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ON 2-HOMOGENEITY OF MONOUNARY ALGEBRAS

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Abstract. Fraïssé introduced the notion of a k-set-homogeneous relational structure. In the present paper the following classes of monounary algebras are described: $\mathscr{S}h_2(S)$, $\mathscr{S}h_2(S^c)$, $\mathscr{S}h_2(P^c)$ —the class of all algebras which are 2-set-homogeneous with respect to subalgebras, connected subalgebras, connected partial subalgebras, respectively, and $\mathscr{H}_2(S)$, $\mathscr{H}_2(S^c)$, $\mathscr{H}_2(P^c)$ —the class of all algebras which are 2-homogeneous with respect to subalgebras, connected subalgebras, connected partial subalgebras, respectively.

Keywords: monounary algebra, homogeneous, 2-homogeneous, 2-set-homogeneous

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Several authors have investigated the notion of homogeneity for various types of algebraic structures (cf. [13], [11], [8], [1], [2], [3], [12]).

Further, Fraïssé [7] introduced the notion of a k-set-homogeneous relational structure (where k is a positive integer). Some questions on k-homogeneous and k-set-homogeneous graphs have been studied by Droste, Giraudet, Macpherson, Sauer [4]–[6]; their main results concern the cases $k = 1, 2, 3$.

Homogeneous monounary algebras were investigated in [9]. 1-homogeneous monounary algebras were characterized in [10].

We will apply the following definition (cf. [1]): An algebra A will be called *homogeneous* if for each $x, y \in A$ there is an automorphism φ of A such that $\varphi(x) = y$.

Let $A = (A, F)$ be an algebra, k a positive integer and $S_k(A)$ the system of all k-element subalgebras of A. Let $\emptyset \neq B \subseteq A$ and let $B = (B, F_B)$ be a partial algebra such that whenever $f_B \in F_B$, f_B is n-ary, $x_1, \ldots, x_n \in B$, then $(x_1, \ldots, x_n) \in \operatorname{dom} f_B$ if and only if $f(x_1, \ldots, x_n) \in B$, and then $f_B(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$. We will say that B is a partial subalgebra of A and the system of all k-element partial subalgebras of A will be denoted by the symbol $P_k(A)$.

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The algebra A is said to be *k-set-homogeneous* if, whenever $U, V \in S_k(A)$, $U \cong V$, then there is an automorphism φ of A with $\varphi(U) = V$. Also, A is called *k-homogeneous* if every isomorphism between $U, V \in S_k(A)$ can be extended to an automorphism of A.

These definitions of k-homogeneity and of k-set-homogeneity are in accordance with [5] and [7].

We will say that A is *k-set-homogeneous with respect to partial subalgebras*, if an analogous condition as above is valid, with the distinction that we take $U, V \in P_k(A)$ instead of $S_k(A)$.

Similarly we define the notion of a *k-homogeneous algebra with respect to the partial subalgebras.*

Further, if A is a monounary algebra, then we denote by $S_k^c(A)$ ($P_k^c(A)$) the system of all k-element connected monounary algebras (connected partial monounary algebras) belonging to $S_k(A)$ (or $P_k(A)$, respectively). Then, analogously as above, we introduce the notions of the *k-set-homogeneous* (*k-homogeneous*) *algebra with respect to the connected subalgebras* or *with respect to the connected partial subalgebras.*

Let us denote by

$\mathscr{H}$—the class of all homogeneous monounary algebras;

$\mathscr{H}_k(S)$—the class of all k-homogeneous monounary algebras;

$\mathscr{H}_k(P)$—the class of all monounary algebras which are k-homogeneous with respect to partial subalgebras;

$\mathscr{H}_k(S^c)$—the class of all monounary algebras which are k-homogeneous with respect to connected subalgebras;

$\mathscr{H}_k(P^c)$—the class of all monounary algebras which are k-homogeneous with respect to connected partial subalgebras.

The symbols $\mathscr{S}h_k(S)$, $\mathscr{S}h_k(P)$, $\mathscr{S}h_k(S^c)$ and $\mathscr{S}h_k(P^c)$ will have an analogous meaning with the distinction that instead of "homogeneous" we take "set-homogeneous."

In the present paper we will describe the following classes: $\mathscr{S}h_2(S)$ (cf. Thm. 5.3), $\mathscr{H}_2(S)$ (cf. Thm. 5.4), $\mathscr{S}h_2(S^c)$ (Thm. 5.5), $\mathscr{H}_2(S^c)$ (Thm. 5.6), $\mathscr{S}h_2(P^c)$ (Thm. 4.1) and $\mathscr{H}_2(P^c)$ (Thm. 4.2).

The remaining classes for $k = 2$, i.e., the classes $\mathscr{S}h_2(P)$ and $\mathscr{H}_2(P)$ will be dealt with elsewhere.

1. Preliminaries

First we give three lemmas which immediately follow from the above definitions.

1.1. Lemma. *If $k \in \mathbb{N}$, $K \in \{S, P, S^c, P^c\}$, then $\mathscr{H}_k(K) \subseteq \mathscr{S}h_k(K)$.*

1.2. Lemma. *Let $k \in \mathbb{N}$, $K \in \{S, P\}$. Then*

$$\mathscr{S}h_k(K) \subseteq \mathscr{S}h_k(K^c), \quad \mathscr{H}_k(K) \subseteq \mathscr{H}_k(K^c).$$

1.3. Lemma. *If $k \in \mathbb{N}$, then $\mathscr{H}_k(P) \subseteq \mathscr{H}_k(S)$ and $\mathscr{S}h_k(P) \subseteq \mathscr{S}h_k(S)$.*

Thus we obtain the following partially ordered set (ordered by inclusion of classes):

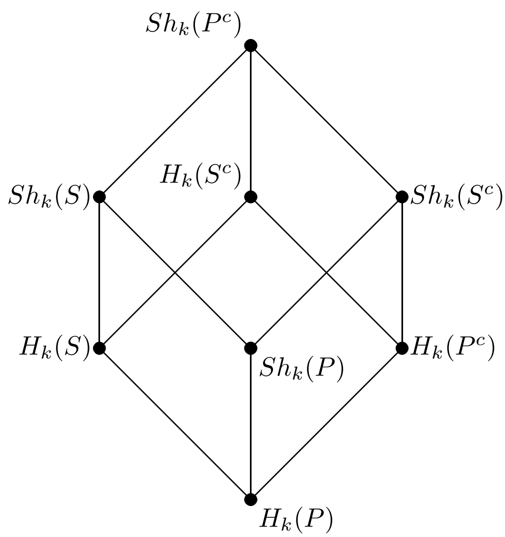

Fig. 1

1.4. Notation. For $\alpha \in \mathbb{N}$ let (Z_α, f) be a monounary algebra such that $Z_\alpha = \{0, 1, \ldots, \alpha - 1\}$, $f(i) \equiv i + 1 \pmod{\alpha}$ for each $i \in Z_\alpha$.

1.5. Notation. Let λ, α be cardinals, $\lambda > 0$. We denote by $M_{\lambda\alpha} = (M_{\lambda\alpha}, f)$ a fixed monounary algebra such that

(a) there is $c \in M_{\lambda\alpha}$ with $f(c) = c$,
(b) if $x \in M_{\lambda\alpha}$, then $f^2(x) = c$,
(c) $\operatorname{card} f^{-1}(c) - \{c\} = \lambda$,
(d) if $a \in f^{-1}(c) - \{c\}$, then $\operatorname{card} f^{-1}(a) = \alpha$.

We will write also M_λ instead of $M_{\lambda 0}$.

1.6. Definition. Let $A = (A, f)$ be a connected monounary algebra possessing a 2-element cycle $\{c_1, c_2\}$. For $i \in \{1, 2\}$ let $A_i = \{x \in A\colon$ there is $n \in \mathbb{N} \cup \{0\}$ such that $f^n(x) = c_i$, $f^m(x) \notin \{c_1, c_2\}$ for each $m \in \mathbb{N} \cup \{0\}$, $m < n\}$. Then the partial subalgebras (A_1, f) and (A_2, f) of (A, f) will be called the ears of A.

1.7. Notation. Let α be a cardinal, $\alpha > 0$. We denote by $K_\alpha = (K_\alpha, f)$ a fixed connected monounary algebra such that K_α contains a cycle $\{c_1, c_2\}$ with 2 elements and with ears C_1, C_2 such that $f(x) = c_1$ for each $x \in C_1$, $f(x) = c_2$ for each $x \in C_2$, $\operatorname{card} C_1 = \operatorname{card} C_2 = \alpha + 1$. Further, we put $K_0 = Z_2$.

If $\alpha \geqslant 0$, then we denote $L_\alpha = K_\alpha - (C_1 - \{c_1\})$.

1.8. Remark. Let $\alpha > 0$ be a cardinal. In [9] a connected monounary algebra B_α without a cycle and such that $\operatorname{card} f^{-1}(x) = \alpha$ for each $x \in B_\alpha$ was constructively described.

For a cardinal γ, let $I(\gamma)$ be a set of indices such that $\operatorname{card} I(\gamma) = \gamma$.

1.9. Notation. Let γ be a cardinal, $\gamma > 0$ and let (A, f) be a monounary algebra. We denote by $\gamma \cdot (A, f)$ a monounary algebra (B, f) such that

$$B = \{(\lambda, a)\colon\ \lambda \in I(\gamma),\ \ a \in A\},$$
$$f((\lambda, a)) = (\lambda, f(a)) \text{ for each } \lambda \in I(\gamma),\ \ a \in A;$$

i.e., $\gamma \cdot (A, f)$ consists of γ copies of (A, f).

1.10. Lemma. *Let φ be an automorphism of a monounary algebra (A, f), let B, C be connected components of (A, f). If $\varphi(u) \in C$ for some $u \in B$, then $\varphi(B) = C$.*

Proof. Suppose that $\varphi(u) \in C$. Since φ is a homomorphism, we obtain that $\varphi(B) \subseteq C$. Notice that $\varphi(B)$ is a connected subalgebra of C. Let $c \in C$. Then there is $n \in \mathbb{N}$ such that $f^n(c) \in \varphi(B)$ and there is $b \in B$ with $\varphi(b) = f^n(c)$. The mapping φ is bijective, thus there is $d \in A$ such that $\varphi(d) = c$. We have

$$\varphi(f^n(d)) = f^n(\varphi(d)) = f^n(c) = \varphi(b).$$

Hence $f^n(d) = b$ and $d \in B$, i.e., $c \in \varphi(B)$, $C = \varphi(B)$. $\square$

We will apply the following result proved in [9]:

1.11. Theorem. *Let (A, f) be a monounary algebra. Then $(A, f) \in \mathscr{H}$ if and only if there are cardinals $\alpha > 0$, $\gamma > 0$ such that either*

(i) $\alpha \in \mathbb{N}$ *and* $(A, f) \cong \gamma \cdot (Z_\alpha, f)$,

or

(ii) $(A, f) \cong \gamma \cdot (B_\alpha, f)$.

2. The class $\mathscr{S}h_2(P^c)$

In this section we suppose that $A = (A, f)$ is a monounary algebra belonging to the class $\mathscr{S}h_2(P^c)$.

2.1. Lemma. *Let B, C be connected components of A, let $b \in B$, $c \in C$ be such that $f^2(b) = f(b) \neq b$, $f^2(c) = f(c) \neq c$. Then $B \cong C$.*

Proof. Let $U = \{b, f(b)\}$, $V = \{c, f(c)\}$. Then $U, V \in P_2^c(A)$, $U \cong V$. This implies that there is $\varphi \in \operatorname{Aut} A$ such that $\varphi(U) = V$. Then $\varphi(B) = C$ by 1.10, therefore $B \cong C$. $\square$

2.2. Lemma. *Let B, C be connected components of A, let $b \in B$, $c \in C$ be such that $b = f^2(b) \neq f(b)$, $c = f^2(c) \neq f(c)$. Then $B \cong C$.*

Proof. Put $U = \{b, f(b)\}$, $V = \{c, f(c)\}$. Then $U, V \in P_2^c(A)$, $U \cong V$, which yields that there is $\varphi \in \operatorname{Aut} A$ such that $\varphi(U) = V$. Hence $\varphi(B) = C$ according to 1.10 and $B \cong C$. $\square$

2.3. Lemma. *Assume that $a, b, c \in A$ are distinct and such that $f(a) = b = f(c)$, $f(b) = c$. If $x \in A$, $f^2(x) \neq x$, then either $f(x) = f^2(x)$ or $f^3(x) = x$.*

Proof. Suppose that there is $x \in A$ such that $x \neq f^2(x) = f(x)$. Let $U = \{a, b\}$, $V = \{x, f(x)\}$. Then $U, V \in P_2^c(A)$, $U \cong V$, thus there is $\varphi \in \operatorname{Aut} A$ such that $\varphi(U) = V$. Then $\varphi(a) = x$ and we obtain

$$f^3(x) = f^3(\varphi(a)) = \varphi(f^3(a)) = \varphi(b) = \varphi(f(a)) = f(\varphi(a)) = f(x).$$

$\square$

2.4. Lemma. *Let B be a connected component of A such that B possesses neither* 1- *nor* 2-*element cycles. Then $B \in \mathscr{H}$, i.e., either $B \cong B_\lambda$ for some $\lambda > 0$ or $B \cong Z_\lambda$ for some $\lambda \in \mathbb{N} - \{1, 2\}$.*

Proof. We have $f(x) \neq x \neq f^2(x)$ for each $x \in B$. Let $a, b \in B$. Then $U = \{a, f(a)\}$, $V = \{b, f(b)\}$ are isomorphic members of P_2^c, thus there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. This implies $\varphi(a) = b$. Therefore for each $a, b \in B$ there is $\psi \in \operatorname{Aut} B$ with $\psi(a) = b$ and hence $B \in \mathscr{H}$. According to 1.10, either $B \cong B_\lambda$ for some $\lambda > 0$ or $B \cong Z_\lambda$ for some $\lambda \in \mathbb{N}$. The assumption yields that if $B \cong Z_\lambda$, then $\lambda \notin \{1, 2\}$. $\square$

2.5. Lemma. *Let B, C be connected components of A such that they possess neither* 1- *nor* 2-*element cycles. Then $B \cong C$.*

Proof. Let $b \in B$, $c \in C$, $U = \{a, f(a)\}$, $V = \{b, f(b)\}$. Then $U, V \in P_2^c(A), U \cong V$, thus there is $\varphi \in \operatorname{Aut} A$ such that $\varphi(U) = V$. Then $\varphi(B) = C$ and $B \cong C$. □

2.6. Lemma. *Let B be a connected component of A possessing neither* 1- *nor* 2-*element cycles and let C be a connected component with a cycle having at most* 2 *elements. Then either* $\operatorname{card} C \leqslant 2$ *or $C \cong M_\alpha$ for some $\alpha > 0$.*

Proof. By way of contradiction, assume that $\operatorname{card} C > 2$ and that $C \ncong M_\alpha$ for $\alpha > 0$. Then there is $u \in C$ such that the elements u, $f(u)$, $f^2(u)$ are mutually distinct. Take $v \in B$, $U = \{u, f(u)\}$, $V = \{v, f(v)\}$. Then $U, V \in P_2^c(A)$, $U \cong V$, but for no automorphism φ of A we have $\varphi(u) \in B$, which is a contradiction. □

2.7. Lemma. *Let B be a connected component of A such that B has a* 2-*element cycle and ears C_1, C_2 with* $\operatorname{card} C_1 > 1$, $\operatorname{card} C_2 > 1$. *Then $B \cong K_\lambda$ for some $\lambda > 0$.*

Proof.
Let C be the cycle of B. The assumption yields that there are $a \in C_1 - C$, $b \in C_2 - C$ such that $f(a) \in C$, $f(b) \in C$. Put $U = \{a, f(a)\}$, $V = \{b, f(b)\}$. Then $U, V \in P_2^c(A)$, $U \cong V$, thus there is $\varphi \in \operatorname{Aut} A$ such that $\varphi(U) = V$. This implies that the ears C_1 and C_2 are isomorphic. According to 2.3 we obtain that there is $\lambda > 0$ such that $B \cong K_\lambda$. □

2.8. Lemma. *Let B be a connected component of A such that B contains a* 2-*element cycle,* $\operatorname{card} B > 2$, *and let C be a connected component such that* $\operatorname{card} C > 1$, $C \ncong B$. *Then there is $\alpha > 0$ with $C \cong M_\alpha$.*

Proof. Suppose that C contains neither 1- nor 2-element cycles. Since B possesses a cycle with 2 elements, we obtain by 2.6 (if we take B, C instead of C, B) that $\operatorname{card} B \leqslant 2$, which is a contradiction. Then 2.2 and the assumption $C \ncong B$ yields that there is $c \in C$ with $f(c) = c$. Next, according to 2.3, $f(x) = f^2(x)$ for each $x \in C$, therefore there is $\alpha > 0$ such that $C \cong M_\alpha$. □

2.9. Lemma. *Let B be a connected component of A, let $c \in B$ satisfy $f(c) = c$,* $\operatorname{card} B > 1$. *Then there are cardinals $\lambda > 0$, $\alpha \geqslant 0$ such that $B \cong M_{\lambda\alpha}$.*

Proof. If $f(x) = c$ for each $x \in B$, then $B \cong M_\lambda \cong M_{\lambda 0}$. Let there be $a, b \in B$ such that $f(a) = b \neq c = f(b)$. By way of contradiction, suppose that $B \ncong M_{\lambda\alpha}$ for any $\lambda > 0$, $\alpha \geqslant 0$. First let there be $d \in B$ such that $f^2(d) \neq c$. Put $U = \{a, b\}$,

$V = \{d, f(d)\}$. Then $U, V \in P_2^c(A)$, $U \cong V$, but there is no $\varphi \in \operatorname{Aut} A$ with $\varphi(a) = d$, a contradiction. Thus $f^2(x) = c$ for each $x \in B$. Denote $\lambda = \operatorname{card}(f^{-1}(c) - \{c\})$. The assumption $B \ncong M_{\lambda\alpha}$ now yields that there are $u, v \in f^{-1}(c) - \{c\}$ such that $\operatorname{card} f^{-1}(u) \neq \operatorname{card} f^{-1}(v)$. Take $U = \{u, f(u)\}$, $V = \{v, f(v)\}$. Then $U, V \in P_2^c(A)$, $U \cong V$, but there is no automorphism of A mapping u into v, which is a contradiction.

$\square$

3. The class $\mathscr{H}_2(P^c)$

Let $A = (A, f)$ be a monounary algebra. In 3.1–3.4 we prove some sufficient conditions under which A belongs to the class $\mathscr{H}_2(P^c)$. Next we deal with a condition under which $A \in \mathscr{S}h_2(P^c) - \mathscr{H}_2(P^c)$.

3.1. Lemma. *Let there be cardinals $k, \lambda, \alpha > 0$, $l, m, n \geqslant 0$ such that*

$$A \cong k \cdot B_\lambda + l \cdot Z_2 + m \cdot Z_1 + n \cdot M_\alpha.$$

Then $A \in \mathscr{H}_2(P^c)$.

Proof. Let $U, V \in P_2^c(A)$, $U \cong V$. Let φ be an isomorphism of U onto V, $\varphi \neq \mathrm{id}_U$. One of the following conditions is satisfied:

(1) U, V are 2-element cycles,

(2) $U = \{u, f(u)\}$, $V = \{v, f(v)\}$, $f^2(u) = f(u)$, $f^2(v) = f(v)$,

(3) $U = \{u, f(u)\}$, $V = \{v, f(v)\}$, U, V are subsets of connected components without cycles.

If (1) is valid, then U and V are connected components of A and it is obvious that φ can be extended to an automorphism of A.

Let (2) hold. Then $\varphi(u) = v$, $\varphi(f(u)) = f(v)$. If $f(u) = f(v)$, then put

$$\overline{\varphi}(x) = \begin{cases} u & \text{if } x = v, \\ v & \text{if } x = u, \\ x & \text{otherwise.} \end{cases}$$

The mapping $\overline{\varphi}$ is an extension of φ and $\overline{\varphi} \in \operatorname{Aut} A$. Suppose that $f(u) \neq f(v)$. Since the connected components B, C containing u, v, respectively, are both isomorphic to M_α, hence obviously there exists $\psi \in \operatorname{Aut}(B \cup C)$ such that $\psi(u) = v$, $\psi(f(u)) = f(v)$. Moreover, ψ can be extended to an automorphism $\overline{\psi}$ of A, thus φ can be extended to $\overline{\psi} \in \operatorname{Aut} A$.

Let (3) hold. Then $\varphi(u) = v$, $\varphi(f(u)) = f(v)$. By the assumption, the connected components B, C containing u, v, respectively, are isomorphic to B_λ. Then 1.10 implies that $B \cup C \in \mathscr{H}$, thus there is $\psi \in \operatorname{Aut}(B \cup C)$ such that $\psi(u) = v$. Then $\psi(f(u)) = f(v)$. Further, ψ can be extended to $\overline{\varphi} \in \operatorname{Aut} A$, therefore $A \in \mathscr{H}_2(P^c)$.

□

Repeating the steps of the proof of 3.1, only with the distinction that we take Z_λ instead of B_λ, we obtain

3.2. Lemma. *Let there be cardinals $k, \alpha > 0$, $l, m, n \geqslant 0$, $\lambda \in \mathbb{N} - \{1, 2\}$ such that*

$$A \cong k \cdot Z_\lambda + l \cdot Z_2 + m \cdot Z_1 + n \cdot M_\alpha.$$

Then $A \in \mathscr{H}_2(P^c)$.

3.3. Lemma. *Let there be cardinals $k, \alpha, \lambda > 0$, $m, n \geqslant 0$ such that*

$$A \cong k \cdot K_\lambda + m \cdot Z_1 + n \cdot M_\alpha.$$

Then $A \in \mathscr{H}_2(P^c)$.

Proof. Let $U, V \in P_2^c(A)$, $U \cong V$. Let φ be an isomorphism of U onto V, $\varphi \neq \mathrm{id}_U$. We have the following possibilities:

(1) U, V are 2-element cycles,

(2) $U = \{u, f(u)\}$, $V = \{v, f(v)\}$, $f^2(u) = f(u), f^2(v) = f(v)$,

(3) $U = \{u, f(u)\}$, $V = \{v, f(v)\}$, u, v do not belong to cycles, $f(u), f(v)$ belong to 2-element cycles.

Let B and C be connected components containing U, V, respectively. Let (1) hold. There are $u \in U$, $v \in V$ with $\varphi(u) = v$, $\varphi(f(u)) = f(v)$. The assumption yields that $B \cong K_\lambda \cong C$, thus there is $\psi \in \operatorname{Aut}(B \cup C)$ such that $\psi(u) = v$. Then $\psi(f(u)) = f(v)$. Further, ψ can be extended to $\overline{\varphi} \in \operatorname{Aut} A$, therefore $\overline{\varphi} \in \operatorname{Aut} A$ is an extension of φ.

For the case (2) the same steps as in the proof of 3.1 can be applied.

Suppose that (3) is valid. We have $C \cong B \cong K_\lambda$, thus there is an automorphism ψ of $B \cup C$ such that $\psi(u) = v$, $\psi(f(f(u)) = f(v)$. Then ψ can be extended to $\overline{\varphi} \in \operatorname{Aut} A$, hence φ can be extended to $\overline{\varphi} \in \operatorname{Aut} A$.

Therefore $A \in \mathscr{H}_2(P^c)$. □

3.4. Lemma. *Let there be cardinals $k, \lambda > 0$, $m, n, \beta \geqslant 0$ such that*

$$A \cong k \cdot M_{\lambda\beta} + m \cdot Z_2 + n \cdot Z_1.$$

Then $A \in \mathscr{H}_2(P^c)$.

P r o o f. Let $U, V \in P_2^c(A)$, $U \cong V$. Let φ be an isomorphism of U onto V, $\varphi \neq \mathrm{id}_U$. Then $U = \{u, f(u)\}$, $V = \{v, f(v)\}$, $\varphi(u) = v$, $\varphi(f(u)) = f(v)$ and one of the following conditions is satisfied:

(1) $f^2(u) = f(u)$, $f^2(v) = f(v)$,

(2) $f^3(u) = f^2(u) \neq f(u)$, $f^3(v) = f^2(v) \neq f(v)$,

(3) U, V are 2-element cycles.

Let B, C be connected components containing U, V, respectively. First suppose that either (1) or (2) is valid. Then $B \cong M_{\lambda\beta} \cong C$. This implies that in the both cases φ can be extended to $\overline{\varphi} \in \operatorname{Aut} A$. Now suppose that (3) holds. Then it is obvious that φ can be extended to $\overline{\varphi} \in \operatorname{Aut} A$. Therefore $A \in \mathscr{H}_2(P^c)$. □

3.5. Lemma. *Let there be cardinals $k, \lambda, \alpha > 0$, $m, n \geqslant 0$ such that*

$$A \cong k \cdot L_\lambda + m \cdot Z_1 + n \cdot M_\alpha.$$

Then $A \in \mathscr{S}h_2(P^c) - \mathscr{H}_2(P^c)$.

P r o o f. We have $k > 0$, thus there is a connected component $D \cong L_\lambda$, i.e., D contains a 2-element cycle $\{d_1, d_2\}$ such that $f(x) = d_1$ for each $x \in D - \{d_1\}$, $\operatorname{card}(D - \{d_1, d_2\}) = \lambda$. Put $U_0 = \{d_1, d_2\} = V_0$, $\varphi(d_1) = d_2$, $\varphi(d_2) = d_1$. Then φ is an isomorphism of U_0 onto V_0 and U_0, $V_0 \in P_2^c(A)$. Since $\lambda > 0$, φ cannot be extended to an automorphism of A, therefore $A \notin \mathscr{H}_2(P^c)$.

Let $U, V \in P_2^c(A)$, $U \cong V$ and $U \neq V$. We have (1), (2) or (3) as in the previous proof. Let (1) hold and let B and C be connected components containing U, V, respectively. Then $B \cong L_\lambda \cong C$ and there are $u \in U$, $v \in V$ such that $f(x) = u$ for each $x \in B - \{u\}$, $f(x) = v$ for each $x \in B - \{v\}$, $\operatorname{card}(B - \{u, f(u)\}) = \lambda = \operatorname{card}(C - \{v, f(v)\})$. Then there is $\psi \in \operatorname{Aut}(B \cup C)$ such that $\psi(u) = v$, $\psi(f(u)) = f(v)$. We can extend ψ into $\overline{\varphi} \in \operatorname{Aut} A$, therefore $\overline{\varphi}(U) = V$.

If (2) is valid, then there is $\overline{\varphi} \in \operatorname{Aut} A$ with $\overline{\varphi}(U) = V$ analogously as in 3.1. Further, if (3) holds, then we can proceed similarly as in the case (1). Hence $A \in \mathscr{S}h_2(P^c)$. □

4. Characterization of the classes $\mathscr{S}h_2(P^c)$ and $\mathscr{H}_2(P^c)$

4.1. Theorem. *A monounary algebra A belongs to the class $\mathscr{S}h_2(P^c)$ if and only if there are cardinals $k, \lambda, \alpha > 0$, $l, m, n, \beta \geqslant 0$ such that one of the following conditions is satisfied:*

(i) $A \cong k \cdot B_\lambda + l \cdot Z_2 + m \cdot Z_1 + n \cdot M_\alpha$,

(ii) $A \cong k \cdot Z_\lambda + l \cdot Z_2 + m \cdot Z_1 + n \cdot M_\alpha$,

(iii) $A \cong k \cdot K_\lambda + m \cdot Z_1 + n \cdot M_\alpha$,
(iv) $A \cong k \cdot M_{\lambda\beta} + m \cdot Z_2 + n \cdot Z_1$,
(v) $A \cong m \cdot Z_2 + n \cdot Z_1$, $(m, n) \neq (0, 0)$,
(vi) $A \cong k \cdot L_\lambda + m \cdot Z_1 + n \cdot M_\alpha$.

Proof. I. Let $A \in \mathscr{S}h_2(P^c)$.

(1) First let there be a connected component B of A such that B contains neither 1- nor 2-element cycles. Then 2.5 implies that each connected component which possesses neither 1- nor 2-element cycles is isomorphic to B; let k be the number of such components. By 2.4, there is a cardinal $\lambda > 0$ such that either $B \cong B_\lambda$ or $\lambda \in \mathbb{N} - \{1, 2\}$, $B \cong Z_\lambda$. We obtain in view of 2.6 that if there is a connected component having a 2-element cycle, then it is a cycle; let l be the number of 2-element cycles. Then 2.6 and 2.1 imply that either the remaining connected components are 1-element or there is $\alpha > 0$ such that the remaining connected components with more that one element are isomorphic to M_α. Hence either (i) or (ii) is valid.

(2) Suppose that each connected component of A possesses a cycle with at most two elements.

a) Let there be a connected component B with a 2-element cycle and with ears C_1, C_2 such that $\operatorname{card} C_1 > 1$, $\operatorname{card} C_2 > 1$. By 2.7, $B \cong K_\lambda$ for some $\lambda > 0$ and 2.2 implies that any two connected components which possess 2-element cycles are isomorphic to K_λ. Let k be the number of such components. Then 2.8 and 2.1 imply that (iii) is valid.

b) Now suppose that there is a connected component B with a 2-element cycle and with ears C_1, C_2 such that $\operatorname{card} C_1 > 1$, $\operatorname{card} C_2 = 1$. By 2.3, $B \cong L_\lambda$ for some $\lambda > 0$ and in view of 2.2, any two connected components with a 2-element cycle are isomorphic to L_λ. Then 2.8 and 2.1 yield that (vi) is valid.

c) If each connected component of A contains a cycle with one element, then 2.9 and 2.1 imply that either (iv) or (v) is satisfied.

II. Conversely, suppose that some of the conditions (i)–(vi) is fulfilled. If (i), (ii), (iii) or (iv) is valid, then $A \in \mathscr{H}_2(P^c)$ by 3.1–3.4, thus $A \in \mathscr{S}h_2(P^c)$. The case (v) is trivial, $A \in \mathscr{H}_2(P^c)$. If (vi) holds, then 3.5 implies that $A \in \mathscr{S}h_2(P^c)$. $\square$

4.2. Theorem. *A monounary algebra A belongs to the class $\mathscr{H}_2(P^c)$ if and only if some of the conditions* (i)–(v) *of* 4.1 *is satisfied.*

Proof. The assertion follows from the relation $\mathscr{H}_2(P^c) \subseteq \mathscr{S}h_2(P^c)$, according to 4.1 and 3.5. $\square$

5. Characterization of $\mathscr{S}h_2(S)$, $\mathscr{H}_2(S)$, $\mathscr{S}h_2(S^c)$ and $\mathscr{H}_2(S^c)$

The aim of this section is to find necessary and sufficient conditions under which a monounary algebra belongs to the classes mentioned.

Let $A = (A, f)$ be a monounary algebra.

5.1. Lemma. *Let B and C be connected components of A such that B and C contain 2-element cycles. If $A \in \mathscr{S}h_2(S^c)$, then $B \cong C$.*

P r o o f. Let $A \in \mathscr{S}h_2(S^c)$. Assume that U, V are cycles of B, C, respectively. Then $U, V \in S_2^c(A)$, $U \cong V$, therefore there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. By 1.10 this yields that $\varphi(B) = C$, hence $B \cong C$. □

5.2. Lemma. *Let a, b, c be distinct elements of A with $f(a) = a$, $f(b) = b$, $f(c) = c$. If $A \in \mathscr{S}h_2(S)$, then all connected components with a 1-element cycle are isomorphic.*

P r o o f. Let $A \in \mathscr{S}h_2(S)$. Take $U = \{a, b\}$, $V = \{a, c\}$. Then $U, V \in S_2(A)$, $U \cong V$, thus there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. Denote by C_1, C_2, C_3 the connected components containing a, b, c, respectively.

First assume that $\varphi(a) = a$, $\varphi(b) = c$. Then $C_2 \cong C_3$. Consider $U' = \{a, b\}$, $V' = \{b, c\}$. We have $U', V' \in S_2(A)$, $U' \cong V'$, hence there is $\psi \in \operatorname{Aut} A$ with $\psi(U') = V'$. This implies that $\psi(a) \in \{b, c\}$ and $\psi(C_1) = C_2$ or $\psi(C_1) = C_3$, thus $C_1 \cong C_2 \cong C_3$.

Now let $\varphi(a) = c$, $\varphi(b) = a$. Then $C_1 \cong C_3$ and $C_2 \cong C_1$. Thus $C_1 \cong C_2 \cong C_3$.

If $d \in A - \{a, b, c\}$, $f(d) = d$, then we can consider a, b, d to prove that any two connected components which possess 1-element cycles are isomorphic. □

5.3. Notation. For distinct $b, c \in A$ with $f(b) = f(c) = c$ we denote

$$C(b, c) = \{x \in A\colon \text{ there is } n \in \mathbb{N} \cup \{0\} \text{ with } f^n(x) = b\} \cup \{c\}.$$

Let $Q(A)$ be the system of all subalgebras of A of the form $C(b, c)$ and $\overline{Q}(A)$ be the system of all connected components K such that $K \cap Q(A) \neq \emptyset$.

5.4. Lemma. *Let $A \in \mathscr{S}h_2(S^c)$. Then any two elements of $Q(A)$ and of $\overline{Q}(A)$, respectively, are isomorphic.*

P r o o f. Let $C(b, c)$ and $C(a, d)$ be distinct elements of $Q(A)$. Then $f(b) = f(c) = c$, $f(a) = f(d) = d$. Take $U = \{b, c\}$, $V = \{a, d\}$. We have $U, V \in S_2^c(A)$, $U \cong V$, thus there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$; then $\varphi(b) = a$, $\varphi(c) = d$. We obtain that $C(b, c) \cong C(a, d)$. If $c \neq d$, then obviously the corresponding components are isomorphic, i.e., any two elements of $\overline{Q}(A)$ are isomorphic. □

5.5. Theorem. *A monounary algebra belongs to $\mathscr{S}h_2(S)$ if and only if it satisfies the following conditions:*

(a) *any two connected components which have 2-element cycles are isomorphic,*

(b) *if there are at least 3 distinct connected components which possess 1-element cycles, then all connected components with 1-element cycles are isomorphic,*

(c) *any two elements of $Q(A)$ and of $\overline{Q}(A)$, respectively, are isomorphic.*

Proof. Let $A \in \mathscr{S}h_2(S)$. Then $A \in \mathscr{S}h_2(S^c)$, thus we obtain by 5.1 and 5.4 that the conditions (a) and (c) are satisfied. Further, (b) is valid in view of 5.2.

Assume that (a)–(c) are valid. Let $U, V \in S_2(A)$ be distinct, $U \cong V$. Then we obtain one of the following cases:

(1) U, V are 2-element cycles,

(2) U and V consist of two 1-element cycles,

(3) $U = \{b, c\}$, $V = \{a, d\}$, $f(b) = f(c) = c$, $f(a) = f(d) = d$.

In the first case, (a) implies that there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. In the second case there are at least three connected components with 1-element cycles, thus we can apply (b) and then there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. If (3) is valid, then $C(b, c)$ and $C(a, d)$ belong to $Q(A)$, thus they are isomorphic by (c) and then there is $\varphi \in \operatorname{Aut} A$ with $\varphi(U) = V$. Therefore $A \in \mathscr{S}h_2(S)$. □

5.6. Theorem. *A monounary algebra belongs to the class $\mathscr{H}_2(S)$ if and only if it satisfies the following conditions:*

(a) *any two connected components with 1-element cycles are isomorphic,*

(b) *any two connected components with 2-element cycles are isomorphic,*

(c) *the two ears of a connected component with a 2-element cycle are isomorphic,*

(d) *any two elements of $Q(A)$ and of $\overline{Q}(A)$, respectively, are isomorphic.*

Proof. Let $A \in \mathscr{H}_2(S)$. Then $A \in \mathscr{S}h_2(S)$, hence 5.5 implies that (b) and (d) are valid.

First let us prove (c). Let B be a connected component possessing a 2-element cycle $\{c_1, c_2\}$ and let C_1, C_2 be the ears of B. Take $U = \{c_1, c_2\} = V$ and let $\varphi(c_1) = c_2$, $\varphi(c_2) = c_1$. Then φ is an isomorphism of U onto V and $U, V \in S_2(A)$, hence φ can be extended to an automorphism $\overline{\varphi}$ of A. We obtain $\overline{\varphi}(C_1) = C_2$, therefore $C_1 \cong C_2$, i.e., (c) is valid.

Now let $a, b \in A$, $f(a) = a$, $f(b) = b$. Put $U' = \{a, b\} = V'$, $\psi(a) = b$, $\psi(b) = a$. Then $U', V' \in S_2(A)$ and ψ is an isomorphism of U' onto V', thus there is $\overline{\psi} \in \operatorname{Aut} A$ such that $\overline{\psi}$ is an extension of ψ. This yields that the connected components containing a and b are isomorphic, thus (a) holds.

Conversely, assume that the conditions (a)–(d) are satisfied. Let $U, V \in S_2(A)$ be such that there is an isomorphism $\varphi \neq \mathrm{id}_U$ of U onto V. One of the following possibilities occurs:

(1) $U = V$ is a 2-element cycle,

(2) $U \neq V$ and U, V are 2-element cycles,

(3) U, V consist of two 1-element cycles,

(4) $U = \{b, c\}$, $V = \{a, d\}$, $f(b) = f(c) = c$, $f(a) = f(d) = d$.

If (1) is valid, then (c) implies that φ can be extended to $\overline{\varphi} \in \operatorname{Aut} A$. If (2) is valid, then (b) and (c) yield that φ can be extended to $\overline{\varphi} \in \operatorname{Aut} A$.

Let (3) hold, $U = \{u_1, u_2\}$, $V = \{v_1, v_2\}$, $\varphi(u_1) = v_1$, $\varphi(u_2) = v_2$. Denote by U_1, V_1, U_2, V_2 the connected components containing u_1, v_1, u_2, v_2, respectively. By (a) there exist isomorphisms $\varphi_{U_1 \to V_1}$, $\varphi_{U_1 \to V_2}$, etc. Put

$$\overline{\varphi}(x) = \begin{cases} \varphi_{U_1 \to V_1}(x) & \text{if } x \in U_1, \\ \varphi_{V_1 \to U_1}(x) & \text{if } x \in V_1, \\ \varphi_{U_2 \to V_2}(x) & \text{if } x \in U_2, \\ \varphi_{V_2 \to U_2}(x) & \text{if } x \in V_2, \\ x & \text{otherwise.} \end{cases}$$

Then $\overline{\varphi}$ is an extension of φ and $\overline{\varphi} \in \operatorname{Aut} A$.

Suppose that (4) holds. Then $\varphi(b) = a$, $\varphi(c) = d$. Further, (d) implies that $C(b, c) \cong C(a, d)$, thus there is $\psi \in \operatorname{Aut} A$ such that $\psi(C(b, c)) = \psi(C(a, d))$. We have $\psi(b) = a$, $\psi(c) = d$, i.e., ψ is an extension of φ.

Therefore $A \in \mathscr{H}_2(S)$. □

5.7. Theorem. *A monounary algebra A belongs to $\mathscr{S}h_2(S^c)$ if and only if any two connected components with 2-element cycles are isomorphic and any two elements of $Q(A)$ and of $\overline{Q}(A)$, respectively, are isomorphic.*

P r o o f. If $A \in \mathscr{S}h_2(S^c)$, then the above condition is satisfied according to 5.1 and 5.4. The converse implication is obvious. □

5.8. Theorem. *A monounary algebra belongs to $\mathscr{H}_2(S^c)$ if and only if the conditions* (b)–(d) *of* 5.6 *are satisfied.*

P r o o f. It is analogous to 5.6. □

References

[1] *B. Csákány*: Homogeneous algebras. In: Contributions to General Algebra. Proc. Klagenfurt Conference, 1978. Verlag J. Heyn, Klagenfurt, 1979, pp. 77–81.

[2] *B. Csákány*: Homogeneous algebras are functionally complete. Algebra Universalis *11* (1980), 149–158.

[3] *B. Csákány and T. Gavalcová*: Finite homogeneous algebras I. Acta Sci. Math. *42* (1980), 57–65.

[4] *M. Droste and H. D. Macpherson*: On k-homogeneous posets and graphs. J. Comb. Theory Ser. A *56* (1991), 1–15.

[5] *M. Droste, M. Giraudet, H. D. Macpherson and N. Sauer*: Set-homogeneous graphs. J. Comb. Theory Ser. B *62* (1994), 63–95.

[6] *M. Droste, M. Giraudet and D. Macpherson*: Set-homogeneous graphs and embeddings of total orders. Order *14* (1997), 9–20.

[7] *R. Fraïssé*: Theory of Relations. North-Holland, Amsterdam, 1986.

[8] *B. Ganter, J. Płonka and H. Werner*: Homogeneous algebras are simple. Fund. Math. *79* (1973), 217–220.

[9] *D. Jakubíková-Studenovská*: Homogeneous monounary algebras. Czechoslovak Math. J. *52* (2002), 309–317.

[10] *D. Jakubíková-Studenovská*: On homogeneous and 1-homogeneous monounary algebras. In: Contributions to General Algebra 12. Proceedings of the Vienna Conference, June, 1999. Verlag J. Heyn, Klagenfurt, 2000, pp. 222–224.

[11] *E. Marczewski*: Homogeneous algebras and homogeneous operations. Fund. Math. *56* (1964), 81–103.

[12] *A. H. Mekler*: Homogeneous partially ordered sets. In: Finite and Infinite Combinatorics in Sets and Logic. Proceeding NATO ASI conference in Banf 1991 (N. W. Sauer, R. E. Woodrow and B. Sands, eds.). Kluwer, Dordrecht, 1993, pp. 279–288.

[13] *R. S. Pierce*: Some questions about complete Boolean algebras. In: Lattice Theory, Proc. Symp. Pure Math, Vol. II. AMS, Providence, 1961, pp. 129–140.

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