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Cheng Dong Zhao; Qi-Min He

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ON THE CENTER OF THE GENERALIZED LIÉNARD SYSTEM

Cheng-Dong Zhao, Beijing, and Qi-Min He, Suzhou

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Abstract. In this paper, we discuss the conditions for a center for the generalized Liénard system

$$(\mathrm{E})_1 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \varphi(y) - F(x), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -g(x),$$

or

$$(\mathrm{E})_2 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \psi(y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -f(x)h(y) - g(x),$$

with $f(x)$, $g(x)$, $\varphi(y)$, $\psi(y)$, $h(y)\colon \mathbb{R} \to \mathbb{R}$, $F(x) = \int_0^x f(x)\,\mathrm{d}x$, and $xg(x) > 0$ for $x \neq 0$. By using a different technique, that is, by introducing auxiliary systems and using the differential inquality theorem, we are able to generalize and improve some results in [1], [2].

Keywords: generalized Liénard system, local center, global center, the differetial inequality theorem, the first approximation

MSC 2000: 34C05, 34C25

1. Introduction

In this paper, we discuss the conditions for a center for the generalized Liénard system

$$(\mathrm{E})_1 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \varphi(y) - F(x), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -g(x),$$

or

$$(\mathrm{E})_2 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \psi(y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -f(x)h(y) - g(x),$$

with $f(x)$, $g(x)$, $\varphi(y)$, $\psi(y)$, $h(y)\colon \mathbb{R} \to \mathbb{R}$, $F(x) = \int_0^x f(x)\,\mathrm{d}x$, and $xg(x) > 0$ for $x \neq 0$.

We assume throughout this paper that $f(x)$ and $g(x)$ are continuous, $\varphi(y)$, $\psi(y)$, $h(y)$ are of class C^1. These guarantee the existence of a unique solution to the initial value problem for the system $(\mathrm{E})_1$ or $(\mathrm{E})_2$. Moreover, we always assume that $\varphi(0) = 0$, $\varphi'(y) > 0$ for $-\infty < y < +\infty$ and $\varphi(\pm\infty) = \pm\infty$. We write $G(x) = \int_0^x g(x)\,\mathrm{d}x$ as usual.

If $\varphi(y) \equiv y$ and $h(y) \equiv \psi(y) \equiv y$ (namely the well-known Liénard system), by $F(0) = 0$ and $xg(x) > 0$ for $x \neq 0$, the origin is the unique critical point for this system. Many authors have proposed the conditions which guarantee the origin is a local or a global center (e.g., see [1]–[5]). However, as far as we know, up to now, few results were given for the more general system $(\mathrm{E})_1$ or $(\mathrm{E})_2$. In this paper, by using a different technique, that is, by introducing auxiliary systems and using the differential inquality theorem, we are able to generalize and improve some results in [1], [2] (see the Sections 2, 3, 4 below).

In the Section 2 and the Section 3, we give some sufficient conditions for a local center for the system $(\mathrm{E})_1$ or $(\mathrm{E})_2$. In the Section 4, we give some sufficient conditions for a global center for the system $(\mathrm{E})_1$ or $(\mathrm{E})_2$. In the Section 5, we give some examples.

2. The local center of $(\mathrm{E})_1$

Consider the generalized Liénard system $(\mathrm{E})_1$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \varphi(y) - F(x), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -g(x),$$

where $\varphi(0) = 0$, $\varphi'(y) > 0$, $\varphi(\pm\infty) = \pm\infty$, $F(x) = \int_0^x f(x)\,\mathrm{d}x$, and $xg(x) > 0$ for $x \neq 0$.

We now first introduce the following lemma.

Lemma 1. *If $xg(x) > 0$ for $x \neq 0$ in $(\mathrm{E})_1$, then the trajectory of $(\mathrm{E})_1$ originating from any point $(x_0, \varphi^{-1}(F(x_0)))$ $(x_0 > 0)$ on the characteristic curve $y = \varphi^{-1}(F(x))$ either intersects the positive y-axis as t decreases or tends to the origin as $t \to -\infty$ remaining in the region*

$$\{(x,y) \mid x > 0,\ \ y > \varphi^{-1}(F(x))\},$$

and intersects the negative y-axis as t increases or tends to the origin as $t \to +\infty$ remaining in the region

$$\{(x,y) \mid x > 0,\ \ y < \varphi^{-1}(F(x))\}.$$

P r o o f. We only prove the case of the region $\{(x,y) \mid x>0,\ y>\varphi^{-1}(F(x))\}$ (the other case can be proved in the same way). Suppose the contrary. Note that $x(t)$ and $y(t)$ are decreasing and increasing respectively for decreasing time in this region, hence if the trajectory is bounded, it must have a limit point and this limit point is just the origin, thus we obtain a contradiction to the above assumption; if this trajectory is unbounded, there is a vertical asymptote $x=a\geqslant 0$ and as $x\to a+$, $y\to+\infty$ and $\mathrm{d}y/\mathrm{d}x\to+\infty$. But in fact, $\mathrm{d}y/\mathrm{d}x=-g(x)/\varphi(y)-F(x)\to 0$ as $x\to a+$, $y\to+\infty$, thus we obtain a contradiction again. Hence, Lemma 1 is proved.

In the present paper we prove the following theorem.

Theorem 1. *In the system* $(\mathrm{E})_1$, *assume*

(i) $F(-x)=F(x),\quad g(-x)=-g(x)$.

(ii) $g(x)>0$ *for* $x>0$.

(iii) *There exist constants* $k_2\leqslant k_1$, $r\geqslant\frac{1}{2}$ *and* $\bar{x}>0$ *such that*

$$k_2G^r(x)\leqslant F(x)\leqslant k_1G^r(x)\quad \textit{for}\quad 0<x<\bar{x},$$

in addition, $|k_j|<\sqrt{8\varphi'(0)}$ $(j=1,2)$ *if* $r=\frac{1}{2}$.

Then the system $(\mathrm{E})_1$ *has a local center at the origin.*

P r o o f. By the conditions (i) and (ii), the origin is the unique critical point for $(\mathrm{E})_1$, $F(x)$ is an even function and $g(x)$ is an odd function, hence if $(x(t),y(t))$ is a solution of $(\mathrm{E})_1$, $(-x(-t),y(-t))$ is also a solution of $(\mathrm{E})_1$; that is, the trajectories defined by $(\mathrm{E})_1$ have mirror symmetry about the y-axis. So we can only consider the case in the region $x>0$.

If we write $u(x)=\sqrt{2G(x)}$ for $x>0$, where $G(x)=\int_0^x g(x)\,\mathrm{d}x$, then by the system $(\mathrm{E})_1$ and using the definition of u, we get

$$\frac{\mathrm{d}u}{\mathrm{d}\tau}=\varphi(y)-F_1(u),\quad \frac{\mathrm{d}y}{\mathrm{d}\tau}=-u\quad \text{for } u>0, \tag{1}$$

or

$$\frac{\mathrm{d}u}{\mathrm{d}y}=\frac{F_1(u)-\varphi(y)}{u}\qquad \text{for } u>0, \tag{2}$$

where $F_1(u)\equiv F\bigl(G^{-1}\bigl(\frac{1}{2}u^2\bigr)\bigr)$ with $F_1(0)=0$ and $\mathrm{d}t/\mathrm{d}\tau=\sqrt{2G(x)}/g(x)>0$ for $x>0$. Moreover, from the condition (iii) we get

$$\frac{k_2u^{2r}}{2^r}\leqslant F_1(u)\leqslant\frac{k_1u^{2r}}{2^r}\quad \text{for } 0<u<\bar{u}=\sqrt{2G(\bar{x})}. \tag{3}$$

Now we introduce two auxiliary systems ($j = 1, 2$)

$$(4)_j \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = \varphi(y) - \frac{k_j u^{2r}}{2^r}, \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = -u \quad \text{for } u > 0,$$

or

$$(5)_j \qquad \frac{\mathrm{d}u}{\mathrm{d}y} = \frac{k_j u^{2r}/2^r - \varphi(y)}{u} \qquad \text{for } u > 0.$$

For $r > \frac{1}{2}$, the first approximation to $(4)_j$ $(j = 1, 2)$ is

$$(6) \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = \varphi'(0)y, \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = -u.$$

It is clear that the origin is a center for the system (6). Hence, by [6], p. 142, it follows that the origin is either a center or a focus or a center-focus for $(4)_j$.

For $r = \frac{1}{2}$, the first approximations to $(4)_j$ $(j = 1, 2)$ are respectively

$$(7)_j \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = \frac{-k_j u}{\sqrt{2}} + \varphi'(0)y, \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = -u.$$

Since $|k_j| < \sqrt{8\varphi'(0)}$ $(j = 1, 2)$, we know that the origin is either a focus (as $k_j \neq 0$) or a center (as $k_j = 0$) for $(7)_j$ $(j = 1, 2)$. Hence, by [6], p. 142, it follows that the origin is either a focus or a center or a center-focus for $(4)_j$ $(j = 1, 2)$ respectively.

Thus, for $r \geqslant \frac{1}{2}$, the trajectory of $(4)_j$ originating from any point $N(u_N, y_N)$ $(y_N = \varphi^{-1}(F_1(u_N)),\ 0 < u_N < \bar{u})$ must intersect the positive y-axis as t decreases and the negative y-axis as t increases.

In what follows, for the sake of convenience, let $u(y)$ and $u_j(y)$ denote respectively the trajectories of (1) and $(4)_j$ $(j = 1, 2)$ originating from the same point $N(u_N, y_N)$, that is, $u(y)$ and $u_j(y)$ are respectively the solutions of (2) and $(5)_j$ $(j = 1, 2)$ satisfying the initial condition $u(y_N) = u_j(y_N) = u_N$. Noting (3), we have

$$\frac{\frac{k_2 u^{2r}}{2^r} - \varphi(y)}{u} \leqslant \frac{F_1(u) - \varphi(y)}{u} \leqslant \frac{\frac{k_1 u^{2r}}{2^r} - \varphi(y)}{u} \quad \text{for } 0 < u < \bar{u}.$$

This implies that

$$0 < \left.\frac{\mathrm{d}u(y)}{\mathrm{d}y}\right|_{(2)} \leqslant \left.\frac{\mathrm{d}u(y)}{\mathrm{d}y}\right|_{(5)_1} \qquad \text{for } y < y_N$$

and

$$\left.\frac{\mathrm{d}u(y)}{\mathrm{d}y}\right|_{(5)_2} \leqslant \left.\frac{\mathrm{d}u(y)}{\mathrm{d}y}\right|_{(2)} < 0 \quad \text{for } y > y_N \quad (0 < u < u_N < \bar{u}).$$

Hence, by the differential inequality theorem, we obtain

$$u(y) \geqslant u_1(y) \quad \text{for } y < y_N \tag{8}$$

and

$$u_2(y) \leqslant u(y) \quad \text{for } y > y_N \quad (0 < u < u_N < \bar{u}). \tag{9}$$

Thus, by (8) and (9), the trajectory $u(y)$ of (1) originating from any point $N(u_N, y_N)$ $(0 < u_N < \bar{u})$ is bounded away from the origin by the trajectory $u_1(y)$ of $(4)_1$ in the region $y < y_N$ and by the trajectory $u_2(y)$ of $(4)_2$ in the region $y > y_N$ respectively. Therefore, the trajectory $u(y)$ of (1) originating from any point $N(u_N, y_N)$ $(0 < u_N < \bar{u})$ cannot tend to the origin. Further, according to Lemma 1, the trajectory $u(y)$ of (1) must intersect the y-axis at two points $A(0, y_A)$ with $y_A > 0$ and $C(0, y_C)$ with $y_C < 0$.

Returning to the (x, y) plane, we know that the trajectory of $(\mathrm{E})_1$ originating from any point $N(x_N, y_N)$ $(y_N = \varphi^{-1}(F(x_N)),\ 0 < x_N < \bar{x})$ must intersect the y-axis at two points $A(0, y_A)$ with $y_A > 0$ and $C(0, y_C)$ with $y_C < 0$. Since the trajectory of $(\mathrm{E})_1$ has mirror symmetry about the y-axis and the point $N(x_N, y_N)$ $(0 < x_N < \bar{x})$ is arbitrary, the origin must be a local center of $(\mathrm{E})_1$. Hence, Theorem 1 is proved. □

Corollary 1.1. *In the system* $(\mathrm{E})_1$, *assume*

(i) $F(-x) = F(x), \quad g(-x) = -g(x)$.

(ii) $g(x) > 0$ *for* $x > 0$.

(iii) *There exist constants* $k_2 \leqslant k_1$ *and* $\bar{x} > 0$ *such that*

$$k_2 G(x) \leqslant F(x) \leqslant k_1 G(x) \quad \text{for } 0 < x < \bar{x}.$$

Then the system $(\mathrm{E})_1$ *has a local center at the origin.*

Remark 1.1. Theorem 2.2 in [2] $(\varphi(y) \equiv y)$ is a special case of Corollary 1.1. Moreover, we use a slightly weaker condition "$k_2 \leqslant k_1$" instead of the condition "$k_2 < 0 < k_1$" in [2], Theorem 2.2.

Corollary 1.2. *In the system* $(\mathrm{E})_1$, *assume*

(i) $F(-x) = F(x), \quad g(-x) = -g(x)$.

(ii) $g(x) > 0$ *for* $x > 0$.

(iii) *There exist constants* $k_2 \leqslant k_1$ *and* $\bar{x} > 0$ *such that*

$$k_2 \leqslant \frac{f(x)}{g(x)} \leqslant k_1 \quad \text{for } 0 < x < \bar{x}.$$

Then the system $(\mathrm{E})_1$ *has a local center at the origin.*

P r o o f. It is clear that from Corollary 1.1 this corollary follows. $\square$

Remark 1.2. Theorem 2.2 in [1] ($\varphi(y) \equiv y$) is a special case of Corollary 1.2. Moreover, the condition "$k_2 < 0 < k_1$" in [1] is not required in the present paper.

Corollary 1.3. *In the system* $(\mathrm{E})_1$*, assume that* $\varphi(y) \equiv y$ *and*

(i) $F(-x) = F(x), \quad g(-x) = -g(x)$.

(ii) $g(x) > 0$ *for* $x > 0$.

(iii) *There exist constants* $k_2 \leqslant k_1$, $r \geqslant \frac{1}{2}$ *and* $\bar{x} > 0$ *such that*

$$k_2 G^r(x) \leqslant F(x) \leqslant k_1 G^r(x) \quad \text{for } 0 < x < \bar{x},$$

in addition, $|k_j| < \sqrt{8}$ $(j = 1, 2)$ *if* $r = \frac{1}{2}$.

Then the system $(\mathrm{E})_1$ *has a local center at the origin.*

Remark 1.3. Corollary 3.3 in [2] is a special case of this corollary. It is because we use a slightly weaker condition "$r \geqslant \frac{1}{2}$ and $k_1 \geqslant k_2$" instead of the condition "$\frac{1}{2} \leqslant r < 1$ and $k_2 = -\alpha < 0 < \alpha = k_1$."

3. The local center of $(\mathrm{E})_2$

Consider the generalized Liénard system $(\mathrm{E})_2$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \psi(y), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -f(x)h(y) - g(x).$$

We now prove the following theorems.

Theorem 2. *In the system* $(\mathrm{E})_2$*, assume*

(i) $f(-x) = -f(x)$, $g(-x) = -g(x)$.

(ii) $g(x) > 0$ *for* $x > 0$, $y\psi(y) > 0$ *for* $y \neq 0$ *and* $yh(y) > 0$ *for* $y \neq 0$.

(iii) $g'(0) > 0, \psi'(0) > 0$.

Then the system $(\mathrm{E})_2$ *has a local center at the origin.*

P r o o f. By the conditions (i) and (ii), the origin is the unique critical point for $(\mathrm{E})_2$. It is easy to see that the first approximation to $(\mathrm{E})_2$ is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \psi'(0)y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -g'(0)x. \tag{10}$$

Because $g'(0) > 0$ and $\psi'(0) > 0$, the origin is a center for (10). By [6], p. 142, it follows that the origin is either a center or a focus or a center-focus for $(\mathrm{E})_2$. Note that the trajectories defined by $(\mathrm{E})_2$ have mirror symmetry about the y-axis, thus, by [6], p. 144, the origin must be a local center of $(\mathrm{E})_2$. Hence, Theorem 2 is proved. $\square$

Corollary 2.1. *In the system* $(\mathrm{E})_2$*, assume that* $h(y) \equiv \psi(y)$ *and that the conditions of Theorem 2 hold. Then the system* $(\mathrm{E})_2$ *has a local center at the origin.*

Remark 2.1. Theorem 2.1 in [1] ($h(y) \equiv \psi(y) \equiv y$) is a special case of Corollary 2.1.

Letting $h(y) \equiv \psi(y)$, one obtains from $(\mathrm{E})_2$ that

$$(\mathrm{E})_3 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \psi(y), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -f(x)\psi(y) - g(x).$$

In the following, we shall give the conditions of a local center for the system $(\mathrm{E})_3$. For this purpose, we first prove the following lemma.

Lemma 2. *If* $xg(x) > 0$ *for* $x \neq 0$*,* $y\psi(y) > 0$ *for* $y \neq 0$ *and* $\psi(\pm\infty) = \pm\infty$ *in* $(\mathrm{E})_3$*, then the trajectory of* $(\mathrm{E})_3$ *originating from any point* $(x_0, 0)$ $(x_0 > 0)$ *either intersects the positive* y*-axis as* t *decreases or tends to the origin as* $t \to -\infty$ *remaining in the region* $\{(x,y) \mid x > 0,\ y > 0\}$*, and intersects the negative* y*-axis as* t *increases or tends to the origin as* $t \to +\infty$ *remaining in the region* $\{(x,y) \mid x > 0,$ $y < 0\}$*.*

P r o o f. By introducing a new variable $v = y + F(x)$, $(E)_3$ becomes

$$(11) \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \psi(v - F(x)), \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -g(x),$$

the regions

$$\{(x,y) \mid x > 0,\ y > 0\} \quad \text{and} \quad \{(x,y) \mid x > 0,\ y < 0\}$$

become the regions

$$\{(x,v) \mid x > 0,\ v > F(x)\} \quad \text{and} \quad \{(x,v) \mid x > 0,\ v < F(x)\}$$

respectively, the y-axis becomes the v-axis, the x-axis becomes the curve $v = F(x)$, and the point $(x_0, 0)$ $(x_0 > 0)$ on (x,y) plane becomes $(x_0, F(x_0))$ $(x_0 > 0)$ on (x,v) plane.

Since $y\psi(y) > 0$ for $y \neq 0$, that is, since $(v - F(x))\psi(v - F(x)) > 0$ for $v \neq F(x)$, the curve $v = F(x)$ $(x \neq 0)$ is the vertical isocline of (11). The trajectory of $(\mathrm{E})_3$ originating from any point $(x_0, 0)$ $(x_0 > 0)$ becomes that of (11) originating from the point $(x_0, F(x_0))$ $(x_0 > 0)$. Therefore, to prove Lemma 2, we only need to prove the following conclusion: The trajectory of (11) originating from any point $(x_0, F(x_0))$ $(x_0 > 0)$ either intersects the positive v-axis as t decreases or tends to the origin as $t \to -\infty$ remaining in the region $\{(x, v) \mid x > 0,\ \ v > F(x)\}$, and intersects the negative v-axis as t increases or tends to the origin as $t \to +\infty$ remaining in the region $\{(x, v) \mid x > 0,\ \ v < F(x)\}$.

Now, we only prove the case in the region $\{(x, v) \mid x > 0,\ \ v > F(x)\}$ (the other case can be proved in the same way). Suppose the contrary. Note that $x(t)$ and $v(t)$ are decreasing and increasing respectively for decreasing time in this region, hence if the trajectory is bounded, it must have a limit point and this limit point is just the origin, thus we obtain a contradiction to the above assumption; if this trajectory is unbounded, there is a vertical asymptote $x = a \geqslant 0$ and as $x \to a+$, $v \to +\infty$ and $\mathrm{d}v/\mathrm{d}x \to +\infty$. But, in fact

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-g(x)}{\psi(v - F(x))} \to 0 \quad \text{as} \quad x \to a+,\ \ v \to +\infty,$$

thus we obtain a contradiction again. Hence, Lemma 2 is proved. □

Theorem 3. *In the system* $(\mathrm{E})_3$, *assume*

(i) $f(-x) = -f(x)$, $g(-x) = -g(x)$.

(ii) $g(x) > 0$ *for* $x > 0$, $\psi(0) = 0$, $\psi'(y) > 0$ *and* $\psi(\pm\infty) = \pm\infty$.

(iii) *There exist constants* $k_2 < 0 < k_1$ *and* $\bar{x} > 0$ *such that*

$$k_2 \leqslant \frac{f(x)}{g(x)} \leqslant k_1 \quad \text{for } 0 < x < \bar{x}.$$

Then the system $(\mathrm{E})_3$ *has a local center at the origin.*

Proof. By the conditions (i) and (ii), the origin is the unique critical point for $(\mathrm{E})_3$ and if $(x(t), y(t))$ is a solution of $(\mathrm{E})_3$, $(-x(-t), y(-t))$ is also a solution of $(\mathrm{E})_3$; that is, the trajectories defined by $(\mathrm{E})_3$ have mirror symmetry about the y-axis. So we can only consider the case in the region $x > 0$.

Because $\psi(0) = 0$, $\psi'(y) > 0$ and $\psi(\pm\infty) = \pm\infty$, there exist a unique $y_1 > 0$ and a unique $y_2 < 0$ respectively such that $\psi(y_1) = 1/|k_2| > 0$ and $\psi(y_2) = -1/k_1 < 0$.

First we define the region $D_1 = \{(x, y) \mid 0 < x < \bar{x},\ \ 0 < y < y_1\}$ and consider

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{-\psi(y)}{g(x)\Big(\dfrac{f(x)}{g(x)}\,\psi(y) + 1\Big)} \quad \text{for } (x, y) \in D_1. \tag{12}$$

Noting that $\psi'(y) > 0$ and $f(x)/g(x) \geqslant k_2$ $(k_2 < 0)$, we have

$$\frac{f(x)}{g(x)}\,\psi(y) + 1 \geqslant k_2\psi(y) + 1 > k_2\psi(y_1) + 1 = 0 \quad \text{for } (x, y) \in D_1.$$

Further, we get $\mathrm{d}x/\mathrm{d}y\big|_{(12)} > 0$ for $(x,y) \in D_1$. Therefore, the trajectory $x(y)$ of $(\mathrm{E})_3$ originating from any point $A(x_0,0)$ $(0 < x_0 < \bar{x})$ can not tend to the origin in D_1.

Next we define the region $D_2 = \{(x,y) \mid 0 < x < \bar{x},\ y_2 < y < 0\}$ and similarly consider

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{-\psi(y)}{g(x)\Big(\dfrac{f(x)}{g(x)}\,\psi(y)+1\Big)} \quad \text{for } (x,y) \in D_2. \tag{13}$$

Noting that $\psi'(y) > 0$ and $f(x)/g(x) \leqslant k_1$ $(k_1 > 0)$, we have

$$\frac{f(x)}{g(x)}\,\psi(y) + 1 \geqslant k_1\psi(y) + 1 > k_1\psi(y_2) + 1 = 0 \quad \text{for } (x,y) \in D_2.$$

Further, we get $\mathrm{d}x/\mathrm{d}y\big|_{(13)} > 0$ for $(x,y) \in D_2$. Therefore, the trajectory $x(y)$ of $(\mathrm{E})_3$ originating from any point $A(x_0,0)$ $(0 < x_0 < \bar{x})$ can not tend to the origin in D_2.

Thus, by Lemma 2, the trajectory of $(\mathrm{E})_3$ originating from any point $A(x_0,0)$ $(0 < x_0 < \bar{x})$ must intersect the positive y-axis and the negative y-axis respectively. Therefore, by symmetry, the trajectory of $(\mathrm{E})_3$ passing through the point $A(x_0,0)$ $(0 < x_0 < \bar{x})$ is a closed orbit surrounding the origin. Since the point $A(x_0,0)$ $(0 < x_0 < \bar{x})$ is arbitrary, it means that the origin must be a local center of $(\mathrm{E})_3$. Hence Theorem 3 is proved. □

Remark 3.1. Theorem 2.2 in [1] ($h(y) \equiv \psi(y) \equiv y$) is a special case of Theorem 3.

4. The global center of $(\mathrm{E})_1$ and $(\mathrm{E})_2$

We now prove the following theorems.

First, we generalize the related results in [1] (Section 3).

Theorem 4. *In the system* $(\mathrm{E})_1$, *assume*

(i) $F(-x) = F(x)$, $g(-x) = -g(x)$, *and* $G(\pm\infty) = +\infty$.

(ii) $g(x) > 0$ *for* $x > 0$, $\varphi'(y) \geqslant l$ *(a positive constant).*

(iii) *There exist constants* $k_2 \leqslant k_1$ *and* $\bar{x} > 0$ *such that*

$$k_2 G(x) \leqslant F(x) \leqslant k_1 G(x) \quad \text{for } 0 < x < \bar{x}.$$

(iv) $|F(x)| \leqslant A$ *(a constant), for all* x.

Then the system $(\mathrm{E})_1$ *has a global center at the origin.*

P r o o f. By the conditions (i), (ii), (iii) and Corollary 1.1, we know that the origin is not only the unique critical point but also a local center for $(\mathrm{E})_1$, and if $(x(t), y(t))$ is a solution of $(\mathrm{E})_1$, $(-x(-t), y(-t))$ is also a solution of $(\mathrm{E})_1$; that is, the trajectories defined by $(\mathrm{E})_1$ have mirror symmetry about the y-axis.

Now we consider the following two families of closed curves

$$\lambda_1(x,y) = \Phi\Big(y + \frac{A}{l}\Big) + G(x) = c_1 \quad \text{for } x \geqslant 0, \tag{14}$$

$$\lambda_2(x,y) = \Phi\Big(y - \frac{A}{l}\Big) + G(x) = c_2 \quad \text{for } x \leqslant 0, \tag{15}$$

where $\Phi(y) = \int_0^y \varphi(u)\,\mathrm{d}u$, c_1 and c_2 are arbitrary positive constants. Because $\varphi'(y) \geqslant l > 0$ and $|F(x)| \leqslant A$, we have

$$\begin{aligned}\frac{\mathrm{d}\lambda_1}{\mathrm{d}t}\Big|_{(\mathrm{E})_1} &= g(x)\Big[\varphi(y) - \varphi\Big(y+\frac{A}{l}\Big) - F(x)\Big] \\ &= -\,g(x)\Big[F(x) + \varphi'(\xi)\frac{A}{l}\Big] \\ &\leqslant -\,g(x)[F(x)+A] \leqslant 0 \qquad \text{for } x \geqslant 0\end{aligned} \tag{16}$$

with $y < \xi < y + A/l$, and

$$\begin{aligned}\frac{\mathrm{d}\lambda_2}{\mathrm{d}t}\Big|_{(\mathrm{E})_1} &= g(x)\Big[\varphi(y) - \varphi\Big(y-\frac{A}{l}\Big) - F(x)\Big] \\ &= -\,g(x)\Big[F(x) - \varphi'(\zeta)\frac{A}{l}\Big] \\ &\leqslant -\,g(x)[F(x)-A] \leqslant 0 \quad \text{for } x \leqslant 0\end{aligned} \tag{17}$$

with $y - A/l < \zeta < y$.

Therefore, any trajectory of $(\mathrm{E})_1$ crosses these closed curves (14) and (15) from their exteriors to their interiors as t increases. Let γ_P^+ denote the positive half-trajectory of $(\mathrm{E})_1$ originating from any point $P(x_0, y_0)$ $(x_0 > 0)$. Since the origin is a local center, γ_P^+ can not tend to the origin. Hence, by (14) and (16), γ_P^+ must intersect the y-axis at $M(0, y_M)$ with $y_M < 0$ as t increases. Further, after M as t increases, by (15) and (17), γ_P^+ must intersect the y-axis again at $N(0, y_N)$ with $y_N > 0$. Noting that the trajectory has the mirror symmetry about the y-axis, it follows that γ_P^+ must return to the point $P(x_0, y_0)$, so γ_P^+ must be a closed trajectory. Since the point $P(x_0, y_0)$ is arbitrary, the origin must be a global center of $(\mathrm{E})_1$. Hence, Theorem 4 is proved. □

Remark 4.1. If the condition (iii) of Theorem 4 is replaced by the slightly stronger condition "There exist constants $k_2 < 0 < k_1$ and $\overline{x} > 0$ such that $k_2 \leqslant f(x)/g(x) \leqslant k_1$ for $0 < x < \overline{x}$" and specially letting $\varphi(y) \equiv y$ in $(\mathrm{E})_1$, Theorem 4 is just Theorem 3.1 in [1].

Next, let $\varphi(y) \equiv y$ in $(\mathrm{E})_1$. By introducing auxiliary systems and using the differential inequality theorem, we give some conditions under which $(\mathrm{E})_1$ has a global center at the origin.

Theorem 5. *In the system* $(\mathrm{E})_1$, *assume that* $\varphi(y) \equiv y$ *and*

(i) $F(-x) = F(x)$, $g(-x) = -g(x)$.

(ii) $g(x) > 0$ *for* $x > 0$, $G(\pm\infty) = +\infty$.

(iii) *There exist constants* $k_2 \leqslant k_1$ *with* $|k_j| < 2$ $(j = 1, 2)$ *and* $0 < \overline{x} \leqslant d$ *such that*

$$k_2\sqrt{2G(x)} \leqslant F(x) \leqslant k_1\sqrt{2G(x)} \quad \text{for } 0 < x < \overline{x} \quad \text{and} \quad x > d.$$

Then the system $(\mathrm{E})_1$ *has a global center at the origin.*

P r o o f. By the conditions (i), (ii) and (iii) for $0 < x < \overline{x}$ and from Theorem 1, we know that the origin is not only the unique critical point but also a local center for $(\mathrm{E})_1$, and if $(x(t), y(t))$ is a solution of $(\mathrm{E})_1$, $(-x(-t), y(-t))$ is also a solution of $(\mathrm{E})_1$; that is, the trajectories defined by $(\mathrm{E})_1$ have mirror symmetry about the y-axis. So we can only consider the case in the region $x > 0$.

If we write $u(x) = \sqrt{2G(x)}$ for $x > 0$, where $G(x) = \int_0^x g(x)\,\mathrm{d}x$, then by the system $(\mathrm{E})_1$ and using the definition of u, we get

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = y - F_1(u), \quad \frac{\mathrm{d}y}{\mathrm{d}\tau} = -u \quad \text{for } u > 0, \tag{18}$$

or

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{F_1(u) - y}{u} \qquad \text{for } u > 0, \tag{19}$$

where $F_1(u) \equiv F\bigl(G^{-1}(\frac{1}{2}u^2)\bigr)$ with $F_1(0) = 0$ and $\mathrm{d}t/\mathrm{d}\tau = \sqrt{2G(x)}/g(x) > 0$ for $x > 0$. Moreover, from the condition (iii) for $x > d$, we get

$$k_2 u \leqslant F_1(u) \leqslant k_1 u \quad \text{for } u > \sqrt{2G(d)}. \tag{20}$$

Now we introduce two auxiliary systems $(j = 1, 2)$

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = -\,k_j u + y, \quad \frac{\mathrm{d}y}{\mathrm{d}\tau} = -u \quad \text{for } u > 0, \tag{21$_j$}$$

or

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{k_j u - y}{u} \qquad \text{for } u > 0. \tag{22$_j$}$$

Since $|k_j| < 2$ $(j = 1, 2)$, it means that the origin is either a focus (as $k_j \neq 0$) or a center (as $k_j = 0$) for $(21)_j$ $(j = 1, 2)$ respectively.

In what follows, for the sake of convenience, let $u(y)$ and $u_j(y)$ $(j = 1, 2)$ denote the trajectories of (18) and $(21)_j$ $(j = 1, 2)$ originating from the same point $P(u_0, y_0)$ with $u_0 > \sqrt{2G(d)}$ respectively, that is, $u(y)$ and $u_j(y)$ are respectively the solutions of (19) and $(22)_j$ $(j = 1, 2)$ satisfying the initial condition $u(y_0) = u_j(y_0) = u_0$. If $y_0 < F_1(u_0)$, we get from (20)

$$0 < \left.\frac{\mathrm{d}u(y)}{\mathrm{d}y}\right|_{(19)} = \frac{F_1(u) - y}{u} \leqslant \left.\frac{\mathrm{d}u(y)}{\mathrm{d}y}\right|_{(22)_1} = \frac{k_1 u - y}{u}$$

for $F_1(u) > y \geqslant y_0$ $(u > u_0 > \sqrt{2G(d)})$. Further, by the differential inequality theorem, we get

$$u(y) \leqslant u_1(y) \quad \text{for } F_1(u) > y \geqslant y_0 \quad (u > u_0 > \sqrt{2G(d)}). \tag{23}$$

Since the origin is either a focus or a center for $(21)_1$, $u_1(y)$ must intersect $y = F_1(u)$. Hence, by (23) the trajectory $u(y)$ of (18) must also intersect $y = F_1(u)$. Similarly, if $y_0 \geqslant F_1(u_0)$, we get from (20)

$$\left.\frac{\mathrm{d}u}{\mathrm{d}y}\right|_{(22)_2} = \frac{k_2 u - y}{u} \leqslant \left.\frac{\mathrm{d}u}{\mathrm{d}y}\right|_{(19)} = \frac{F_1(u) - y}{u} < 0$$

for $F_1(u) < y \leqslant y_0$ $(u > u_0 > \sqrt{2G(d)})$. Further, by the differential inequality theorem, we get

$$u_2(y) \geqslant u(y) \quad \text{for } F_1(u) < y \leqslant y_0 \quad (u > u_0 > \sqrt{2G(d)}). \tag{24}$$

Since the origin is either a focus or a center for $(21)_2$, $u_2(y)$ must intersect $y = F_1(u)$. Hence, by (24) the trajectory $u(y)$ of (18) must also intersect $y = F_1(u)$.

Thus, the trajectory $u(y)$ of (18) originating from any point $P(u_0, y_0)$ with $u_0 > \sqrt{2G(d)}$ must intersect $y = F_1(u)$. According to Lemma 1 and the fact that the origin is a local center, it means that this trajectory must intersect the positive y-axis and the negative y-axis respectively. Returning to the (x, y) plane, the trajectory $x(y)$ of $(\mathrm{E})_1$ originating from any point $P(x_0, y_0)$ with $x_0 > d$ must intersect the positive y-axis and the negative y-axis respectively. By the mirror symmetry of the trajectory about the y-axis, this trajectory must be a closed trajectory. Letting γ_B^+ denote the positive trajectory of $(\mathrm{E})_1$ originating from the point $B(x, y)$ with $\bar{x} \leqslant x \leqslant d$, we know that γ_B^+ is bounded and cannot tend to the origin since the origin is a local center. Because the trajectory has mirror symmetry about the y-axis and the origin is the unique critical point of $(\mathrm{E})_1$, it follows that γ_B^+ must be a closed trajectory. Thus the origin is a global center of $(\mathrm{E})_1$. Hence, Theorem 5 is proved. $\square$

Finally, by using the same method as in the proof of Theorem 4, we can give a new result for $(\mathrm{E})_2$.

Theorem 6. *In the system* $(\mathrm{E})_2$*, assume that* $h(y) \equiv \psi(y)$ *and*

(i) $f(-x) = -f(x)$, $g(-x) = -g(x)$ *and* $G(\pm\infty) = +\infty$.

(ii) $g(x) > 0$ *for* $x > 0, \psi(0) = 0, \psi'(y) > 0$ *and* $\psi(\pm\infty) = \pm\infty$.

(iii) *There exist constants* $k_2 < 0 < k_1$ *and* $\bar{x} > 0$ *such that*

$$k_2 \leqslant \frac{f(x)}{g(x)} \leqslant k_1 \quad \text{for } 0 < x < \bar{x}.$$

(iv) $|F(x)| \leqslant A$ *(a constant) for all* x.

Then the system $(\mathrm{E})_2$ *has a global center at the origin.*

5. Examples

Example 1. Taking $\varphi(y) = y^3 + 2y$, $F(x) = \arctan x^2$ and $g(x) = 2x$ in the system $(\mathrm{E})_1$, we get the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y^3 + 2y - \arctan x^2, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -2x. \tag{25}$$

It is easy to see that $\varphi(0) = 0$, $\varphi'(y) = 3y^2 + 2$, $F(x) = F(-x)$ and $G(x) = x^2 = G(-x)$. By an easy computation, we get that

$$\left(\frac{F(x)}{G(x)}\right)' = \frac{2H(x)}{x^3} < 0 \quad \text{for } x > 0,$$

where $H(x) = x^2/(1+x^4) - \arctan x^2 < 0$ for $x > 0$ (because $H'(x) < 0$ for $x > 0$). This indicates that $F(x)/G(x)$ is a decreasing function for $x > 0$. Since

$$\frac{F(x)}{G(x)} = \frac{\arctan x^2}{x^2} \to 1 \quad \text{as } x \to 0,$$

we have $\frac{1}{4}\pi \leqslant F(x)/G(x) \leqslant 1$ for $0 < x \leqslant 1$. Hence, the system (25) satisfies all the conditions of Theorem 1 with $k_1 = 1 > k_2 = \frac{1}{4}\pi$, $r = 1$ and $\bar{x} = 1$, so (25) has a local center at the origin.

However, since $\frac{1}{4}\pi \leqslant F(x)/G(x) \leqslant 1$ for $0 < x \leqslant 1$, the system (25) does not satisfy the condition "$k_2 \leqslant F(x)/G(x) \leqslant k_1$ $(k_2 < 0)$" of Theorem 2.2 in [2]. Moreover, we have $\frac{1}{2} \leqslant f(x)/g(x) = 1/(1+x^4) \leqslant 1$ for $0 < x \leqslant 1$. This implies that the system (25) does not satisfy the condition "$k_2 \leqslant f(x)/g(x) \leqslant k_1$ $(k_2 < 0)$" of Theorem 2.2 in [1] either.

Noting that $|F(x)| = \arctan x^2 \leqslant \frac{1}{2}\pi$ for all x, it is easy to check that the system (25) satisfies all the conditions of Theorem 4. Thus (25) has a global center at the origin.

Example 2. Taking $\varphi(y) \equiv y$, $F(x) = x^2 \arctan x^2$ and $g(x) = 2x^3$ in the system $(\mathrm{E})_1$, we get the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y - x^2 \arctan x^2, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -2x^3. \tag{26}$$

It is clear that $F(x) = F(-x)$, $G(x) = G(-x)$, $F(x)/\sqrt{2G(x)} = \arctan x^2$. Then, it follows that $0 \leqslant F(x)/\sqrt{2G(x)} \leqslant \frac{1}{2}\pi$ for all x. Thus the system (26) satisfies all the conditions of Theorem 5 with $k_2 = 0$ and $k_1 = \frac{1}{2}\pi < 2$. Hence, (26) has a global center at the origin.

However, since $F(x) = x^2 \arctan x^2$ is bounded, the system (26) does not satisfy the condition "$|F(x)| \leqslant A$ for all x" in [1].

Example 3. Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y - F(x), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -2x^3, \tag{27}$$

where

$$F(x) = \begin{cases} x^6(x^2-5)(x^2-6) & \text{for } 0 \leqslant |x| \leqslant \sqrt{6}, \\ \frac{1}{6}x^2 \sin 6^3(x^2-6) & \text{for } |x| > \sqrt{6}. \end{cases}$$

It is easy to see that $F(x) = F(-x)$ and $G(x) = \frac{x^4}{2} = G(-x)$. Letting $u = x^2$, we get

$$F(\sqrt{u}) = F_1(u) = \begin{cases} u^3(u-5)(u-6) & \text{for } 0 \leqslant u \leqslant 6, \\ \frac{1}{6}u \sin 6^3(u-6) & \text{for } u > 6, \end{cases}$$

$$G(\sqrt{u}) = G_1(u) = \frac{u^2}{2},$$

$$F_1'(u) = f_1(u) = \begin{cases} u^3(u-5) + u^3(u-6) + 3u^2(u-5)(u-6) & \text{for } 0 \leqslant u \leqslant 6, \\ \frac{1}{6}\sin 6^3(u-6) + 6^2 u \cos 6^3(u-6) & \text{for } u > 6, \end{cases}$$

and

$$\frac{F(\sqrt{u})}{\sqrt{2G(u)}} = \frac{F_1(u)}{\sqrt{2G_1(u)}} = \begin{cases} u^2(u-5)(u-6) & \text{for } 0 \leqslant u \leqslant 6, \\ \frac{1}{6}\sin 6^3(u-6) & \text{for } u > 6. \end{cases}$$

It is clear that $f_1(u)$ is continuous. Therefore, $f(x)$ is also continuous. Moreover, an easy computation shows that

$$\left|\frac{F_1(u)}{\sqrt{2G_1(u)}}\right| < \frac{15}{8} < 2 \qquad \text{for } 0 < u \leqslant \frac{1}{4} \quad \text{and} \quad u > 6$$

and

$$\left|\frac{F_1(u)}{\sqrt{2G_1(u)}}\right| > 2 \qquad \text{for } \frac{1}{3} < u \leqslant 4.9,$$

that is,

$$\left|\frac{F(x)}{\sqrt{2G(x)}}\right| < \frac{15}{8} < 2 \qquad \text{for } 0 < |x| \leqslant \frac{1}{2} \quad \text{and} \quad |x| > \sqrt{6}$$

and

$$\left|\frac{F(x)}{\sqrt{2G(x)}}\right| > 2 \qquad \text{for } \frac{\sqrt{3}}{3} < |x| \leqslant \frac{7\sqrt{10}}{10}.$$

Thus, (27) satisfies all the conditions of Theorem 5 with $k_1 = \frac{15}{8} < 2$ and $k_2 = -\frac{1}{6}$. Hence, (27) has a global center at the origin.

Example 4. Taking $\psi(y) = y^5 + 3y$, $f(x) = 2x\cos x^2$ and $g(x) = 4x$ in the system $(\mathrm{E})_3$, we get the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y^5 + 3y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -2x(\cos x^2)(y^5 + 3y) - 4x. \tag{28}$$

It is clear that

$$\frac{f(x)}{g(x)} = \frac{\cos x^2}{2} \quad \text{for } x \neq 0.$$

Then it follows that

$$-\frac{1}{2} \leqslant \frac{f(x)}{g(x)} \leqslant \frac{1}{2} \quad \text{for } 0 < x \leqslant \sqrt{\pi}.$$

Moreover, $|F(x)| = |\sin x^2| \leqslant 1$ for all x. Thus (28) satisfies all the conditions of Theorem 6 with $k_1 = \frac{1}{2} > k_2 = -\frac{1}{2}$ and $A = 1$. Hence (28) has a global center at the origin.

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Authors' addresses: C.-D. Zhao, Institute of Mathematics, Chinese Academy of Sciences, Beijing, 100080 P. R. China, e-mail: `zhaocd42@hotmail.com`; Q.-M. He, Department of Mathematics, Suzhou University, Suzhou, 215006 P. R. China, e-mail: `heqim@sina.com`.