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OSCILLATION OF FORCED NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS OF FIRST ORDER

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Abstract. Necessary and sufficient conditions are obtained for every solution of

$$\Delta(y_n + p_n y_{n-m}) \pm q_n G(y_{n-k}) = f_n$$

to oscillate or tend to zero as $n \to \infty$, where p_n, q_n and f_n are sequences of real numbers such that $q_n \geqslant 0$. Different ranges for p_n are considered.

Keywords: neutral difference equations, oscillation, nonoscillation, asymptotic behaviour

MSC 2000: 39A10, 39A12

1. Introduction

In this paper we study the oscillatory and asymptotic behaviour of solutions of a class of forced nonlinear neutral difference equations of first order with variable coefficients of the form

$$\Delta(y_n + p_n y_{n-m}) + q_n G(y_{n-k}) = f_n, \tag{1}$$

where Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, p_n, q_n and f_n $(n = 0, 1, 2, \ldots)$ are sequences of real numbers such that $q_n \geqslant 0$, $G \in C(\mathbb{R}, \mathbb{R})$ satisfies $xG(x) > 0$ for $x \neq 0$ and $m \geqslant 0$, $k \geqslant 0$. We assume

$$G(x) \text{ is nondecreasing and } \left|\sum_{n=0}^{\infty} f_n\right| < \infty. \tag{H$_1$}$$

We discuss the problem in various ranges of p_n, viz.

(i) $-1 < b_1 \leqslant p_n \leqslant 0$,
(ii) $0 \leqslant p_n \leqslant b_2 < 1$,
(iii) $1 < b_3 \leqslant p_n \leqslant b_4 < \infty$ and
(iv) $-\infty < b_5 \leqslant p_n \leqslant b_6 < -1$,

where b_i, $1 \leqslant i \leqslant 6$, is a constant. We have obtained conditions which are necessary and sufficient for every solution of (1) to be oscillatory or tend to zero as $n \to \infty$. Equation (1) is studied in Section 2. In section 3, the same problem is considered for

$$\Delta(y_n + p_n y_{n-m}) - q_n G(y_{n-k}) = f_n. \tag{2}$$

By a solution of (1) (or (2)) on $[0, \infty]$ we mean a sequence $\{y_n\}$ of real numbers which is defined for $n \geqslant -r$ and which satisfies (1) (or (2)) for $n = 0, 1, 2, \ldots$, where $r = \max\{k, m\}$. If

$$y_n = A_n \quad \text{for } n = -r, \ldots, 0 \tag{3}$$

are given, then (1) (or (2)) has a unique solution satisfying the initial conditions (3). A solution $\{y_n\}$ of (1) (or (2)) is said to be oscillatory if for every $N > 0$ there exists an $n \geqslant N$ such that $y_n y_{n+1} \leqslant 0$; otherwise, it is called nonoscillatory.

In recent years, several papers on oscillation of solutions of neutral delay difference equations have appeared (see [1]–[3], [5]–[7], [9], [10]). In [1], Cheng and Lin have provided a complete characterization of oscillation of solutions of

$$\Delta(y_n + p y_{n-m}) + q y_{n-k} = 0, \quad n = 0, 1, 2, \ldots \tag{4}$$

where p and q are real numbers, $m > 0$ and $k \geqslant 0$ are integers. Their study depends on the theory of envelopes and on the characteristic equation of (4). However, the method depending on the characteristic equation does not work for equations with variable coefficients. In [5], Lalli et al have considered oscillation of

$$\Delta(y_n + p y_{n-m}) + q_n y_{n-k} = 0,$$

where $q_n \geqslant 0$, and some of their results generalize the results in [2]. They have also considered the forced equation of the form

$$\Delta(y_n + p_n y_{n-m}) + q_n y_{n-k} = f_n.$$

However, there are examples to which their results cannot be applied but where our results hold. The method developed in this work is different from those in [3],

[5], [6]. Our work heavily depends on a lemma which may be regarded as the discrete analogue of Lemma 1.5.2 in [4]. It seems that not much work has been done on equations of the form (1). In [9], Thandapani et al have considered m-th order nonlinear equations of neutral type. However, equation (1) or (2) does not follow from those equations for $m = 1$ due to their assumptions on the nonlinear term $F(n, u)$. Our assumptions cannot always be compared with those in [3], [5], [6] because the approaches are different. However, some of our results extend the results in [3], [5], [6]. In an earlier work [7], we have studied (1) with $p_n = p$. Equations (1) and (2) may be regarded as a discrete analogoue of

$$(y(t) + p(t)y(t-\tau))' \pm q(t)G(y(t-\delta)) = f(t).$$

Oscillatory and asymptotic behaviour of solutions of such equations are studied in [8] with help of Lemma 1.5.2 in [4].

2. Oscillation of solutions of equation (1)

In this section we obtain necessary and sufficient conditions for every solution of (1) to be oscillatory or tending to zero as $n \to \infty$. The following lemma, which may be regarded as the discrete analogue of Lemma 1.5.2 in [4], plays a key role in this work. For completeness, its proof is given.

Lemma 2.1. *Let $\{f_n\}$, $\{g_n\}$ and $\{p_n\}$ be sequences of real numbers defined for $n \geqslant n_0 \geqslant 0$ such that*

$$f_n = g_n + p_n g_{n-m}, \quad n \geqslant n_0 + m, \tag{5}$$

where $m \geqslant 0$ is an integer. Suppose that there exist real numbers b_1, b_2, b_3, b_4 such that p_n is in one of the following ranges:

(I) $-\infty < b_1 \leqslant p_n \leqslant 0$,

(II) $0 \leqslant p_n \leqslant b_2 < 1$ *or*

(III) $1 < b_3 \leqslant p_n \leqslant b_4 < \infty$.

If $g_n > 0$ for $n \geqslant n_0$, $\liminf\limits_{n\to\infty} g_n = 0$ and $\lim\limits_{n\to\infty} f_n = L$ exists, then $L = 0$.

Proof. We may write (5) as

$$f_{n+m} - f_n = g_{n+m} + (p_{n+m} - 1)g_n - p_n g_{n-m}, \quad n \geqslant n_0 + m. \tag{6}$$

Since $\liminf\limits_{n\to\infty} g_n = 0$, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $\lim\limits_{k\to\infty} g_{n_k} = 0$. Suppose that (I) holds. As the sequence $\{p_{n_k+m} - 1\}$ is bounded, we have

$\lim\limits_{k\to\infty}(p_{n_k+m}-1)g_{n_k}=0$ and hence (6) yields that

$$\lim_{k\to\infty}[g_{n_k+m}-p_{n_k}g_{n_k-m}]=0.$$

Since $g_{n_k+m}>0$ for large k, we have $\lim\limits_{k\to\infty}p_{n_k}g_{n_k-m}=0$. From (5) it follows that

$$L=\lim_{k\to\infty}f_{n_k}=\lim_{k\to\infty}[g_{n_k}+p_{n_k}g_{n_k-m}]=0.$$

Next suppose that (II) holds. Replacing n by n_k-m in (6) and then taking limit as $k\to\infty$, we obtain

$$\lim_{k\to\infty}[(1-p_{n_k})g_{n_k-m}+p_{n_k-m}g_{n_k-2m}]=0.$$

Since $1-b_2>0$, we have

$$0\leqslant(1-b_2)\liminf_{k\to\infty}g_{n_k-m}\leqslant\liminf_{k\to\infty}[(1-p_{n_k})g_{n_k-m}+p_{n_k-m}g_{n_k-2m}]=0$$

and

$$0\leqslant(1-b_2)\limsup_{k\to\infty}g_{n_k-m}\leqslant\limsup_{k\to\infty}[(1-p_{n_k})g_{n_k-m}+p_{n_k-m}g_{n_k-2m}]=0.$$

Hence $\lim\limits_{k\to\infty}g_{n_k-m}=0$. From (5) we get

$$L=\lim_{k\to\infty}f_{n_k}=\lim_{k\to\infty}[g_{n_k}+p_{n_k}g_{n_k-m}]=0.$$

Finally, let (III) hold. Putting n_k+m in place of n in (6) and letting $k\to\infty$, one obtains

$$\lim_{k\to\infty}[g_{n_k+2m}+(p_{n_k+2m}-1)g_{n_k+m}-p_{n_k+m}g_{n_k}]=0.$$

As the sequence $\{p_{n_k+m}\}$ is bounded, we have

$$\lim_{k\to\infty}[g_{n_k+2m}+(p_{n_k+2m}-1)g_{n_k+m}]=0.$$

Since $g_{n_k+2m}>0$ for large k and $\{p_{n_k+2m}-1\}$ is a positive bounded sequence, we conclude that $\lim\limits_{k\to\infty}g_{n_k+m}=0$. Thus from (5) we obtain

$$L=\lim_{k\to\infty}f_{n_k+m}=\lim_{k\to\infty}[g_{n_k+m}+p_{n_k+m}g_{n_k}]=0.$$

Hence the lemma is proved. □

Corollary 2.2. *Suppose that the conditions of Lemma* 2.1 *hold. If* $g_n < 0$ *for* $n \geqslant n_0$, $\limsup\limits_{n\to\infty} g_n = 0$ *and* $\lim\limits_{n\to\infty} f_n = L$ *exists, then* $L = 0$.

P r o o f. Setting $h_n = -g_n$ for $n \geqslant n_0$, we get $-f_n = h_n + p_n h_{n-m}$, $h_n > 0$ for $n \geqslant n_0$ and $\liminf\limits_{n\to\infty} h_n = 0$. The conclusion follows from Lemma 2.1. □

Theorem 2.3. *Let* $-1 < b_1 \leqslant p_n \leqslant 0$ *and let* (H_1) *hold. Every solution of equation* (1) *oscillates or tends to zero as* $n \to \infty$ *if and only if*

$$(\mathrm{H}_2) \qquad \sum_{n=0}^{\infty} q_n = \infty.$$

P r o o f. Suppose the (H_2) holds. Let $\{y_n\}$ be a solution of (1) on $[0, \infty)$. If $\{y_n\}$ is oscillatory, then there is nothing to prove. Suppose that $\{y_n\}$ is nonoscillatory. Hence there exists $N_1 > 0$ such that $y_n < 0$ or > 0 for $n \geqslant N_1$. We show that $\lim\limits_{n\to\infty} y_n = 0$ in either case. Let $y_n < 0$ for $n \geqslant N_1$. Setting

$$z_n = y_n + p_n y_{n-m} \tag{7}$$

and

$$w_n = z_n - \sum_{i=0}^{n-1} f_i \tag{8}$$

for $n \geqslant N_1 + m$, we obtain

$$\Delta w_n = -q_n G(y_{n-k}) \geqslant 0 \tag{9}$$

for $n \geqslant N_1 + m + k$. Hence there exists $N_2 > N_1 + m + k$ such that $w_n > 0$ or < 0 for $n \geqslant N_2$. Let $w_n > 0$ for $n \geqslant N_2$. We claim that $\{y_n\}$ is bounded. If not, then there is a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $n_j \to \infty$ and $y_{n_j} \to -\infty$ as $j \to \infty$ and

$$y_{n_j} = \min\{y_n \colon\ N_2 \leqslant n \leqslant n_j\}.$$

We may choose n_j sufficiently large so that $n_j - m > N_2$ and hence

$$w_{n_j} = y_{n_j} + p_{n_j} y_{n_j-m} - \sum_{i=0}^{n_j-1} f_i \leqslant (1+p_{n_j}) y_{n_j} - \sum_{i=0}^{n_j-1} f_i \leqslant (1+b_1) y_{n_j} - \sum_{i=0}^{n_j-1} f_i. \tag{10}$$

Thus $w_{n_j} < 0$ for large n_j, a contradiction. Thus our claim holds and hence $\{w_n\}$ is bounded. Consequently, $\lim\limits_{n\to\infty} w_n$ exists. If $\limsup\limits_{n\to\infty} y_n = \alpha$, $-\infty < \alpha < 0$, then there exists $\beta < 0$ such that $y_n < \beta$ for $n \geqslant N_3 > N_2$. From (9) we get

$$\sum_{n=N_3+k}^{r-1} q_n G(y_{n-k}) = -\sum_{n=N_3+k}^{r-1} \Delta w_n = w_{N_3+K} - w_r \geqslant -w_r,$$

which implies that

$$\sum_{n=N_3+k}^{\infty} q_n G(y_{n-k}) > -\infty.$$

However,

$$\sum_{n=N_3+k}^{\infty} q_n G(y_{n-k}) < G(\beta) \sum_{n=N_3+k}^{\infty} q_n = -\infty$$

by (H_2), a contradiction. Hence $\limsup\limits_{n\to\infty} y_n = 0$. As $\lim\limits_{n\to\infty} z_n$ exists, Corollary 2.2 implies that $\lim\limits_{n\to\infty} z_n = 0$. Next suppose that $w_n < 0$ for $n \geqslant N_2$. Hence $\lim\limits_{n\to\infty} w_n$ exists. If $\{y_n\}$ is unbounded, then proceeding as above we obtain from (10) that $\lim\limits_{j\to\infty} w_{n_j} = -\infty$, a contradiction. Thus $\{y_n\}$ is bounded and hence $\limsup\limits_{n\to\infty} y_n$ exists. Proceeding as above we may show that $\limsup\limits_{n\to\infty} y_n = 0$. Since $\lim\limits_{n\to\infty} z_n$ exists, we have $\lim\limits_{n\to\infty} z_n = 0$ by Corollary 2.2. Hence in either case $w_n > 0$ or < 0 for $n \geqslant N_2$, we have $\limsup\limits_{n\to\infty} y_n = 0$ and $\lim\limits_{n\to\infty} z_n = 0$. As $z_n \leqslant y_n + b_1 y_{n-m}$ for $n \geqslant N_2$, we infer that

$$\begin{aligned} 0 = \liminf_{n\to\infty} z_n \leqslant \liminf_{n\to\infty} [y_n + b_1 y_{n-m}] &\leqslant \liminf_{n\to\infty} y_n + \limsup_{n\to\infty} (b_1 y_{n-m}) \\ &= \liminf_{n\to\infty} y_n + b_1 \liminf_{n\to\infty} y_{n-m} \\ &= (1 + b_1) \liminf_{n\to\infty} y_n \end{aligned}$$

implies that $\liminf\limits_{n\to\infty} y_n = 0$. Hence $\lim\limits_{n\to\infty} y_n = 0$. Suppose that $y_n > 0$ for $n \geqslant N_1$. Setting $\tilde{y}_n = -y_n$, the sequence $\{\tilde{y}_n\}$ is a solution of

$$\Delta(\tilde{y}_n + p_n \tilde{y}_{n-m}) + q_n \tilde{G}(\tilde{y}_{n-k}) = \tilde{f}_n \tag{11}$$

where $\tilde{f}_n = -f_n$ and $\tilde{G}(y) = -G(-y)$. As all conditions of the theorem are satisfied for (11), $\lim\limits_{n\to\infty} \tilde{y}_n = 0$ and hence $\lim\limits_{n\to\infty} y_n = 0$.

For the proof of the necessity part of the theorem, we assume that

$$\sum_{n=0}^{\infty} q_n < \infty$$

and show that (1) admits a positive solution $\{y_n\}$ such that $\liminf\limits_{n\to\infty} y_n > 0$. It is possible to choose an integer $N > 0$ such that

$$\left| \sum_{n=N}^{\infty} f_n \right| < \frac{1 + b_1}{10} \text{ and } G(1) \sum_{n=N}^{\infty} q_n < \frac{1 + b_1}{5}, \text{ beacuse } \lim_{n\to\infty} \sum_{i=n}^{\infty} f_i = 0. \tag{12}$$

Let $X = \ell_\infty^N$ be the Banach space of all real bounded sequences $x = \{x_n\}$ with the sup norm

$$\|x\| = \sup\{|x_n|\colon\ n \geqslant N\}.$$

Let $K = \{x \in X\colon\ x_n \geqslant 0 \text{ for } n \geqslant N\}$. For $x, y \in X$ we define $x \leqslant y$ if and only if $y - x \in K$. Thus X is a partially ordered Banach space. Let

$$W = \Big\{x \in X\colon\ \frac{1+b_1}{10} \leqslant x_n \leqslant 1,\ n \geqslant N\Big\}.$$

If $x^0 = \{x_n^0\}$, where $x_n^0 = \frac{1}{10}(1+b_1)$ for $n \geqslant N$, then $x^0 = \inf W$ and $x^0 \in W$. Let W^* be a nonempty subset of W. The supremum of W^* is the sequence $x^* = \{x_n^*\colon\ n \geqslant N\}$, where $x_n^* = \sup\{x_n\colon\ x = \{x_i\colon\ i \geqslant N\} \in W^*\}$. Clearly, $x^* \in W$. For $y \in W$, we define

$$(Ty)_n = \begin{cases} (Ty)_{N+r}, & N \leqslant n \leqslant N+r, \\ -p_n y_{n-m} + \dfrac{1+b_1}{5} + \displaystyle\sum_{i=n}^{\infty} q_i G(y_{i-k}) - \sum_{i=n}^{\infty} f_i, & n \geqslant N+r, \end{cases}$$

where $r = \max\{k, m\}$. Hence using (12) we obtain, for $n \geqslant N$,

$$(Ty)_n \leqslant -b_1 + \frac{2(1+b_1)}{5} + \frac{1+b_1}{10} < 1$$

and

$$(Ty)_n \geqslant \frac{1+b_1}{5} - \frac{1+b_1}{10} = \frac{1+b_1}{10}.$$

Thus $T\colon\ W \to W$. Clearly, for $x, y \in W$, $x \leqslant y$ implies that $Tx \leqslant Ty$. Hence T has a fixed point in W by the Knaster-Tarski fixed point theorem (see Theorem 1.7.3 in [4]). If $y = \{y_n\} \in W$ is this fixed point of T, then

$$y_n = \begin{cases} y_{N+r}, & N \leqslant n \leqslant N+r, \\ -p_n y_{n-m} + \dfrac{1+b_1}{5} + \displaystyle\sum_{i=n}^{\infty} q_i G(y_{i-k}) - \sum_{i=n}^{\infty} f_i, & n \geqslant N+r. \end{cases}$$

Hence y is a positive solution of (1) with $\liminf\limits_{n\to\infty} y_n \geqslant \frac{1}{10}(1+b_1) > 0$. Thus the theorem is proved. $\square$

Remark. Theorem 2.3 extends Theorem 3.4 in [5] and Lemma 11.4.4 in [4].

Example. Consider

$$\Delta\Big(y_n + \Big(\mathrm{e}^{-(n+1)} - \frac{1}{2}\Big)y_{n-1}\Big) + \frac{1}{2}\mathrm{e}^{-6}(3\mathrm{e}^{2n} + 2\mathrm{e}^{n})y_{n-2}^3$$
$$= \mathrm{e}^{-2(n+1)} + \Big(\mathrm{e}^{-1} + \frac{1}{2}\mathrm{e}\Big)\mathrm{e}^{-n}, \quad n \geqslant 0.$$

As all conditions of Theorem 2.3 are satisfied, every nonoscillatory solution of the equation tends to zero as $n \to \infty$. In particular, $y = \{\mathrm{e}^{-n}\}$ is a positive solution of the equation with $y_n \to 0$ as $n \to \infty$.

Theorem 2.4. *Let* $0 \leqslant p_n \leqslant b_2 < 1$ *and let* (H_1) *hold.*

(i) *If* (H_2) *holds, then every solution of* (1) *oscillates or tends to zero as* $n \to \infty$.

(ii) *Suppose that G satisfies the Lipschitz condition on intervals of the form* $[a, b]$, $0 < a < b < \infty$. *If every solution of equation* (1) *oscillates or tends to zero as* $n \to \infty$, *then* (H_2) *holds.*

Proof. (i) Assume that (H_2) holds. Let $y = \{y_n\}$ be a nonoscilatory solution of (1) on $[0, \infty)$. Hence $y_n > 0$ or < 0 for $n \geqslant N_1 > 0$. We show that $\lim_{n\to\infty} y_n = 0$ in either case. Let $y_n < 0$ for $n \geqslant N_1$. Setting z_n and w_n for $n \geqslant N_1 + m$ as in (7) and (8) respectively, we obtain (9) for $n \geqslant N_1 + m + k$. Then $w_n > 0$ or < 0 for $n \geqslant N_2 > N_1 + k + m$. Let $w_n > 0$ for $n \geqslant N_2$. If $\{y_n\}$ is unbounded, then there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $n_j \to \infty$ and $y_{n_j} \to -\infty$ as $j \to \infty$. Chossing n_j sufficiently large so that $n_j - m > N_2$, we get

$$w_{n_j} = y_{n_j} + p_{n_j} y_{n_j - m} - \sum_{i=0}^{n_j - 1} f_i < y_{n_j} - \sum_{i=0}^{n_j - 1} f_i.$$

Hence $w_{n_j} < 0$ for large n_j, a contradiction. Thus $\{y_n\}$ is bounded. This implies that $\{w_n\}$ is bounded and hence $\lim_{n\to\infty} w_n$ exists. Proceeding as in Theorem 2.3 we obtain $\limsup_{n\to\infty} y_n = 0$. As $\lim_{n\to\infty} z_n$ exists, from Corollary 2.2 it follows that $\lim_{n\to\infty} z_n = 0$. Next let $w_n < 0$ for for $n \geqslant N_2$. Hence $\lim_{n\to\infty} w_n$ exists. Consequently, $\lim_{n\to\infty} z_n$ exists. Since $z_n \leqslant y_n$ for $n \geqslant N_2$, the sequence $\{y_n\}$ is bounded. One may proceed as in Theorem 2.3 to show that $\limsup_{n\to\infty} y_n = 0$ and hence $\lim_{n\to\infty} z_n = 0$ by Corollary 2.2. Thus in either case $w_n > 0$ of $w_n < 0$ we obtain $\limsup_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} z_n = 0$. Further, $z_n \leqslant y_n$ for $n \geqslant N_2$ implies that $\liminf_{n\to\infty} y_n = 0$ and hence $\lim_{n\to\infty} y_n = 0$. If $y_n > 0$ for $n \geqslant N_1$, then we may proceed as above to obtain $\lim_{n\to\infty} y_n = 0$. Thus the proof of part (i) is complete.

(ii) We assume that

$$\sum_{n=0}^{\infty} q_n < \infty.$$

We may choose $N > 0$ sufficiently large such that

$$\left| \sum_{n=N}^{\infty} f_n \right| < \frac{1 - b_2}{10} \quad \text{and} \quad L \sum_{n=N}^{\infty} q_n < \frac{1 - b_2}{5},$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $[\frac{1}{10}(1-b_2), 1]$. Let $X = \ell_\infty^N$ and

$$S = \Big\{x \in X\colon\ \frac{1-b_2}{10} \leqslant x_n \leqslant 1,\ \ n \geqslant N\Big\}.$$

Since S is a closed subset of X, we infer that S is a complete metric space, where the metric is induced by the norm on X. For $y \in S$, define

$$(Ty)_n = \begin{cases} (Ty)_{N+r}, & N \leqslant n \leqslant N+r,\\ -p_n y_{n-m} + \dfrac{1+4b_2}{5} + \displaystyle\sum_{i=n}^{\infty} q_i G(y_{i-k}) - \sum_{i=n}^{\infty} f_i, & n \geqslant N+r. \end{cases}$$

Clearly, for $n \geqslant N$,

$$(Ty)_n < \frac{1+4b_2}{5} + L\sum_{i=N}^{\infty} q_i + \frac{1-b_2}{10} < \frac{1+4b_2}{5} + \frac{1-b_2}{5} + \frac{1-b_2}{10} < 1$$

and

$$(Ty)_n >\ -b_2 + \frac{1+4b_2}{5} - \frac{1-b_2}{10} = \frac{1-b_2}{10}$$

imply that $T\colon\ S \to S$. Further, for $u, v \in S$ and $n \geqslant N+r$,

$$|(Tu)_n - (Tv)_n| \leqslant b_2\|u-v\| + \frac{1-b_2}{5}\|u-v\| = \mu\|u-v\|$$

implies that

$$\|Tu - Tv\| \leqslant \mu\|u-v\|,$$

where $0 < \mu = b_2 + \frac{1}{5}(1-b_2) = \frac{1}{5}(1+4b_2) < 1$. Thus T is a contraction and hence it has a unique fixed point $y = \{y_n\}$ in S. Clearly, y is a positive solution of (1) with $\liminf\limits_{n\to\infty} y_n > 0$. Thus part (ii) of the theorem is proved. □

Corollary 2.5. *Let* $0 \leqslant p_n \leqslant b_2 < 1$ *and let* (H_1) *hold. Suppose that* G *satisfies the Lipschitz condition on intervals of the form* $[a, b]$, $0 < a < b < \infty$. *Then every solution of Eq.* (1) *oscillates or tends to zero as* $n \to \infty$ *if and only if* (H_2) *holds.*

Example. Consider

$$\begin{aligned}\Delta\Big[y_n + \Big(\frac{1}{2} + \mathrm{e}^{-(n+1)}\Big)y_{n-1}\Big] + \frac{\mathrm{e}^{-3}}{2}(\mathrm{e}^{2n+1} + \mathrm{e}^{2n} + 2\mathrm{e}^{n})y_{n-1}^3\\ = \mathrm{e}^{-2(n+1)} + \mathrm{e}^{-(n+1)}, \quad n \geqslant 1.\end{aligned}$$

From Corollary 2.5 it follows that every nonoscillatory solution of the equation tends to zero as $n \to \infty$. In particular, $\{\mathrm{e}^{-n}\}$ is such a solution. However, Theorem 6.1 in [5] cannot be applied to this example since

$$F_n = \frac{1-\mathrm{e}^{-2n}}{\mathrm{e}^2-1} + \frac{1-\mathrm{e}^{-n}}{\mathrm{e}-1} > 0$$

implies that $F_n^- = 0$, where $F_n = \sum\limits_{i=0}^{n-1} f_i$ and $F_n^- = \max\{-F_n, 0\}$. Furhter, Theorem 6.2 in [5] fails to hold for this equation, because $\lim\limits_{n\to\infty} F_n$ exists finitely.

Theorem 2.6. (i) *If* $1 < b_3 \leqslant p_n \leqslant b_4 < \infty$ *and* (H_1) *and* (H_2) *hold, then every solution of* (1) *oscillates or tends to zero as* $n \to \infty$.

(ii) *If* $1 < b_3 \leqslant p_n \leqslant b_4 \leqslant \frac{1}{2}b_3^2$, (H_1) *holds, G satisfies Lipschitz condition on intervals of the form* $[a,b]$, $0 < a < b < \infty$ *and every solution of* (1) *oscillates or tends to zero, then* (H_2) *holds.*

P r o o f. The proof is similar to that of Theorem 2.4. However, in this case we choose N sufficiently large such that

$$\Bigl|\sum_{n=N}^{\infty} f_n\Bigr| < \frac{b_3-1}{8b_4} \quad \text{and} \quad L\sum_{n=N}^{\infty} q_n < \frac{b_3-1}{4b_3},$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $[\frac{b_3-1}{8b_3b_4}, 1]$. We set

$$S = \Bigl\{x \in X \colon \frac{b_3-1}{8b_3b_4} \leqslant x_n \leqslant 1, \quad n \geqslant N\Bigr\}$$

and define $T\colon S \to S$ by

$$(Ty)_n = \begin{cases} (Ty)_{N+r}, \quad N \leqslant n \leqslant N+r, \\ -\dfrac{1}{p_{n+m}} y_{n+m} + \dfrac{1}{p_{n+m}}\Bigl[\sum\limits_{i=n+m}^{\infty} q_i G(y_{i-k}) - \sum\limits_{i=n+m}^{\infty} f_i\Bigr] + \dfrac{2b_3^2+b_3-1}{4b_3p_{n+m}}, \\ \qquad n \geqslant N+r. \end{cases}$$

Thus, $1 < b_3 \leqslant p_n \leqslant b_4 < \frac{1}{2}b_3^2$ implies that

$$\begin{aligned} (Ty)_n &\leqslant \frac{b_3-1}{4b_3^2} + \frac{b_3-1}{8b_4b_3} + \frac{2b_3^2+b_3-1}{4b_3^2} \\ &\leqslant \frac{b_3-1}{4b_3^2} + \frac{b_3-1}{8b_3^2} + \frac{2b_3^2+b_3-1}{4b_3^2} = \frac{4b_3^2+5b_3-5}{8b_3^2} < 1 \end{aligned}$$

and

$$(Ty)_n \geqslant -\frac{1}{b_3} - \frac{b_3-1}{8b_4b_3} + \frac{2b_3^2+b_3-1}{4b_3b_4} = \frac{4b_3^2+b_3-8b_4-1}{8b_3b_4} > \frac{b_3-1}{8b_3b_4}.$$

Clearly, T is a contraction. $\square$

Theorem 2.7. *Let* $-\infty < b_5 \leqslant p_n \leqslant b_6 < -1$ *and let* (H_1) *hold.*

(i) *If*

$$(\mathrm{H}_3) \qquad \sum_{j=0}^{\infty} q_{n_j} = \infty \quad \text{for every subsequence } \{n_j\} \text{ of } \{n\},$$

then every solution of (1) *oscillates or tends to zero or tends to* $\pm\infty$ *as* $n \to \infty$.

(ii) *Suppose that* G *is Lipschitzian on every interval of the form* $[a,b]$, $0 < a < b < \infty$. *If every solution of* (1) *oscillates or tends to zero or tends to* $\pm\infty$ *as* $n \to \infty$, *then* (H_2) *holds.*

Proof. (i) Let (H_3) hold. Let $\{y_n\}$ be a nonoscillatory solution of (1) on $[0,\infty)$. Hence $y_n > 0$ or < 0 for $n \geqslant N_1 > 0$. Let $y_n < 0$ for $n \geqslant N_1$. Setting z_n and w_n for $n \geqslant N_1 + m$ as in (7) and (8) respectively, we get (9) for $n \geqslant N_1 + m + k$. Thus $w_n > 0$ or < 0 for $n \geqslant N_2 > N_1 + m + k$. Let $w_n > 0$ for $n \geqslant N_2$. If $\lambda = \lim\limits_{n\to\infty} w_n$, then $0 < \lambda \leqslant \infty$. Suppose that $0 < \lambda < \infty$. Then $\lim\limits_{n\to\infty} z_n$ exists. We claim that $\{y_n\}$ is bounded. If not, then there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $n_j \to \infty$ and $y_{n_j} \to -\infty$ as $j \to \infty$. Hence for every $M > 0$ there exists $N_3 > N_2$ such that $n_j \geqslant N_3$ implies $y_{n_j} < -M$. Let $N_4 \geqslant N_3 + k$. Hence

$$\sum_{n_j=N_4}^{\infty} q_{n_j} G(y_{n_j-k}) < G(-M) \sum_{n_j=N_4}^{\infty} q_{n_j} = -\infty$$

by (H_3). From (9) we get

$$\sum_{n_j=N_4}^{r-1} q_{n_j} G(y_{n_j-k}) = -\sum_{n_j=N_4}^{r-1} \Delta w_{n_j} = -w_r + w_{N_4} \geqslant -w_r,$$

which implies that

$$\sum_{n_j=N_4}^{\infty} q_{n_j} G(y_{n_j-k}) \geqslant -\lambda > -\infty,$$

a contradiction. Hence $\{y_n\}$ is bounded. Proceeding as in the proof of Theorem 2.3 we obtain $\limsup\limits_{n\to\infty} y_n = 0$. Hence $\lim\limits_{n\to\infty} z_n = 0$ by Corollary 2.2. Clearly,

$$\begin{aligned} 0 = \limsup_{n\to\infty} z_n &= \limsup_{n\to\infty} [y_n + p_n y_{n-m}] \\ &\geqslant \limsup_{n\to\infty} [y_n + b_6 y_{n-m}] \\ &\geqslant \liminf_{n\to\infty} y_n + \limsup_{n\to\infty} (b_6 y_{n-m}) \\ &= \liminf_{n\to\infty} y_n + b_6 \liminf_{n\to\infty} y_{n-m} = (1 + b_6) \liminf_{n\to\infty} y_n \end{aligned}$$

implies that $\liminf_{n\to\infty} y_n \geqslant 0$ since $1+b_6<0$. Then $\liminf_{n\to\infty} y_n = 0$. Consequently, $\lim_{n\to\infty} y_n = 0$. If $\lambda=\infty$, then $\lim_{n\to\infty} z_n = \infty$. Since $z_n < p_n y_{n-m} \leqslant b_5 y_{n-m}$, we have $\liminf_{n\to\infty}(b_5 y_{n-m}) = \infty$, that is, $\limsup_{n\to\infty} y_{n-m} = -\infty$. Hence $\lim_{n\to\infty} y_n = -\infty$. Suppose that $w_n<0$ for $n \geqslant N_2$. Then $\lim_{n\to\infty} w_n$ exists and hence $\lim_{n\to\infty} z_n$ exists. Proceeding as above we may show that $\lim_{n\to\infty} y_n = 0$. If $y_n>0$ for $n\geqslant N_1$, it may be shown similarly that $\lim_{n\to\infty} y_n = 0$ or $+\infty$. Thus part (i) of the theorem is proved.

We claim that (H_2) holds. If not, then

$$\sum_{n=0}^{\infty} q_n < \infty.$$

Choose

$$M > \max\Big\{-b_5, b_6 + \frac{b_6}{1+b_6}\Big\} \quad \text{and} \quad L = \frac{2M - b_6(M+1)}{(b_6-M)(b_6+1)} > 0.$$

It is possible to find $N>0$ sufficiently large such that

$$\Big|\sum_{n=N}^{\infty} f_n\Big| < \frac{-b_6}{M-b_6} \quad \text{and} \quad K\sum_{n=N}^{\infty} q_n < \frac{-b_6}{M-b_6},$$

where $K=\max\{K_1, G(L)\}$ and K_1 is the Lipschitz constant of G on $\big[\frac{-b_6}{M-b_6}, L\big]$. As usual we take $X=\ell_\infty^N$. We set

$$S = \Big\{x\in X\colon\ \frac{-b_6}{M-b_6} \leqslant x_n \leqslant L, \quad n\geqslant N\Big\}$$

and, for $y\in S$, define

$$(Ty)_n = \begin{cases} (Ty)_{N+r}, \quad N\leqslant n\leqslant N+r, \\ -\dfrac{1}{p_{n+m}} y_{n+m} - \dfrac{M(2-b_6)}{p_{n+m}(M-b_6)} + \dfrac{1}{p_{n+m}} \displaystyle\sum_{i=n+m}^{\infty} q_i G(y_{i-k}) \\ \qquad\qquad - \dfrac{1}{p_{n+m}} \displaystyle\sum_{i=n+m}^{\infty} f_i, \quad n\geqslant N+r. \end{cases}$$

Since, for $n\geqslant N$,

$$(Ty)_n \geqslant -\frac{M(2-b_6)}{b_5(M-b_6)} - \frac{2}{M-b_6} \geqslant \frac{-b_6}{M-b_6}$$

and

$$(Ty)_n \leqslant -\frac{L(M-b_6)+2M-b_6(M+1)}{b_6(M-b_6)} = L,$$

we have $T\colon S\to S$. It may be verified that T is a contraction. Thus the theorem is proved. $\square$

Remark. Clearly, (H_3) implies (H_2). It will be interesting to prove Theorem 2.7 (i) with (H_2) in place of (H_3). We may note that (H_2) does not imply (H_3). Indeed, considering

$$\Delta\Big[y_n + \Big(\frac{1}{n} - 3\Big) y_{n-1}\Big] + q_n y_{n-1}^3 = f_n, \quad n \geqslant 1,$$

where

$$q_n = (B_{n-1}/n^2) + n^2 B_n > 0,$$

$$B_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

and

$$f_n = \frac{1}{(n+2)^2} - \frac{4}{(n+1)^2} + \frac{1}{(n+1)^3} - \frac{1}{n^3} + \frac{3}{n^2} + \frac{B_{n-1}}{n^8} + \frac{B_n}{n^4}$$

we obtain

$$\sum_{n=1}^{\infty} q_n = \sum_{m=1}^{\infty} q_{2m} + \sum_{m=0}^{\infty} q_{2m+1} = \frac{1}{4}\sum_{m=1}^{\infty} \frac{1}{m^2} + \sum_{m=0}^{\infty} (2m+1)^2 = \infty.$$

However,

$$\sum_{m=1}^{\infty} q_{2m} = \frac{1}{4}\sum_{m=1}^{\infty} \frac{1}{m^2} < \infty.$$

Clearly, $\sum\limits_{n=1}^{\infty} |f_n| < \infty$, $-\infty < p < 1/n - 3 \leqslant -2 < -1$ for $p \leqslant -3$ and $\{y_n\} = \{1/(n+1)^2\}$ is a nonoscillatory solution of the equation with $y_n \to 0$ as $n \to \infty$. Thus this example strengthens our belief that Theorem 2.7 can be proved with (H_2) instead of (H_3) when $\Big|\sum\limits_{n=0}^{\infty} f_n\Big| < \infty$ is replaced by the stronger condition $\sum\limits_{n=0}^{\infty} |f_n| < \infty$. If

$$(\mathrm{H}_3') \qquad \liminf_{n\to\infty} q_n > 0,$$

then $(\mathrm{H}_3') \Rightarrow (\mathrm{H}_3) \Rightarrow (\mathrm{H}_2)$. But (H_3) does not imply (H_3'). Indeed, taking $q_n = 1/n$, we observe that $\liminf\limits_{n\to\infty} q_n = 0$ and $\sum\limits_{j=1}^{\infty} q_{n_j} = \infty$ for every subsequence $\{n_j\}$ of $\{n\}$.

Theorem 2.8. *Suppose that* $-\infty < b_5 \leqslant p_n \leqslant b_6 < -1$ *and* (H_1) *holds.*

(i) *If* (H_2) *holds, then every bounded solution of* (1) *oscillates or tends to zero as* $n \to \infty$.

(ii) *Let* G *be Lipschitzian on every interval of the form* $[a, b]$, $0 < a < b < \infty$. *If every bounded solution of* (1) *oscillates or tends to zero as* $n \to \infty$, *then* (H_2) *holds.*

Proof. The proof is similiar to that of Theorem 2.7 and hence is omitted. $\square$

Remark. Theorems 2.7 and 2.8 extend Theorem 4.3 in [5]. It seems that the forcing terms change substantially the qualitative behaviour of the solutions of the equation.

Example. Theorem 2.8 implies that every bounded solution of

$$\begin{aligned}\Delta[y_n + (\mathrm{e}^{-n} - 3)y_{n-2}] + \mathrm{e}^{-3}(3\mathrm{e}^{2n+1} + \mathrm{e}^{2n} + \mathrm{e}^{n+2})y_{n-1}^3 \\ = (\mathrm{e}^{-1} + 3\mathrm{e}^2)\mathrm{e}^{-n} + \mathrm{e}^{-2n}, \quad n \geqslant 1,\end{aligned}$$

oscillates or tends to zero as $n \to \infty$. Clearly, $\{\mathrm{e}^{-n}\}$ is a positive solution of the equation which tends to zero as $n \to \infty$.

Example. As all conditions of Theorem 2.7 are satisfied, every solution of

$$\begin{aligned}\Delta[y_n - (1 + \mathrm{e} + \mathrm{e}^{-2n})y_{n-1}] + (2\mathrm{e}^{-(2n-1)} + \mathrm{e}^{-(4n+1)} + e - 1)y_{n-1} \\ = \mathrm{e}^{-5n} + (2\mathrm{e}^2 + \mathrm{e})\mathrm{e}^{-3n} + (2\mathrm{e}^{-1} - \mathrm{e}^{-2} + 2\mathrm{e}^2 - \mathrm{e})\mathrm{e}^{-n}, \quad n \geqslant 1,\end{aligned}$$

oscillates or tends to zero or tends to $\pm\infty$ as $n \to \infty$. In particular, $\{\mathrm{e}^n + \mathrm{e}^{-n}\}$ is an unbounded positive solution of the equation which tends to $+\infty$ as $n \to \infty$.

Theorem 2.9. *Let $0 \leqslant p_n \leqslant b < \infty$. If*

$$(\mathrm{H}_4) \qquad \liminf_{n\to\infty} F_n = -\infty \quad \textit{and} \quad \limsup_{n\to\infty} F_n = +\infty,$$

where $F_n = \sum\limits_{i=0}^{n-1} f_i$, then every solution of (1) *oscillates.*

Proof. If possible, let $\{y_n\}$ be a nonoscillatory solution of (1) on $[0, \infty)$. Hence $y_n > 0$ or < 0 for $n \geqslant N_1 > 0$. Setting z_n as in (7) for $n \geqslant N_1 + m$, we obtain

$$\Delta(z_n - F_n) = -q_n G(y_{n-k}).$$

If $y_n > 0$ for $n \geqslant N_1$, then $z_n > 0$ and $\Delta(z_n - F_n) \leqslant 0$ for $n \geqslant N_1 + m + k$. Hence $z_n - F_n \leqslant 0$ or $\geqslant 0$ for $n \geqslant N_2 > N_1 + m + k$. However, $(z_n - F_n) \leqslant 0$ for $n \geqslant N_2$ implies that $0 < z_n \leqslant F_n$ and hence $\liminf\limits_{n\to\infty} F_n \geqslant 0$, a contradiction to (H_4). If $(z_n - F_n) \geqslant 0$ for $n \geqslant N_2$, then $\lim\limits_{n\to\infty}(z_n - F_n)$ exists. However,

$$z_n = (z_n - F_n) + F_n$$

implies that $\liminf\limits_{n\to\infty} z_n \leqslant \lim\limits_{n\to\infty}(z_n - F_n) + \liminf\limits_{n\to\infty} F_n < -\infty$, a contradiction to the fact that $z_n > 0$ for $n \geqslant N_2$. Hence $y_n < 0$ for $n \geqslant N_1$. Consequently, $z_n < 0$ and $\Delta(z_n - F_n) \geqslant 0$ for $n \geqslant N_1 + m + k$. If $z_n - F_n \geqslant 0$ for $n \geqslant N_2 > N_1 + m + k$, then $\limsup\limits_{n\to\infty} F_n \leqslant 0$, a contradiction to (H_4). Thus $z_n - F_n \leqslant 0$ for $n \geqslant N_2$ and $\lim\limits_{n\to\infty}(z_n - F_n)$ exists. Writing $z_n = (z_n - F_n) + F_n$, we obtain $\limsup\limits_{n\to\infty} z_n = \infty$, which contradicts the fact that $z_n < 0$ for $n \geqslant N_2$. Thus the theorem is proved. □

Example. Theorem 2.9 implies that all solutions of

$$\Delta[y_n + (1+\mathrm{e}^{-n})y_{n-1}] + (\mathrm{e}^{-(2n-2)} + 2\mathrm{e}^{-(3n-2)})y_{n-1}^3 = (-1)^{n+1}\mathrm{e}^{n+1}, \quad n \geqslant 1,$$

oscillate. In particular, $\{(-1)^n \mathrm{e}^n\}$ is an oscillatory solution of the equation. Here $F_n = (\mathrm{e}+1)^{-1}((-1)\mathrm{e}^{n+1} - \mathrm{e})$.

Remark. Theorem 2.9 extends Theorem 6.2 in [5].

3. Oscillation of solutions of equation (2)

Oscillation and asymptotic behaviour of solutions of equation (2) are studied in this section.

Theorem 3.1. *Let $-1 < b_1 \leqslant p_n \leqslant 0$ and let* (H_1) *hold.*

(i) *If* (H_2) *holds, then every solution of* (2) *oscillates or tends to zero or tends to $\pm\infty$ as $n \to \infty$.*

(ii) *Suppose that G satisfies the Lipschitz condition on intervals of the form $[a, b]$, $0 < a < b < \infty$. If every solution of* (2) *oscillates or tends to zero or $\pm\infty$ as $n \to \infty$, then* (H_2) *holds.*

Proof. (i) Let $\{y_n\}$ be a nonoscillatory solution of (2) on $[0, \infty)$. Then $y_n > 0$ or < 0 for $n \geqslant N_1 > 0$. Let $y_n < 0$ for $n \geqslant N_1$. Setting z_n and w_n for $n \geqslant N_1 + m$ as in (7) and (8), respectively, we obtain

$$\Delta w_n = q_n G(y_{n-k}) \leqslant 0$$

for $n \geqslant N_1 + m + k$. Hence $w_n > 0$ or < 0 for $n \geqslant N_2 > N_1 + m + k$. If $w_n > 0$ for $n \geqslant N_2$, then $\lim\limits_{n\to\infty} w_n$ exists and hence $\lim\limits_{n\to\infty} z_n$ exists. We claim that $\{y_n\}$ is bounded. Otherwise, there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $n_j \to \infty$ and $y_{n_j} \to -\infty$ as $j \to \infty$ and

$$y_{n_j} = \min\{y_n \colon N_2 \leqslant n \leqslant n_j\}.$$

Hence

$$w_{n_j} = y_{n_j} + p_{n_j} y_{n_j - m} - \sum_{i=0}^{n_j-1} f_i \leqslant (1 + p_{n_j}) y_{n_j} - \sum_{i=0}^{n_j-1} f_i \leqslant (1+b_1) y_{n_j} - \sum_{i=0}^{n_j-1} f_i$$

implies that $w_{n_j} < 0$ for large n_j, a contradiction. Thus the claim holds. Proceeding as in Theorem 2.3, one may show that $\limsup_{n\to\infty} y_n = 0$. Hence $\lim_{n\to\infty} z_n = 0$ by Corollary 2.2. Because $z_n \leqslant y_n + b_1 y_{n-m}$, we have

$$\begin{aligned}
0 = \liminf_{n\to\infty} z_n &\leqslant \liminf_{n\to\infty} (y_n + b_1 y_{n-m}) \\
&\leqslant \liminf_{n\to\infty} y_n + \limsup_{n\to\infty} (b_1 y_{n-m}) \\
&= \liminf_{n\to\infty} y_n + b_1 \liminf_{n\to\infty} y_{n-m} \\
&= (1 + b_1) \liminf_{n\to\infty} y_n
\end{aligned}$$

and hence $\liminf_{n\to\infty} y_n = 0$. Consequently, $\lim_{n\to\infty} y_n = 0$. Let $w_n < 0$ for $n \geqslant N_2$. If $\lambda = \lim_{n\to\infty} w_n$, then $-\infty \leqslant \lambda < 0$. Suppose that $-\infty < \lambda < 0$. Hence $\lim_{n\to\infty} z_n$ exists. Proceeding as above one may show that $\{y_n\}$ is bounded beacuse otherwise $\lim_{j\to\infty} w_{n_j} = -\infty$, a contradiction to the fact that $-\infty < \lambda < 0$. Consequently, $\lim_{n\to\infty} y_n = 0$. If $\lambda = -\infty$, then $\lim_{n\to\infty} z_n = -\infty$ and hence $z_n \geqslant y_n$ implies that $\lim_{n\to\infty} y_n = -\infty$. If $y_n > 0$ for $n \geqslant N_1$, then proceeding as above one may show that $\lim_{n\to\infty} y_n = 0$ or $+\infty$. Hence the first part of the theorem is proved.

The proof of the second part of the theorem is similar to that of Theorem 2.4 and hence it is omitted. □

Corollary 3.2. *If* $-1 < b_1 \leqslant p_n \leqslant 0$ *and* (H_1) *and* (H_2) *hold, then every bounded solution of* (2) *oscillates or tends to zero as* $n \to \infty$.

P r o o f. This follows from Theorem 3.1 (i). □

Remark. Corollary 3.2 extends Corollary 2.1 (v) in [6].

Theorem 3.3. (i) *Let* $0 \leqslant p_n \leqslant b_2 < 1$ *or* $1 < b_3 \leqslant p_n \leqslant b_4 < \infty$ *and let* (H_1) *hold. If* (H_2) *holds, then every solution* $\{y_n\}$ *of* (2) *oscillates or tends to zero as* $n \to \infty$ *or* $\limsup_{n\to\infty} |y_n| = \infty$.

(ii) *Let* $0 \leqslant p_n \leqslant b_2 < 1$ *and let* (H_1) *hold. If* G *satisfies the Lipschitz condition on intervals of the form* $[a, b]$, $0 < a < b < \infty$, *and if every solution* $\{y_n\}$ *of* (2) *oscillates or tends to zero as* $n \to \infty$ *or* $\limsup_{n\to\infty} |y_n| = \infty$, *then* (H_2) *holds.*

P r o o f. The proof is similar to that of Theorem 3.1 and hence it is omitted. □

Remark. The necessity part for the case $1 < b_3 \leqslant p_n \leqslant b_4 < \infty$ is true if $b_4 < b_3^2$. It is required to show that $T\colon\ S \to S$, where

$$S = \{x = \{x_n\} \in \ell_\infty^N\colon\ a \leqslant x_n \leqslant b,\ \ n \geqslant N\} \quad \text{for } y \in S,$$

$$(Ty)_n = \begin{cases} (Ty)_{N+r}, & N \leqslant n \leqslant N+r, \\ -\dfrac{y_{n+m}}{p_{n+m}} + \dfrac{\varrho}{p_{n+m}} - \dfrac{1}{p_{n+m}}\Big[\displaystyle\sum_{i=n+m}^{\infty} q_i G(y_{i-k}) + \sum_{i=n+m}^{\infty} f_i\Big], & n \geqslant N+r, \end{cases}$$

$$\Big|\sum_{n=N}^{\infty} f_n\Big| < \frac{b_3-1}{2},\ \ K\sum_{n=N}^{\infty} q_n < \frac{1}{2}(b_3-1),\ \ K = \max\{K_1, G(b)\},$$

$$\begin{aligned} &a = [2\varrho(b_3^2-b_4) + b_3b_4 + b_4 - 2b_3^2b_4]/2b_3^2b_4 > 0,\\ &b = (2\varrho + b_3 - 1)/2b_3,\\ &\varrho > [2b_3^2b_4 - b_4 + b_3b_4]/2(b_3^2-b_4) > 0 \end{aligned}$$

and K_1 is the Lipschitz constant of G on $[a,b]$. In Theorem 2.6, we have taken $1 < b_3 \leqslant p_n \leqslant b_4 \leqslant \frac{1}{2}b_3^2$ in order to prove that T is a self-mapping on S. This cannot be achieved if we assume $b_4 \leqslant b_3^2$. Indeed, in Theorem 2.6 we used the fact that $4b_3^2 - 8b_4 > 0$, that is, $b_4 < \frac{1}{2}b_3^2$.

Corollary 3.4. *If* $0 \leqslant p_n \leqslant b_2 < 1$ *or* $1 < b_3 \leqslant p_n \leqslant b_4 < \infty$ *and if* (H_1) *and* (H_2) *hold, then every bounded solution of* (2) *oscillates or tends to zero as* $n \to \infty$.

Example. Consider

$$\begin{aligned}\Delta\Big[y_n + \Big(\frac{1}{2} + \mathrm{e}^{-(n+1)}\Big)y_{n-1}\Big] &- \mathrm{e}^{-3}(\mathrm{e}^{2n-1} + \mathrm{e}^{n-2})y_{n-1}^3\\ &= -\Big(\mathrm{e}^{-2n} + \frac{1}{2}\mathrm{e}^{-n} + \frac{1}{2}\mathrm{e}^{-(n-1)}\Big), \quad n \geqslant 1.\end{aligned}$$

Theorem 2.4 in [6] cannot be applied to this example, beacuse $\lim\limits_{n\to\infty} F_n$ exists finitely, where

$$F_n = -\Big[\frac{\mathrm{e}^2(1-\mathrm{e}^{-2n})}{\mathrm{e}^2-1} + \frac{\mathrm{e}(1-\mathrm{e}^{-n})}{2(\mathrm{e}-1)} + \frac{\mathrm{e}^2(1-\mathrm{e}^{-n})}{2(\mathrm{e}-1)}\Big].$$

However, from Corollary 3.4 it follows that every bounded solution of the equation oscillates or tends to zero as $n \to \infty$. In particular, $\{\mathrm{e}^{-n}\}$ is a positive solution of the equation which tends to zero as $n \to \infty$.

Theorem 3.5. *If* $-\infty < b_5 \leqslant p_n \leqslant b_6 < -1$ *and* (H_1) *and* (H_3) *hold, then every solution of* (2) *oscillates or tends to zero or tends to* $\pm\infty$ *as* $n \to \infty$.

Proof. The proof is similar to that of Theorem 2.7 and hence it is omitted. □

Remark. The necessity part for the case $-\infty < b_5 \leqslant p_n \leqslant b_6 < -1$ holds similarly.

Theorem 3.6. *If* $-\infty < b_5 \leqslant p_n \leqslant b_6 < -1$ *and* (H_1) *and* (H_2) *hold, then every bounded solution of* (2) *oscillates or tends to zero as* $n \to \infty$.

P r o o f. The proof is similar to that of Theorem 2.8. □

Remark. Theorem 3.6 extends and improves Theorem 2.3 in [6].

Theorem 3.7. *If* $0 \leqslant p_n \leqslant b < \infty$ *and* (H_4) *holds, then bounded solutions of* (2) *oscillate.*

P r o o f. If possible, let $\{y_n\}$ be a bounded nonoscillatory solution of (2) on $[0, \infty)$. Hence $y_n > 0$ or < 0 for $n \geqslant N_1 > 0$. Setting z_n as in (7) for $n \geqslant N_1 + m$, we obtain

$$\Delta(z_n - F_n) = q_n G(y_{n-k}). \tag{14}$$

Let $y_n > 0$ for $n \geqslant N_1$. Hence $z_n > 0$ and $\Delta(z_n - F_n) \geqslant 0$ for $n \geqslant N_1 + m + k$. Then $\{z_n - F_n\}$ is nondecreasing for $n \geqslant N_2 > N_1 + m + k$. If $z_n - F_n < 0$ for $n \geqslant N_2$, then $0 < z_n < F_n$ leads to a contradiction because $\liminf\limits_{n\to\infty} F_n = -\infty$. Hence there exists $n^* \geqslant N_2$ such that $z_{n^*} - F_{n^*} \geqslant 0$. Consequently, $n \geqslant n^*$ implies that $z_n - F_n \geqslant 0$. Again this leads to a contradiction, since $\{z_n\}$ is bounded and $\limsup\limits_{n\to\infty} F_n = \infty$. Hence $y_n < 0$ for $n \geqslant N_1$. From (14) it follows that $\{z_n - F_n\}$ is nonincreasing and (7) yields $z_n < 0$ for $n \geqslant N_2 > N_1 + m + k$. Clearly, $z_n - F_n > 0$ for $n \geqslant N_2$ contradicts $\limsup\limits_{n\to\infty} F_n = +\infty$. Hence $z_{n^*} - F_{n^*} \leqslant 0$ for some $n^* \geqslant N_2$. Then $n \geqslant n^*$ implies that $z_n - F_n \leqslant 0$. Since $\{z_n\}$ is bounded, we have $\liminf\limits_{n\to\infty} F_n > -\infty$, a contradiction. Hence $\{y_n\}$ oscillates. This completes the proof of the theorem. □

Remark. Theorem 3.7 extends Theorem 2.4 in [6].

Example. Theorem 3.7 implies that every bounded solution of the equation

$$\Delta\Big(y_n + \frac{1}{n} y_{n-2}\Big) - \Big(3 + \frac{1}{n+1} + \frac{1}{n} + 2n\Big) y_{n-1}^3 = (-1)^n (2n+1), \quad n \geqslant 1$$

oscillates. In particular, $\{(-1)^n\}$ is an oscillatory solution of the equation. Here $F_n = (-1)^{n-1} n$.

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