proved the existence of r-cages for $r \geqslant 3$, and for $r = 3, 4, \ldots, 8$ there is a uniqe r-cage.

The *Robertson* graph is the smallest graph of girth 5 and valency 4 (i.e., each vertex is of degree 4). Robertson [6] established that, up to an isomorphism, the Robertson graph is the only smallest graph of girth 5 and valency 4.

The first important result concerning the basis number was given by MacLane [5]. He proved the following theorem:

Theorem 1.1. *A graph G is planar if and only if $b(G) \leqslant 2$.*

Schmeichel [7] proved that for every integer $n \geqslant 5$, $b(K_n) = 3$. Also he proved that for $m, n \geqslant 5$, the basis number $b(K_{m,n})$ of the complete bipartite graph $K_{m,n}$ is equal to 4 except for $K_{6,10}$, $K_{5,n}$ and $K_{6,n}$, with $n = 5, 6, 7, 8$. Moreover, Banks and Schmeichel [2] proved that for $n \geqslant 7$, $b(Q_n) = 4$, where Q_n is the n-cube.

A lower bound for the basis number of a graph is given in the following theorem which is due to Banks and Schmeichel [2].

Theorem 1.2. *For any connected graph G,*

$$\sum_{v \in V(G)} \left\lfloor \frac{b(G)d(v)}{2} \right\rfloor \geqslant g(G) \cdot \gamma(G),$$

where $d(v)$ denotes the degree of the vertex v, $g(G)$ the girth of G, and $\gamma(G)$ the cyclomatic number of G.

Ali and Alsardary [1] investigated the relation between $b(G)$ and $b(G')$ where G' is the graph obtained from a graph G by either adding or deleting an edge in certain ways, or by contracting some edges.

In this note we first investigate the basis number of $K_{5,n}$ and $K_{6,n}$, $n = 5, 6, 7, 8$. We prove that $b(K_{5,n}) = b(K_{6,n}) = 3$ for $n = 5, 6, 7, 8$. Next, we investigate the basis number of r-cages for $r = 5, 6, 7, 8$. We prove that $b(r\text{-cage}) = 3$ for $r = 5, 6, 7$ and $b(8\text{-cage}) = 4$. Finally, we prove that the basis number of the Robertson graph is 4.

2. The basis number of $K_{5,n}$ and $K_{6,n}$, $n = 5, 6, 7, 8$

Schmeichel [7] determined the basis number for all complete bipartite graphs except $K_{5,n}$, $K_{6,n}$, $n = 5, 6, 7, 8$, and $K_{6,10}$. We shall prove the basis number of these graphs is 3, except $K_{6,10}$, for which it seems likely that the basis number is 3.

It is clear that each $K_{m,n}$, $m, n \geqslant 3$, is a non-planar graph; therefore, by Theorem 1.1, we need to find a 3-fold basis for each of the complete bipartite graphs $K_{5,n}$ and $K_{6,n}$, $n = 5, 6, 7, 8$. For each of these graphs, we choose a set B of cycles such that $|B|$ equals the cyclomatic number of the graph and the fold of each edge in B is not more than 3. Then, we show that B is independent.

Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be a partition of the vertices of the complete bipartite graph $K_{m,n}$ into independent sets. Then the set of the edges of $K_{m,n}$ will be

$$E(K_{m,n}) = \{[x_i, y_j]\colon\ i = 1, 2, \ldots, m;\ \ j = 1, 2, \ldots, n\}.$$

Denoting $[x_i, y_j]$ by $e_{i+m(j-1)}$, we have

$$\begin{aligned} E(K_{m,n}) &= \{e_{i+m(j-1)}\colon\ i = 1, 2, \ldots, m;\ \ j = 1, 2, \ldots, n\} \\ &= \{e_1, e_2, \ldots, e_{mn}\}. \end{aligned}$$

For simplicity, the cycles of $K_{m,n}$ will be represented by the sequences of their vertices. To obtain the vector of a cycle C in $K_{m,n}$, we find the edge representation of C using the above notation.

2.1. A 3-fold basis for $\mathcal{C}(K_{5,5})$.

It is clear that the number of edges of $K_{5,5}$ is 25 and the number of vertices is 10. Therefore, $\gamma(K_{5,5}) = 16$.

Schmeichel [7] proved that

$$\{x_i y_j x_{i+1} y_{j+1} \colon\ i = 1, 2, \ldots, m-1;\ \ j = 1, 2, \ldots, n-1\}$$

is a 4-fold basis for $K_{m,n}$. Starting from the 4-fold basis

$$\{x_i y_j x_{i+1} y_{j+1} \colon\ i = 1, 2, 3, 4;\ \ j = 1, 2, 3, 4\}$$

of $K_{5,5}$, we form the cycle matrix M whose rows are the vectors of this basis. Applying some elementary row operations on M, we obtain a cycle matrix M' in which each column contains not more than three non-zero entries. The set of the cycles whose vectors are the rows of M' is found to be

$$\begin{aligned} B(K_{5,5}) = \{&x_1y_1x_2y_2, x_1y_1x_2y_3, x_1y_3x_2y_4, x_1y_4x_2y_5, x_2y_4x_3y_5, x_3y_1x_4y_2,\\ &x_4y_1x_5y_2, x_4y_4x_5y_5, x_1y_1x_3y_2, x_1y_4x_5y_5, x_2y_3x_3y_5, x_3y_2x_4y_4,\\ &x_3y_3x_4y_5, x_4y_1x_5y_3, x_4y_3x_5y_5, x_1y_2x_2y_1x_3y_3\}. \end{aligned}$$

It is clear that $B(K_{5,5})$ is a 3-fold basis for $\mathcal{C}(K_{5,5})$, and we can check that by writing $B(K_{5,5})$ in terms of the edges, we obtain

$$\begin{aligned} B(K_{5,5}) = \{&e_1e_2e_7e_6, e_1e_2e_{12}e_{11}, e_{11}e_{12}e_{17}e_{16}, e_{16}e_{17}e_{22}e_{21}, e_{17}e_{18}e_{23}e_{22}, e_3e_4e_9e_8,\\ &e_4e_5e_{10}e_9, e_{19}e_{20}e_{25}e_{24}, e_1e_3e_8e_6, e_{16}e_{20}e_{25}e_{21}, e_{12}e_{13}e_{23}e_{22}, e_8e_9e_{19}e_{18},\\ &e_{13}e_{14}e_{24}e_{23}, e_4e_5e_{15}e_{14}, e_{14}e_{15}e_{25}e_{24}, e_6e_7e_2e_3e_{13}e_{11}\}. \end{aligned}$$

Moreover, we can easily check that if

$$\sum_{i=1}^{16} a_i C_i = \vec{0}, \tag{1}$$

then $a_i = 0$ for all $i = 1, 2, \ldots, 16$, where $C_1, C_2, \ldots, C_{16}$ are the vectors of the cycles given in $B(K_{5,5})$. This can be done by representing the system of homogeneous equations (1) as

$$AB = \vec{0}, \tag{2}$$

where A is a row matrix $(a_1a_2\ldots a_{16})$, $\vec{0}$ is a 1×25 zero matrix, and B is a 16×25 matrix which is the cycle matrix [4] of $K_{5,5}$ corresponding to the set of cycles $B(K_{5,5})$.

It is clear that $B = [b_{ij}]$, $b_{ij} = 1$ if the edge e_j is in the cycle C_i and zero otherwise.

By an algebraic method, or by computer, we show that B has rank 16 $(= \gamma)$. Thus $A = \vec{0}$ is the only solution for (2). This implies that $B(K_{5,5})$ is an independent set of 16 $(= \gamma(K_{5,5}))$ cycles.

Remark. This procedure is used also in Sections 2.2–2.7, 3.1–3.4 and 4, to show that $B()$ are independent sets of cycles.

2.2. A 3-fold basis for $\mathcal{C}(K_{5,6})$.

Schmeichel [7] proved that

$$B(K_{4,n}) = \begin{cases} x_1y_ix_2y_{i+1} & \text{for } i=1,2,\dots,n-1,\\ x_3y_ix_4y_{i+1} & \text{for } i=1,2,\dots,n-1,\\ x_1y_{2i-1}x_3y_{2i} & \text{for } i=1,2,\dots,\lfloor\frac{n}{2}\rfloor,\\ x_2y_{2i}x_4y_{2i+1} & \text{for } i=1,2,\dots,\lfloor\frac{n-1}{2}\rfloor \end{cases}$$

is a 3-fold basis for $\mathcal{C}(K_{4,n})$.

We want to choose a subset S of $B(K_{4,6})$ such that it is possible to find a set T of $\gamma(K_{5,6})-|S|$ cycles each of length 4 constructed from the edges incident at x_5 together with the edges of $K_{4,6}$ which are of fold less than 3 in S, and such that the fold in $S\cup T$ of each edge of $K_{5,6}$ does not exceed 3. Then, we test the set $S\cup T$ for independence. If $S\cup T$ is independent, then it is a 3-fold basis for $\mathcal{C}(K_{5,6})$, otherwise we consider another S and T, and so on.

The procedure that may be followed to obtain a suitable S is: start taking $S=B(K_{4,6})$, if it does not lead to the required T, take S to be a subset obtained from $B(K_{4,6})$ by omitting one cycle, then by omitting two cycles, and so on, until a suitable S which leads to the required T is obtained.

Following this procedure, we obtain

$$\begin{aligned} S &= B(K_{4,6}) - \{x_1y_2x_2y_3, x_3y_4x_4y_5, x_2y_4x_4y_5\}\\ &= \{x_1y_1x_2y_2, x_1y_3x_2y_4, x_1y_4x_2y_5, x_1y_5x_2y_6, x_3y_1x_4y_2, x_3y_2x_4y_3,\\ &\qquad x_3y_3x_4y_4, x_3y_5x_4y_6, x_1y_1x_3y_2, x_1y_3x_3y_4, x_1y_5x_3y_6, x_2y_2x_4y_3\},\\ T &= \{x_1y_6x_2y_1, x_3y_6x_4y_1, x_5y_1x_2y_2, x_5y_2x_1y_3,\\ &\qquad x_5y_3x_2y_6, x_5y_4x_3y_5, x_5y_5x_4y_6, x_5y_1x_4y_4\}. \end{aligned}$$

Now, we take $B(K_{5,6}) = S\cup T$.

To show that the cycles of $B(K_{5,6})$ are independent, we write them in terms of the edges,

$$\begin{aligned} B(K_{5,6}) = \{&e_1e_2e_7e_6, e_{11}e_{12}e_{17}e_{16}, e_{16}e_{17}e_{22}e_{21}, e_{21}e_{22}e_{27}e_{26}, e_3e_4e_9e_8, e_8e_9e_{14}e_{13},\\ &e_{13}e_{14}e_{19}e_{18}, e_{23}e_{24}e_{29}e_{28}, e_1e_3e_8e_6, e_{11}e_{13}e_{18}e_{16}, e_{21}e_{23}e_{28}e_{26},\\ &e_7e_9e_{14}e_{12}, e_{26}e_{27}e_2e_1, e_{28}e_{29}e_4e_3, e_5e_2e_7e_{10}, e_{10}e_6e_{11}e_{15},\\ &e_{15}e_{12}e_{27}e_{30}, e_{20}e_{18}e_{23}e_{25}, e_{25}e_{24}e_{29}e_{30}, e_5e_4e_{19}e_{20}\}, \end{aligned}$$

and prove that the vectors of these 20 cycles are linearly independent by an algebraic method. Since $\gamma(K_{5,6}) = 20$, it follows that $B(K_{5,6})$ is indeed a basis for $\mathcal{C}(K_{5,6})$. It is a simple matter to verify that it is a 3-fold basis.

2.3. A 3-fold basis for $\mathcal{C}(K_{5,7})$.

To find a 3-fold basis $B(K_{5,7})$ we follow the procedure mentioned in 2.2, i.e., we find a subset S of $B(K_{4,7})$ and a set T of cycles constructed from the edges incident at vertex x_5 together with the edges of $K_{4,7}$ whose fold in S is less than 3 and such that

$$|S| + |T| = \gamma(K_{5,7}),$$

and the fold in $S \cup T$ of each edge of $K_{5,7}$ does not exceed 3.

It is found that

$$\begin{aligned} S &= B(K_{4,7}) - \{x_1y_2x_2y_3, x_3y_3x_4y_4, x_2y_4x_4y_5, x_1y_5x_3y_6\} \\ &= \{x_1y_1x_2y_2, x_1y_3x_2y_4, x_1y_4x_2y_5, x_1y_5x_2y_6, x_1y_6x_2y_7, x_3y_1x_4y_2, x_3y_2x_4y_3, \\ &\qquad x_3y_4x_4y_5, x_3y_5x_4y_6, x_3y_6x_4y_7, x_1y_1x_3y_2, x_1y_3x_3y_4, x_2y_2x_4y_3, x_2y_6x_4y_7\}, \\ T &= \{x_1y_1x_3y_7, x_2y_1x_4y_7, x_5y_1x_2y_2, x_5y_2x_1y_3, x_5y_3x_3y_6, \\ &\qquad x_5y_6x_1y_7, x_5y_7x_3y_4, x_5y_4x_2y_5, x_5y_5x_4y_1, x_5y_3x_4y_4\}. \end{aligned}$$

Writing the cycles of $S \cup T$ in terms of their edges, we arrive at

$$\begin{aligned} S \cup T = \{&e_1e_2e_7e_6, e_{11}e_{12}e_{17}e_{16}, e_{16}e_{17}e_{22}e_{21}, e_{21}e_{22}e_{27}e_{26}, \\ &e_{26}e_{27}e_{32}e_{31}, e_3e_4e_9e_8, e_8e_9e_{14}e_{13}, e_{18}e_{19}e_{24}e_{23}, \\ &e_{23}e_{24}e_{29}e_{28}, e_{28}e_{29}e_{34}e_{33}, e_1e_3e_8e_6, e_{11}e_{13}e_{18}e_{16}, \\ &e_7e_9e_{14}e_{12}, e_{27}e_{29}e_{34}e_{32}, e_1e_3e_{33}e_{31}, e_2e_4e_{34}e_{32}, \\ &e_5e_2e_7e_{10}, e_{10}e_6e_{11}e_{15}, e_{15}e_{13}e_{28}e_{30}, e_{30}e_{26}e_{31}e_{35}, \\ &e_{35}e_{33}e_{18}e_{20}, e_{20}e_{17}e_{22}e_{25}, e_{25}e_{24}e_4e_5, e_{15}e_{14}e_{19}e_{20}\}. \end{aligned}$$

We can easily show that the cycles of $S \cup T$ are independent. Since

$$|S \cup T| = 24 = \gamma(K_{5,7}),$$

it follows that $B(K_{5,7}) = S \cup T$ is indeed a 3-fold basis for $\mathcal{C}(K_{5,7})$.

2.4. A 3-fold basis for $\mathcal{C}(K_{5,8})$.

To find a 3-fold basis for $\mathcal{C}(K_{5,8})$, we choose S from the basis $B(K_{5,7})$ which was obtained in 2.3. Following the procedure given in 2.2, we obtain

$$\begin{aligned} S &= B(K_{5,7}) - \{x_1y_1x_2y_2, x_3y_6x_4y_7, x_1y_3x_3y_4\} \\ &= \{x_1y_3x_2y_4, x_1y_4x_2y_5, x_1y_5x_2y_6, x_1y_6x_2y_7, x_3y_1x_4y_2, x_3y_2x_4y_3, x_3y_4x_4y_5, \\ &\quad x_3y_5x_4y_6, x_1y_1x_3y_2, x_2y_2x_4y_3, x_2y_6x_4y_7, x_1y_1x_3y_7, x_2y_1x_4y_7, x_5y_1x_2y_2, \\ &\quad x_5y_2x_1y_3, x_5y_3x_3y_6, x_5y_6x_1y_7, x_5y_7x_3y_4, x_5y_4x_2y_5x_5y_5x_4y_1, x_5y_3x_4y_4\}, \\ T &= \{x_1y_1x_2y_8, x_2y_2x_5y_8, x_5y_5x_1y_8, x_3y_7x_4y_8, x_4y_4x_3y_8, x_1y_3x_3y_8, x_5y_6x_4y_8\}. \end{aligned}$$

Notice that the cycles of T are constructed from the edges incident at the vertex y_8 together with the edges of $K_{5,7}$ which are of fold less than 3 in S.

We can show that

$$\begin{aligned} B(K_{5,8}) &= S \cup T \\ &= \{e_{11}e_{12}e_{17}e_{16}, e_{16}e_{17}e_{22}e_{21}, e_{21}e_{22}e_{27}e_{26}, e_{26}e_{27}e_{32}e_{31}, \\ &\quad e_3e_4e_9e_8, e_8e_9e_{14}e_{13}, e_{18}e_{19}e_{24}e_{23}, e_{23}e_{24}e_{29}e_{28}, e_1e_3e_8e_6, \\ &\quad e_7e_9e_{14}e_{12}, e_{27}e_{29}e_{34}e_{32}, e_1e_3e_{33}e_{31}, e_2e_4e_{34}e_{32}, e_5e_2e_7e_{10}, \\ &\quad e_{10}e_6e_{11}e_{15}, e_{15}e_{13}e_{28}e_{30}, e_{30}e_{26}e_{31}e_{35}, e_{35}e_{33}e_{18}e_{20}, e_{20}e_{17}e_{22}e_{25}, \\ &\quad e_{25}e_{24}e_4e_5, e_{15}e_{14}e_{19}e_{20}, e_1e_2e_{37}e_{36}, e_7e_{10}e_{40}e_{37}, e_{25}e_{21}e_{36}e_{40}, \\ &\quad e_{33}e_{34}e_{39}e_{38}, e_{19}e_{18}e_{38}e_{39}, e_{11}e_{13}e_{38}e_{36}, e_{30}e_{29}e_{39}e_{40}\} \end{aligned}$$

is a 3-fold basis for $\mathcal{C}(K_{5,8})$ by proving that these 28 $(= \gamma(K_{5,8}))$ cycles are independent.

2.5. A 3-fold basis for $\mathcal{C}(K_{6,6})$.

To find a 3-fold basis for $\mathcal{C}(K_{6,6})$, we choose a suitable subset S from the basis $B(K_{5,6})$ which is obtained in 2.3. Following the procedure mentioned in 2.2, we obtain

$$\begin{aligned} S &= B(K_{5,6}) - \{x_1y_3x_2y_4, x_3y_1x_4y_2, x_1y_5x_3y_6, x_2y_2x_4y_3\} \\ &= \{x_1y_1x_2y_2, x_1y_4x_2y_5, x_1y_5x_2y_6, x_3y_2x_4y_3, x_3y_3x_4y_4, x_3y_5x_4y_6, \\ &\quad x_1y_1x_3y_2, x_1y_3x_3y_4, x_1y_6x_2y_1, x_3y_6x_4y_1, x_5y_1x_2y_2, x_5y_2x_1y_3, \\ &\quad x_5y_3x_2y_6, x_5y_4x_3y_5, x_5y_5x_4y_6, x_5y_1x_4y_4\}, \\ T &= \{x_6y_1x_5y_2, x_6y_2x_2y_4, x_6y_4x_4y_3, x_6y_3x_2y_5, x_6y_5x_5y_6, \\ &\quad x_6y_2x_4y_5, x_1y_3x_6y_6, x_2y_3x_5y_4, x_3y_1x_6y_6\}. \end{aligned}$$

Notice that the cycles of T are constructed from the edges incident at x_6 together with the edges of $K_{5,6}$ which are of fold less than 3 in S.

We can show that

$$\begin{aligned} B(K_{6,6}) &= S \cup T \\ &= \{e_1e_2e_8e_7, e_{19}e_{20}e_{26}e_{25}, e_{25}e_{26}e_{32}e_{31}, e_9e_{10}e_{16}e_{15}, e_{15}e_{16}e_{22}e_{21}, \\ &\quad e_{27}e_{28}e_{34}e_{33}, e_1e_3e_9e_7, e_{13}e_{15}e_{21}e_{19}, e_{31}e_{32}e_2e_1, e_{33}e_{34}e_4e_3, \\ &\quad e_5e_2e_8e_{11}, e_{11}e_7e_{13}e_{17}, e_{17}e_{14}e_{32}e_{35}, e_{23}e_{21}e_{27}e_{29}, e_{29}e_{28}e_{34}e_{35}, \\ &\quad e_5e_4e_{22}e_{23}, e_6e_5e_{11}e_{12}, e_{12}e_8e_{20}e_{24}, e_{24}e_{22}e_{16}e_{18}, e_{18}e_{14}e_{26}e_{30}, \\ &\quad e_{30}e_{29}e_{35}e_{36}, e_{12}e_{10}e_{28}e_{30}, e_{13}e_{18}e_{36}e_{31}, e_{14}e_{17}e_{23}e_{20}, e_3e_6e_{36}e_{33}\} \end{aligned}$$

is a 3-fold basis for $\mathcal{C}(K_{6,6})$ by proving that these 25 $(= \gamma(K_{6,6}))$ cycles are independent.

2.6. A 3-fold basis for $\mathcal{C}(K_{6,7})$.

As for $K_{6,6}$, we obtain $B(K_{6,7})$ by choosing

$$\begin{aligned} S &= B(K_{6,6}) - \{x_1y_3x_6y_6, x_2y_3x_5y_4, x_3y_1x_6y_6\} \\ &= \{x_1y_1x_2y_2, x_1y_4x_2y_5, x_1y_5x_2y_6, x_3y_2x_4y_3, x_3y_3x_4y_4, x_3y_5x_4y_6, x_1y_1x_3y_2, \\ &\quad x_1y_3x_3y_4, x_1y_6x_2y_1, x_3y_6x_4y_1, x_5y_1x_2y_2, x_5y_2x_1y_3, x_5y_3x_2y_6, x_5y_4x_3y_5, \\ &\quad x_5y_5x_4y_6, x_5y_1x_4y_4, x_6y_1x_5y_2, x_6y_2x_2y_4, x_6y_4x_4y_3, x_6y_3x_2y_5, \\ &\quad x_6y_5x_5y_6, x_6y_2x_4y_5\}, \\ T &= \{x_1y_4x_6y_6, x_6y_1x_3y_7, x_5y_3x_1y_7, x_4y_1x_6y_7, x_3y_2x_4y_7, \\ &\quad x_2y_3x_6y_7, x_1y_5x_3y_7, x_2y_4x_5y_7\}, \end{aligned}$$

where the cycles of T are constructed from the edges incident at y_7 together with the edges of $K_{6,6}$ whose fold in S is less than 3. Then it is shown that

$$\begin{aligned} B(K_{6,7}) &= S \cup T \\ &= \{e_1e_2e_8e_7, e_{19}e_{20}e_{26}e_{25}, e_{25}e_{26}e_{32}e_{31}, e_9e_{10}e_{16}e_{15}, e_{15}e_{16}e_{22}e_{21}, \\ &\quad e_{27}e_{28}e_{34}e_{33}, e_1e_3e_9e_7, e_{13}e_{15}e_{21}e_{19}, e_{31}e_{32}e_2e_1, e_{33}e_{34}e_4e_3, \\ &\quad e_5e_2e_8e_{11}, e_{11}e_7e_{13}e_{17}, e_{17}e_{14}e_{32}e_{35}, e_{23}e_{21}e_{27}e_{29}, e_{29}e_{28}e_{34}e_{35}, \\ &\quad e_5e_4e_{22}e_{23}, e_6e_5e_{11}e_{12}, e_{12}e_8e_{20}e_{24}, e_{24}e_{22}e_{16}e_{18}, \\ &\quad e_{18}e_{14}e_{26}e_{30}, e_{30}e_{29}e_{35}e_{36}, e_{12}e_{10}e_{28}e_{30}, e_{19}e_{24}e_{36}e_{31}, \\ &\quad e_6e_3e_{39}e_{32}, e_{17}e_{13}e_{37}e_{41}, e_4e_6e_{42}e_{40}, e_9e_{10}e_{40}e_{39}, \\ &\quad e_{14}e_{18}e_{42}e_{38}, e_{25}e_{27}e_{39}e_{37}, e_{20}e_{23}e_{41}e_{38}\} \end{aligned}$$

is a 3-fold basis for $\mathcal{C}(K_{6,7})$ by proving that the 30 $(= \gamma(K_{6,7}))$ cycles of $B(K_{6,7})$ are independent.

2.7. A 3-fold basis for $\mathcal{C}(K_{6,8})$.

To find a 3-fold basis for $\mathcal{C}(K_{6,8})$, we start from $B(K_{6,7})$ and follow the procedure given in 2.2 to obtain

$$\begin{aligned} S &= B(K_{6,7}) - \{x_1y_1x_2y_2, x_3y_3x_4y_4, x_6y_5x_5y_6\} \\ &= \{x_1y_4x_2y_5, x_1y_5x_2y_6, x_3y_2x_4y_3, x_3y_5x_4y_6, x_1y_1x_3y_2, \\ &\quad x_1y_3x_3y_4, x_1y_6x_2y_1, x_3y_6x_4y_1, x_5y_1x_2y_2, x_5y_2x_1y_3, \\ &\quad x_5y_3x_2y_6, x_5y_4x_3y_5, x_5y_5x_4y_6, x_5y_1x_4y_4, x_6y_1x_5y_2, \\ &\quad x_6y_2x_2y_4, x_6y_4x_4y_3, x_6y_3x_2y_5, x_6y_2x_4y_5, x_1y_4x_6y_6, \\ &\quad x_6y_1x_3y_7, x_5y_3x_1y_7, x_4y_1x_6y_7, x_3y_2x_4y_7, x_2y_3x_6y_7, \\ &\quad x_1y_5x_3y_7, x_2y_4x_5y_7\}, \\ T &= \{x_5y_6x_6y_8, x_5y_5x_6y_8, x_4y_4x_3y_8, x_4y_3x_3y_8, x_2y_2x_1y_8, \\ &\quad x_2y_1x_1y_8, x_5y_7x_1y_8, x_6y_6x_3y_8\}, \end{aligned}$$

$$\begin{aligned} B(K_{6,8}) &= S \cup T \\ &= \{e_{19}e_{20}e_{26}e_{25}, e_{25}e_{26}e_{32}e_{31}, e_9e_{10}e_{16}e_{15}, \\ &\quad e_{27}e_{28}e_{34}e_{33}, e_1e_3e_9e_7, e_{13}e_{15}e_{21}e_{19}, e_{31}e_{32}e_2e_1, \\ &\quad e_{33}e_{34}e_4e_3, e_5e_2e_8e_{11}, e_{11}e_7e_{13}e_{17}, e_{17}e_{14}e_{32}e_{35}, \\ &\quad e_{23}e_{21}e_{27}e_{29}, e_{29}e_{28}e_{34}e_{35}, e_5e_4e_{22}e_{23}, e_6e_5e_{11}e_{12}, \\ &\quad e_{12}e_8e_{20}e_{24}, e_{24}e_{22}e_{16}e_{18}, e_{18}e_{14}e_{26}e_{30}, e_{12}e_{10}e_{28}e_{30}, \\ &\quad e_{19}e_{24}e_{36}e_{31}, e_6e_3e_{39}e_{42}, e_{17}e_{13}e_{37}e_{41}, e_4e_6e_{42}e_{40}, \\ &\quad e_9e_{10}e_{40}e_{39}, e_{14}e_{18}e_{42}e_{38}, e_{25}e_{27}e_{39}e_{37}, e_{20}e_{23}e_{41}e_{38}, \\ &\quad e_{35}e_{36}e_{48}e_{47}, e_{29}e_{30}e_{48}e_{47}, e_{22}e_{21}e_{45}e_{46}, e_{16}e_{15}e_{45}e_{46}, \\ &\quad e_8e_7e_{43}e_{44}, e_2e_1e_{43}e_{44}, e_{41}e_{37}e_{43}e_{47}, e_{36}e_{33}e_{45}e_{48}\} \end{aligned}$$

where the cycles of T are constructed from the edges incident at y_8 together with the edges of $K_{6,7}$ which are of fold less than 3 in S. Then it is shown that $B(K_{6,8})$ is a 3-fold basis for $\mathcal{C}(K_{6,8})$ by proving that the 35 $(= \gamma(K_{6,8}))$ cycles are independent.

Now the proof of the following statement has been completed.

Theorem 2.1. *The basis number of $K_{5,n}$ and $K_{6,n}$ for $n = 5, 6, 7, 8$ is 3.*

3. The basis number of some r-cages

We start this section with a simple result concerning the basis number of cubic graphs.

Let G_c be a cubic graph with n vertices and m edges. Then $m = 3n/2$ and

$$\gamma(G_c) = \frac{3n}{2} - n + 1 = \frac{n}{2} + 1.$$

From Theorem 1.2 we have

$$n \cdot \left\lfloor \frac{3b(G_c)}{2} \right\rfloor \geqslant g(G_c) \cdot \left(\frac{n}{2} + 1\right),$$

where $g(G_c)$ is the girth of G_c. We consider two cases:

Case (I). If $b(G_c)$ is even, then

$$b(G_c) \geqslant \frac{1}{3} g(G_c) \cdot \left(1 + \frac{2}{n}\right). \tag{3}$$

Case (II). If $b(G_c)$ is odd, then

$$b(G_c) \geqslant \frac{1}{3} g(G_c) \cdot \left(1 + \frac{2}{n}\right) + \frac{1}{3}. \tag{4}$$

If $b(G_c) = 2$, then from (1) we get

$$g(G_c) \leqslant \frac{6n}{n+2} < 6.$$

Thus we have the following statement.

Corollary 3.1. *Each cubic graph of girth more than 5 is non-planar.*

If $b(G_c) = 3$, then (2) implies

$$g(G_c) \leqslant \frac{8n}{n+2}.$$

Since the girth is an integer, then $g(G_c) \leqslant 7$ when $b(G_c) = 3$. Therefore, if $g(G_c) \geqslant 8$, then $b(G_c) \geqslant 4$.

Hence, the proof of the following theorem is completed:

Theorem 3.1. *If G_c is a cubic graph of girth not less than 8, then $b(G_c) \geqslant 4$.*

It is mentioned in Section 1 that, for $r = 3, 4, \ldots, 8$, there is a unique r-cage. Since the 3-cage is K_4 and the 4-cage is $K_{3,3}$, we have $b(\text{3-cage}) = 2$ and $b(\text{4-cage}) = 3$.

To determine the basis number of the r-cage for $r = 5, 6, 7, 8$, we divide the remaining part of this section into four subsections.

3.1. The basis number of Petersen graph.

The 5-cage is the graph shown in Fig. 1. It is called Petersen graph, and will be denoted by G_P.

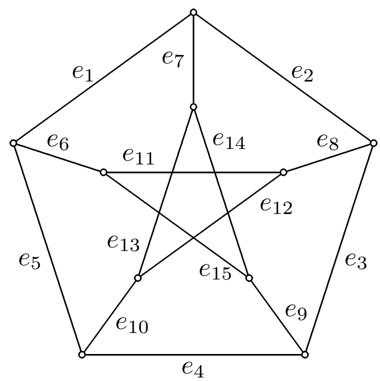

Figure 1. Petersen graph

It is clear that G_P is a non-planar graph, therefore, by Theorem 1.1, $b(G_P) \geqslant 3$.

To find a 3-fold basis for $\mathcal{C}(G_P)$, consider the set of cycles of G_P:

$$B(G_P) = \{e_1e_6e_{15}e_{14}e_{10}, e_2e_7e_{12}e_{11}e_6, e_3e_8e_{14}e_{13}e_7, e_4e_9e_{11}e_{15}e_8, e_5e_{10}e_{13}e_{12}e_9, e_1e_2e_7e_{13}e_{10}\}.$$

We can easily show that the vectors representing the cycles of $B(G_P)$ are linearly independent in $\mathcal{C}(G_P)$. Since

$$|B(G_P)| = 6 = \gamma(G_P) = \dim(\mathcal{C}(G_P)),$$

$B(G_P)$ is a basis for $\mathcal{C}(G_P)$. Moreover, one can easily check that the fold of each edge of G_P in the basis $B(G_P)$ is not more than 3. Thus $b(G_P) \leqslant 3$, and so the basis number of the Petersen graph is 3.

3.2. The basis number of Heawood graph.

The 6-cage is the graph shown in Fig. 2. It is called the Heawood graph, and will be denoted by G_H.

Since G_H is a cubic graph of girth 6, G_H is non-planar by Corollary 3.1 and so $b(G_H) \geqslant 3$.

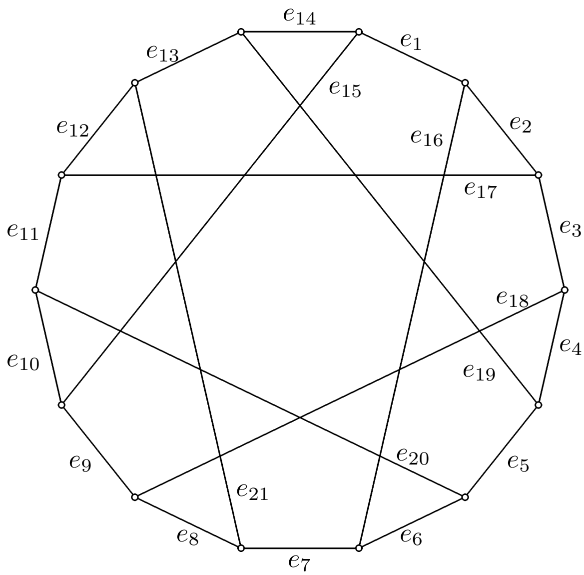

Figure 2. Heawood graph

To find a 3-fold basis for $\mathcal{C}(G_H)$, consider the set of cycles of G_H:

$$B(G_H) = \{e_1e_{16}e_7e_8e_9e_{15}, e_3e_{18}e_9e_{10}e_{11}e_{17}, e_5e_{20}e_{11}e_{12}e_{13}e_{19},\\ e_7e_{21}e_{13}e_{14}e_1e_{16}, e_9e_{15}e_1e_2e_3e_{18}, e_{11}e_{17}e_3e_4e_5e_{20},\\ e_{13}e_{19}e_5e_6e_7e_{21}, e_{14}e_{19}e_4e_{18}e_8e_{21}e_{12}e_{17}e_2e_{16}e_6e_{20}e_{10}e_{15}\}.$$

We can easily show that the vectors representing the cycles of $B(G_H)$ are linearly independent in $\mathcal{C}(G_H)$. Since

$$|B(G_H)| = 8 = \gamma(G_H) = \dim(\mathcal{C}(G_H)),$$

then $B(G_H)$ is a basis for $\mathcal{C}(G_H)$. Moreover, one can easily check that the fold of each edge of G_H in the basis $B(G_H)$ is not more than 3. Thus $b(G_H) \leqslant 3$, and so the basis number of the Heawood graph is 3.

3.3. The basis number of McGee graph.

The 7-cage is the graph shown in Fig. 3. It is called the McGee graph, and will be denoted by G_M.

By Corollary 3.1, G_M is a non-planar graph, and so $b(G_M) \geqslant 3$.

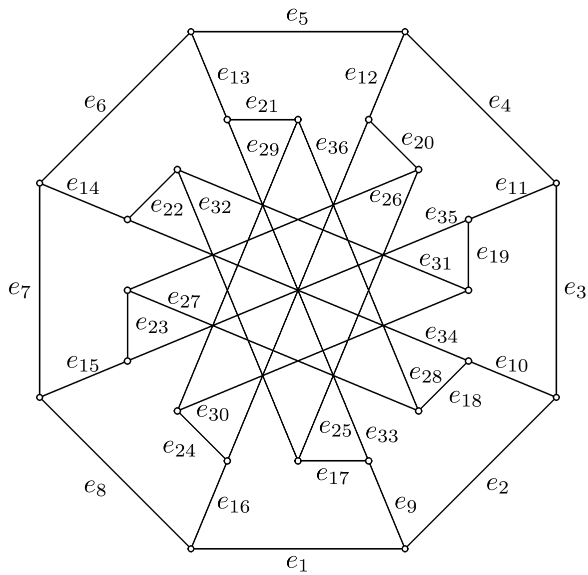

Figure 3. McGee graph

To find a 3-fold basis for $\mathcal{C}(G_M)$, consider the set of cycles of G_M:

$$\begin{aligned} B(G_M) = \{&e_{33}e_9e_2e_3e_4e_5e_{13}, e_{34}e_{10}e_3e_4e_5e_6e_{14}, e_{35}e_{11}e_3e_2e_1e_8e_{15},\\ &e_{36}e_{12}e_5e_6e_7e_8e_{16}, e_9e_1e_{16}e_{24}e_{29}e_{21}e_{33}, e_{16}e_8e_{15}e_{23}e_{26}e_{20}e_{36},\\ &e_{15}e_7e_{14}e_{22}e_{31}e_{19}e_{35}, e_{14}e_6e_{13}e_{21}e_{28}e_{18}e_{34}, e_9e_2e_{10}e_{18}e_{27}e_{26}e_{25}e_{17},\\ &e_{33}e_{17}e_{32}e_{31}e_{30}e_{29}e_{21}, e_{34}e_{18}e_{27}e_{26}e_{25}e_{32}e_{22}, e_4e_{11}e_{19}e_{31}e_{32}e_{25}e_{20}e_{12},\\ &e_{35}e_{19}e_{30}e_{29}e_{28}e_{27}e_{23}\}.\end{aligned}$$

One can easily show that the vectors which represent the cycles of $B(G_M)$ are linearly independent in the vector space $\mathcal{C}(G_M)$. Since

$$|B(G_M)| = 13 = \gamma(G_M) = \dim(\mathcal{C}(G_M)),$$

$B(G_M)$ is a basis for $\mathcal{C}(G_M)$. Moreover, one can easily check that this basis is a 3-fold basis. Thus $b(G_M) \leqslant 3$.

Hence the basis number of the McGee graph is 3.

3.4. The basis number of Levi graph.

The Levi graph is the 8-cage which is shown in Fig. 4. It will be denoted by G_L.

From Theorem 3.1, $b(G_L) \geqslant 4$.

Thus, to prove that $b(G_L) = 4$, we shall give a 4-fold basis for $\mathcal{C}(G_L)$.

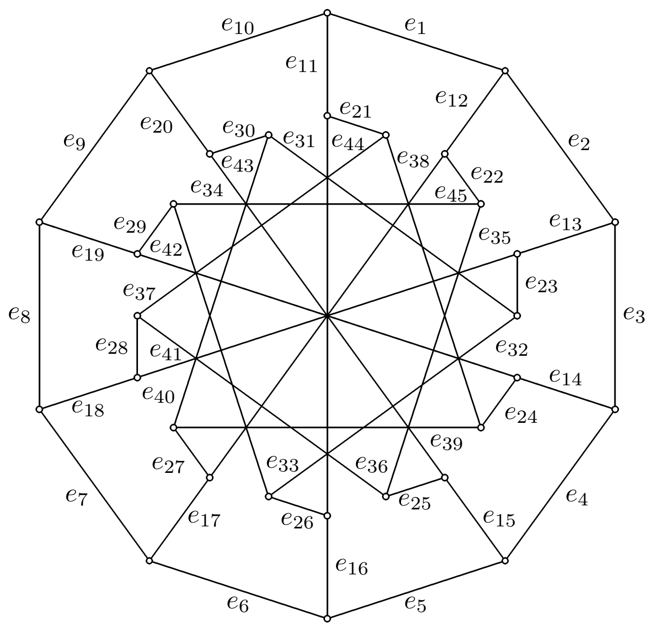

Figure 4. Levi graph

Consider the set of cycles of G_L:

$$\begin{aligned} B(G_L) = \{ & e_1e_{11}e_{21}e_{37}e_{36}e_{35}e_{22}e_{12}, e_2e_{12}e_{22}e_{34}e_{33}e_{32}e_{23}e_{13}, \\ & e_3e_{13}e_{23}e_{31}e_{40}e_{39}e_{24}e_{14}, e_4e_{14}e_{24}e_{38}e_{37}e_{36}e_{25}e_{15}, \\ & e_5e_{15}e_{25}e_{35}e_{34}e_{33}e_{26}e_{16}, e_6e_{16}e_{26}e_{32}e_{31}e_{40}e_{27}e_{17}, \\ & e_7e_{17}e_{27}e_{39}e_{38}e_{37}e_{28}e_{18}, e_8e_{18}e_{28}e_{36}e_{35}e_{34}e_{29}e_{19}, \\ & e_9e_{19}e_{29}e_{33}e_{32}e_{31}e_{30}e_{20}, e_{10}e_{20}e_{30}e_{40}e_{39}e_{38}e_{21}e_{11}, \\ & e_{44}e_{16}e_6e_7e_8e_9e_{10}e_{11}, e_{45}e_{22}e_{35}e_{36}e_{28}e_{18}e_7e_{17}, \\ & e_{41}e_{28}e_{37}e_{38}e_{24}e_{14}e_3e_{13}, e_{42}e_{29}e_{33}e_{32}e_{23}e_{13}e_3e_{14}, \\ & e_{43}e_{15}e_5e_6e_7e_8e_9e_{20}, e_{44}e_{11}e_1e_2e_3e_4e_5e_{16}\}. \end{aligned}$$

Using a simple algebraic method, we prove that the vectors of the cycles of $B(G_L)$ are linearly independent in the vector space $\mathcal{C}(G_L)$. Since

$$|B(G_L)| = 16 = \gamma(G_L) = \dim(\mathcal{C}(G_L)),$$

$B(G_L)$ is a basis for $\mathcal{C}(G_L)$. Moreover, one can easily check that the fold of each edge of G_L in this basis $B(G_L)$ is not more than 4. Therefore, $b(G_L) \leqslant 4$.

Hence the basis number of the Levi graph is 4.

Now, we summarize the results of the three subsections in the following theorem.

Theorem 3.2. *The basis number of an r-cage for $r = 4, 5, 6, 7$ is 3. The basis number of the 8-cage is 4.*

4. The basis number of Robertson graph

It is mentioned in Section 1 that the Robertson graph, which is shown in Fig. 5 and denoted by G_R, is the only smallest graph of girth 5 and valency 4.

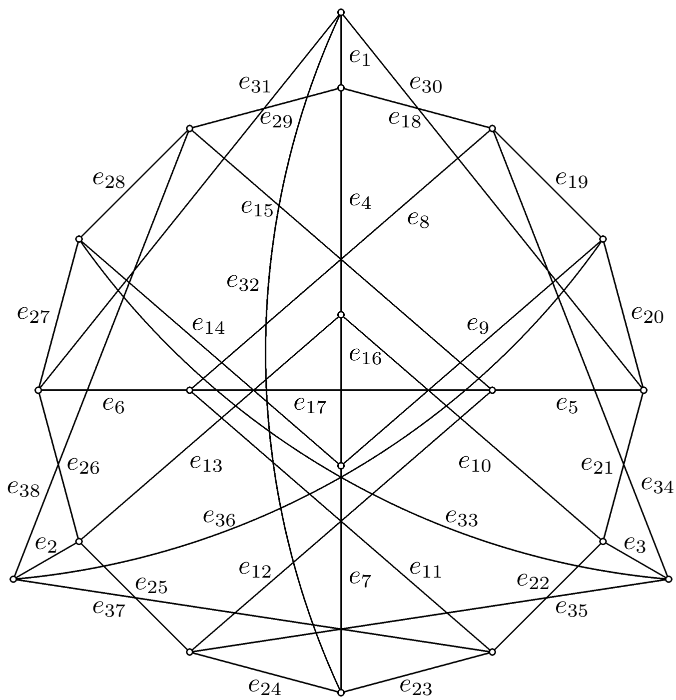

Figure 5. Robertson graph

It is clear that G_R has 19 vertices and 38 edges. Thus $\gamma(G_R) = 20$.

By Theorem 1.2,

$$19\left\lfloor \frac{b(G_R)\cdot 4}{2} \right\rfloor \geqslant (5)(20),$$

that is,

$$b(G_R) \geqslant \frac{50}{19}.$$

This means that $b(G_R) \geqslant 3$.

To prove that the basis number of G_R is 3, we form a 3-fold basis for $\mathcal{C}(G_R)$.

Consider the set of cycles of G_R:

$$B(G_R) = \{e_{33}e_{27}e_6e_8e_{34}, e_{34}e_{19}e_9e_{14}e_{33}, e_{20}e_{21}e_{22}e_{37}e_{36}, e_{37}e_{23}e_7e_{16}e_{13}e_3, e_{37}e_{11}e_6e_{26}e_3, e_{35}e_{12}e_5e_{21}e_2, e_1e_4e_{10}e_{22}e_{23}e_{32}, e_{31}e_6e_{17}e_5e_{30}, e_{29}e_{18}e_8e_{17}e_{15}, e_{24}e_{23}e_{11}e_{17}e_{12}, e_{20}e_{21}e_{10}e_{16}e_9, e_{26}e_{27}e_{14}e_{16}e_{13}, e_{25}e_{13}e_{10}e_2e_{35}, e_{28}e_{14}e_9e_{36}e_{38}, e_{24}e_{25}e_{26}e_{31}e_{32}, e_1e_{18}e_{19}e_{20}e_{30}, e_{29}e_{18}e_{19}e_{36}e_{38}, e_{12}e_{15}e_{28}e_{33}e_{35}, e_8e_{11}e_{22}e_2e_{34}, e_1e_{29}e_{28}e_{27}e_{31}\}.$$

We can easily show that the vectors of the cycles of $B(G_R)$ are linearly independent in the vector space $\mathcal{C}(G_R)$. Since

$$|B(G_R)| = 20 = \gamma(G_R) = \dim(\mathcal{C}(G_R)),$$

$B(G_R)$ is a basis for $\mathcal{C}(G_R)$. Moreover, one can easily check that the fold of each edge of G_R in the basis $B(G_R)$ is not more than 3. Thus $b(G_R) \leqslant 3$. This completes the proof of the following theorem.

Theorem 4.1. *The basis number of the Robertson graph is* 3.

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