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Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 4, 889–896

Persistent URL: http://dml.cz/dmlcz/127773

Czechoslovak Mathematical Journal, 52 (127) (2002), 889–896

CONNECTIONS OF HIGHER ORDER AND PRODUCT PRESERVING FUNCTORS

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(Received January 14, 2000)

Abstract. In this paper we consider a *product preserving functor* $\mathcal{F}$ of order r and a connection Γ of order r on a manifold M. We introduce horizontal lifts of tensor fields and linear connections from M to $\mathcal{F}(M)$ with respect to Γ. Our definitions and results generalize the particular cases of the tangent bundle and the tangent bundle of higher order.

Keywords: connections of higher order, product preserving functors, lifts of tensors and connections

MSC 2000: 53C05, 58A20, 58A32

1. Introduction

Let $\mathcal{F}$ be a product preserving functor (see [4]), then $\mathcal{F}M$ is a fiber bundle, with standard fiber $\mathcal{F}_0(\mathbb{R}^n)$, associated with the principal fiber bundle L^rM of frames of order r, where n is the dimension of M and r is the order of $\mathcal{F}$.

Tangent bundles, tangent bundles of higher order, tangent bundles of p^r-velocities, Weil bundles (bundles of infinitely near points) are examples of product preserving functors. The properties of product preserving functors can be found in [7] and [4].

The horizontal lifts of tensor fields and linear connections to the tangent bundle, with respect to a linear connection, were introduced and studied in [9] and [10]. A similar study for the tangent bundle of higher order is developed in [3] and [5].

The first author was supported by the grant KBN no. 2P 301 030 04; the third author was partially supported by XUNTA DE GALICIA under Project XUGA 20703B98.

In this paper we present the horizontal prolongations of tensor fields of type $(1,1)$ and linear connections from M to $\mathcal{F}M$ with respect to a connection Γ of order r on M, that is a connection on the principal fiber bundle L^rM, which generalize the results given in [10], [3] and [5]. Let us remark that we do not use local coordinates.

2. Product preserving functors and connections of higher order

A *product preserving functor* is a covariant functor $\mathcal{F}$ from the category of all manifolds and all mappings into the category of fibered manifolds satisfying the following conditions:

(1) for each manifold M, $\mathcal{F}(M)$ is a fibered manifold over M;

(2) for each differentiable map $\varphi\colon M \to N$ the induced map $\mathcal{F}(\varphi)\colon \mathcal{F}M \to \mathcal{F}N$ projects on φ and if $\varphi\colon M \to N$ is an immersion between two manifolds with the same dimension, then for each point $x \in M$ the restriction $\mathcal{F}(\varphi)|_{\mathcal{F}_x(M)}\colon \mathcal{F}_x(M) \to \mathcal{F}_{\varphi(x)}(N)$ is a diffeomorphism;

(3) for all pairs of manifolds M_1 and M_2 the map

$$(\mathcal{F}(\pi_1), \mathcal{F}(\pi_2))\colon \mathcal{F}(M_1 \times M_2) \to \mathcal{F}(M_1) \times \mathcal{F}(M_2)$$

is a diffeomorphism, where $\pi_i\colon M_1 \times M_2 \to M_i$ is the projection onto the i-th factor.

From Palais-Terng's theorem (see [8]) we know that there exists an integer r such that $\mathcal{F}$ is of order r. One deduces that $\mathcal{F}(M)$ is an associated bundle with fiber $\mathcal{F}_0(\mathbb{R}^n)$ to the principal fiber bundle L^rM, that is the frame bundle of order r of M with structure group L^r_n where $n = \dim M$.

In this paper we fix a manifold M of dimension n, a product preserving functor $\mathcal{F}$ of order r and a connection Γ of order r on M, that is an arbitrary connection on the principal fiber bundle L^rM of r-frames. Let us denote by $\mathcal{A} = \mathcal{F}(\mathbb{R})$ the Weil algebra of $\mathcal{F}$. We have that $\mathcal{A} = \mathbb{R} \cdot 1 \oplus \mathcal{N}$, where $\mathcal{N} = \mathcal{F}_0(\mathbb{R})$ is the ideal of the nilpotent elements of $\mathcal{A}$ (see [7]).

Γ defines a covariant derivation D_X of sections of each vector bundle associated with L^rM, in particular, a covariant derivation of sections of $J^k(M,\mathbb{R})_0$, $J^k(M,\mathbb{R})$ and $J^{k-1}(TM)$, with $k \leqslant r$. Let us recall this definition.

Let μ be an action of L^r_n on a vector space V, and let E be the vector bundle with fiber V associated with L^rM. Each r-frame $p \in L^rM$ defines an isomorphism $\widetilde{p}\colon V \to E_{\pi(p)}$ of vector spaces. There exists a bijective correpondence between sections of E and equivariant maps $\widetilde{\psi}\colon L^rM \to V$ satisfying the condition $\widetilde{\psi}(p \cdot a) = (\mu_{a^{-1}} \circ \widetilde{\psi})(p)$. If $\psi\colon M \to E$ is a section and $\widetilde{\psi}\colon L^rM \to V$ is the equivariant map associated with ψ, then

$$\widetilde{\psi}(p) = (\widetilde{p}^{-1} \circ \psi \circ \pi_r)(p), \tag{2.1}$$

where $\pi_r\colon\ L^rM \to M$ is the projection.

If X is a vector field on M, we shall denote by X^{H_r} and X^H the horizontal lifts to L^rM and $\mathcal{F}M$, respectively. If $\psi\colon\ M \to E$ is a section, then $X^{H_r}(\widetilde{\psi})\colon\ L^rM \to V$ is an equivariant map and by definition $D_X\psi\colon\ M \to E$ is the section associated with $X^{H_r}(\widetilde{\psi})$, that is

$$D_X\psi(\pi_r(p)) = (\widetilde{p} \circ X^{H^r}(\widetilde{\psi}))(p). \tag{2.2}$$

Let X, Y be two vector fields on M and let $\psi\colon\ M \to E$ be a section; we define $R(X,Y)\psi = (D_X \circ D_Y - D_Y \circ D_X - D_{[X,Y]})(\psi)$. It is not difficult to prove that $R(X,Y)\psi$ is $C^\infty(M)$-linear with respect to ψ, and therefore $R(X,Y)\colon\ E \to E$ is an endomorphism of vector bundles over M. This map $R(X,Y)$ will be called the *curvature transformation* of Γ.

In the case $E = J^r(M,\mathbb{R})_0$ the curvature transformation $R(X,Y)\colon\ J^r(M,\mathbb{R})_0 \to J^r(M,\mathbb{R})_0$ is a derivation (see [2]).

Let us recall that a homomorphism $f\colon\ J^r(M,\mathbb{R})_0 \to J^r(M,\mathbb{R})_0$ is a derivation if $f(y_1y_2) = f(y_1)y_2 + y_1f(y_2)$ for any $y_1, y_2 \in J^r_x(M,\mathbb{R})_0$.

3. Vector fields on $\mathcal{F}(M)$

Let $\lambda\colon\ A \to \mathbb{R}$ be a linear function. If f is a function on M, then we define the λ-lift of f by $f^{(\lambda)} = \lambda \circ \mathcal{F}(f)$.

If $\tau\colon\ M \to J^r(M,\mathbb{R})_0$ is a section, we define $\tau^{(\lambda)}(y) = f^{(\lambda)}_{\pi(y)}(y)$, where $\tau(\pi(y)) = j^r_{\pi(y)}f_{\pi(y)}$. This is a generalization of the λ-lift of functions.

Proposition 3.1. *X^H is the unique vector field on $\mathcal{F}(M)$ such that*

$$X^H(f^{(\lambda)}) = (D_Xj^rf)^{(\lambda)} \tag{3.1}$$

for any function f on M and any linear function $\lambda\colon\ A \to \mathbb{R}$, where j^rf is the section of $J^r(M,\mathbb{R})$ defined by f and D_X is the covariant derivation defined by Γ.

P r o o f. Let φ_t be the 1-parameter group of the vector field X. Let us denote by $\widehat{\varphi}_t$ and $\widetilde{\varphi}_t$ the 1-parameter groups of X^H and X^{Hr}, respectively; then for each $p \in L^rM$ and for each $z \in V = \mathcal{F}_0(\mathbb{R}^n)$ we have

$$\widehat{\varphi}_t(\widetilde{p}(z)) = \widetilde{\varphi_t(p)}(z), \tag{3.2}$$

where $\widetilde{p}\colon V \to \mathcal{F}_{\pi_r(p)}M$ and $\widetilde{\widetilde{\varphi}_t(p)}\colon V \to \mathcal{F}_{\varphi_t(\pi_r(p))}M$ are the diffeomorphisms defined by the r-frames p and $\widetilde{\varphi}_t(p)$, respectively.

Since we shall prove the formula (3.1) locally, without loss of generality we can assume that L^rM is a trivial bundle. We fix a section $\sigma\colon M \to L^rM$ with $\sigma(x) = j_0^r\gamma_x$.

For each point $x \in M$ the two r-frames $\widetilde{\varphi}_t(\sigma(x))$ and $\sigma(\varphi_t(x))$ are at the same fiber of L^rM. Therefore there exists an element $j_0^r\xi_{t,x} \in L_n^r$ such that

$$\widetilde{\varphi}_t(\sigma(x)) = \sigma(\varphi_t(x)) \cdot j_0^r\xi_{t,x} = j_0^r(\gamma_{\varphi_t(x)} \circ \xi_{t,x}). \tag{3.3}$$

Now, from (3.2) and (3.3) we have

$$\widehat{\varphi_t}(\widetilde{\sigma(x)}(z)) = \mathcal{F}(\gamma_{\varphi_t(x)} \circ \xi_{t,x})(z). \tag{3.4}$$

Let us consider a point $y = \widetilde{\sigma(x)}(z) = \mathcal{F}(\gamma_x)(z) \in \mathcal{F}_x(M)$. From the definition of the λ-lift of f and the linearity of the maps $f \to \mathcal{F}(f)$ and λ we deduce that

$$X^H(f^{(\lambda)})(y) = \frac{\mathrm{d}}{\mathrm{dt}}(f^{(\lambda)}(\widehat{\varphi}_t(y)))|_{t=0} = \lambda \circ \mathcal{F}\Big(\frac{\mathrm{d}}{\mathrm{dt}}(f \circ \gamma_{\varphi_t(x)} \circ \xi_{t,x})\big|_{t=0}\Big)(z). \tag{3.5}$$

On the other hand, from (2.2), (3.3) and the linearity of the map $f \to j_0^r f$, we obtain

$$(D_X j^r f)(x) = j_0^r\Big(\frac{\mathrm{d}}{\mathrm{dt}}(f \circ \gamma_{\varphi_t(x)} \circ \xi_{t,x})\big|_{t=0} \circ \gamma_x^{-1}\Big),$$

and therefore from the definition of the λ-lifts of sections we obtain

$$(D_X j^r f)^{(\lambda)}(y) = \lambda \circ \mathcal{F}\Big(\frac{\mathrm{d}}{\mathrm{dt}}(f \circ \gamma_{\varphi_t(x)} \circ \xi_{t,x})\big|_{t=0}\Big)(z). \tag{3.6}$$

So (3.1) follows from (3.5) and (3.6).

We define now a new vector field on $\mathcal{F}(M)$ associated with each derivation of $J^r(M,\mathbb{R})_0$.

Proposition 3.2. *If $S\colon J^r(M,\mathbb{R})_0 \to J^r(M,\mathbb{R})_0$ is a derivation, then there exists one and only one vertical vector field $S^\square$ on $\mathcal{F}(M)$ such that*

$$S^\square(f^{(\lambda)}) = (S \circ j^r f)^{(\lambda)}$$

for any function f on M and any linear function $\lambda\colon A \to \mathbb{R}$.

Proof. Let us denote by $V = J_0^r(\mathbb{R}^n,\mathbb{R})_0$ the fiber of $J^r(M,\mathbb{R})_0$. For each point $p \in L^rM$ we consider $S_p = \widetilde{p} \circ S_{\pi(p)} \circ \widetilde{p}\colon V \to V$, where $S_{\pi(p)} = S|_{J^r_{\pi(p)}(M,\mathbb{R})_0}$ is the restriction of S.

Using the natural identifications (as vector spaces) between V^n and the Lie algebra l_n^r of L_n^r, we define an element $A(S,p)$ of l_n^r by $A(S,p) = (S_p \times \ldots \times S_p)(e)$ where $e = j_0^r(\mathrm{id}_{\mathbb{R}^n})$.

Let $y = \widetilde{p}(z) \in \mathcal{F}(M)$, where $p \in L^r M$ and $z \in \mathcal{F}_0(\mathbb{R}^n)$. Let $\Psi_z\colon L^r M \to \mathcal{F}(M)$ be the map given by $\Psi_z(p) = \widetilde{p}(z)$. Then we have

$$S^{\square}(y) = (\Psi_z)_*(p)(A^*(S,p)_p),$$

where $A^*(S,p)$ is the fundamental vector field defined by the element $A^*(S,p)_p \in l_n^r$.

Let X be a vector field. In [4] the a-lift $X^{(a)}$ of X is defined for each element $a \in \mathcal{A}$. It is the unique vector field on $\mathcal{F}(M)$ such that $X^{(a)}(f^{(\lambda)}) = (Xf)^{(\lambda \circ l_a)}$ for any function f and any λ, where $l_a\colon \mathcal{A} \to \mathcal{A}$ is the translation.

If $a \in A$ is nilpotent, then $X^{(a)}$ is a vertical vector field. For each nilpotent element a, we can generalize the a-lift of functions for sections of $J^{r-1}TM$ setting

$$\Sigma^{(a)}(y) = X^{(a)}_{\pi(y)}(y),$$

where $X_{\pi(y)}$ is a vector field on M such that $\Sigma(\pi(y)) = j^{r-1}_{\pi(y)} X_{\pi(y)}$. This generalization is possible because if a is nilpotent then the vector $X^{(a)}_{\pi(y)}(y)$ depends only on the $(r-1)$-jet $j^{r-1}_{\pi(y)} X_{\pi(y)}$.

Now we can prove

Proposition 3.3. *Let X and Y be vector fields on M and $a \in N$ a nilpotent element of the Weil algebra. Then*

$$[X^H, Y^H] = [X,Y]^H + R(X,Y)^{\square}, \quad [X^H, Y^{(a)}] = (D_X j^{r-1} Y)^{(a)},$$

where $R(X,Y)$ is the curvature transformation of Γ, and $j^{r-1}X\colon x \in M \to j_x^{r-1} X \in J^{r-1}(TM)$ is the section defined by X.

P r o o f. The first formula is an immediate consequence of Propositions 3.1, 3.2 and of the definition of $R(X,Y)$.

To prove the other one we observe that the sections $\tau\colon M \to J^r(M,\mathbb{R})_0$ and $\Sigma\colon M \to J^{r-1}TM$ define a new section $\Sigma \cdot \tau$ of $J^{r-1}(M,\mathbb{R})$ by $(\Sigma \cdot \tau)(x) = j_x^{r-1}(X_x f_x)$, where $\Sigma(x) = j_x^{r-1} X_x$ and $\tau(x) = j_x^r f_x$. Obviously if X is a vector field and f is a function we have $j^{r-1}(fX) = j^r f \cdot j^{r-1} X$. Now we have the formulas

$$X^{(a)}(\tau^{(\lambda)}) = (j_0^{r-1} Y \cdot \tau)^{(\lambda \circ l_a)}, \quad \Sigma^{(a)}(f^{(\lambda)}) = (\Sigma \cdot j^r f)^{(\lambda \circ l_a)}. \tag{3.7}$$

Since the operation $(\Sigma, \tau) \to \Sigma \cdot \tau$ is bilinear we obtain

$$D_X(\Sigma \cdot \tau) = D_X(\Sigma) \cdot \tau + \Sigma \cdot D_X(\tau). \tag{3.8}$$

From Proposition 3.1, the identities (3.8), (3.7) and the definition of the a-lift of vector fields we deduce

$$[X^H, Y^{(a)}](f^{(\lambda)}) = (D_X j^{r-1} Y)^{(a)}(f^{(\lambda)}).$$

Since the vector field is determined by its action on the λ-lifts of functions (see [4]) the above formula give us the second formula of the proposition.

4. Horizontal lifts of tensors fields of type $(1,1)$

For each tensor field t of type $(1,1)$, the horizontal lift t^H of t to $\mathcal{F}(M)$ is the tensor field of type $(1,1)$ on $\mathcal{F}(M)$ defined by

$$t^H(X^H) = (tX)^H, \quad t^H(X^{(a)}) = (tX)^{(a)},$$

where X is any vector field on M and a is any nilpotent element of A. t^H is called the horizontal lift of t with respect to Γ. These formulas determine t^H.

From the definition we deduce that if $w(x)$ is a polynomial with real coefficients and t is a tensor of type $(1,1)$ on M, then $w(t^H) = (w(t))^H$.

In order to study the integrability of the lifted structures we must compute the Nijenhuis tensor of t^H. To compute N_{t^H} we shall use the following operation: given two sections $\Sigma\colon\ M \to J^{r-1}TM$ and $\Phi\colon\ M \to J^{r-1}(TM \otimes T^*M)$ we define a new section

$$\Phi \cdot \Sigma\colon\ M \to J^{r-1}TM$$

by

$$(\Phi \cdot \Sigma)(x) = j_x^{r-1}(t_x X_x),$$

where $\Phi(x) = j_x^{r-1} t_x$ and $\Sigma(x) = j_x^{r-1} X_x$.

If we suppose that $N_t = 0$, then

$$\begin{aligned} N_{t^H}(X^H, Y^H) &= (t^2)^H\big(R(X,Y)^{\square}\big) + R(tX,tY)^{\square} - t^H\big((R(tX,Y) + R(X,tY))^{\square}\big),\\ N_{t^H}(X^H, Y^{(a)}) &= (D_{tX} j^{r-1} t \cdot J^{r-1} Y - j^{r-1} t \cdot D_X j^{r-1} t \cdot J^{r-1} Y)^{(a)},\\ N_{t^H}(X^{(a)}, Y^{(b)}) &= 0,\end{aligned}$$

where X, Y are vector fields on M, $a, b \in \mathcal{N}$ and D denotes the covariant derivation of sections of $TM \otimes T^*M$ with respect to Γ. Using these formulas we easily deduce

Theorem 4.1. *Let J be a complex structure (a tangent structure) and let Γ be a connection of order r on M such that $D_X j^{r-1} J = 0$. If $R(JX, JY) = R(X,Y)$*

($R(JX,Y)=0$), then J^H is a complex structure (a tangent structure, respectively) on $\mathcal{F}(M)$ where $R(\cdot,\cdot)$ denotes the curvature transformation of Γ.

5. Horizontal lifts of linear connections

Proposition 5.1. *Let ∇ be a linear connection and Γ a connection of order r on M. Then there exists one and only one linear connection ∇^H on $\mathcal{F}(M)$ such that*

$$\begin{aligned} \nabla^H_{X^H} Y^H &= (\nabla_X Y)^H, & \nabla^H_{X^H} Y^{(a)} &= [X^H, Y^{(a)}],\\ \nabla^H_{X^{(a)}} Y^H &= 0, & \nabla^H_{X^{(a)}} Y^{(b)} &= (\nabla_X Y)^{(ab)}. \end{aligned}$$

The linear connection ∇^H on $\mathcal{F}(M)$ will be called the horizontal lift of ∇ with respect to Γ.

We point out that in Proposition 5.1 we do not suppose any relationship between ∇ and Γ on M.

In the case $\mathcal{F}(M)=T^rM=J_0^r(\mathbb{R},M)$, the tangent bundle of order r, this proposition was proved in [6]. If $\mathcal{F}(M)$ is the tangent bundle TM and if $\nabla=\Gamma$, this lift coincides with the horizontal lift of linear connections to the tangent bundle introduced by Yano and Ishihara [9], [10].

Let T and $\widetilde{T}$ be torsion tensors of ∇ and ∇^H respectively, then

$$\text{(5.1)}\qquad \begin{cases} \widetilde{T}(X^H,Y^H)=(T(X,Y))^H - R(X,Y)^{\square}, & \widetilde{T}(X^H,Y^{(a)})=0,\\ \widetilde{T}(X^{(a)},Y^{(b)})=(T(X,Y))^{(ab)} \end{cases}$$

where X,Y are vector fields on M, a,b are nilpotent elements of the Weil algebra and $R(X,Y)$ is the curvature transformation of Γ.

From (5.1) we deduce that if ∇ is torsion-free on M and the curvature transformation of Γ vanishes identically, then the horizontal lift ∇^H is a torsion-free connection on $\mathcal{F}(M)$.

The curvature tensor of ∇^H is more difficult to compute because we do not have a formula for $[R(X,Y)^{\square}, Y^{(a)}]$. But it is not hard to check that if ∇ has neither torsion nor curvature and the curvature transformation of Γ vanishes identically, then ∇^H is torsion-free and its curvature vanishes.

One must remark that in the particular case of the tangent bundle $\mathcal{F}(M)=TM$ our horizontal lifts of tensors and linear connections, and their properties, coincide with the results of Yano and Ishihara [9], [10]. Also the results of this paper generalize the results obtained for the tangent bundle of higher order $\mathcal{F}(M)=T^rM$ in [3], [5] and [6].

If we consider our horizontal lifts of tensors and connections to $T^{n,1}M$ and $T^{n,2}M$, their restrictions to LM and L^2M give the horizontal lifts of tensors and connections to the principal fiber bundles LM and L^2M as developed in [1].

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