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ON THE EXISTENCE OF SOLUTIONS FOR SOME NONDEGENERATE NONLINEAR WAVE EQUATIONS OF KIRCHHOFF TYPE

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Abstract. Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary Γ. In this work we study the existence of solutions for the following boundary value problem:

$$
\begin{gathered}
\frac{\partial^2 y}{\partial t^2} - M\Big(\int_\Omega |\nabla y|^2 \,\mathrm{d}x\Big)\Delta y - \frac{\partial}{\partial t}\Delta y = f(y) \quad \text{in } Q = \Omega \times (0,\infty), \\
y = 0 \quad \text{in } \Sigma_1 = \Gamma_1 \times (0,\infty), \\
M\Big(\int_\Omega |\nabla y|^2 \,\mathrm{d}x\Big)\frac{\partial y}{\partial \nu} + \frac{\partial}{\partial t}\Big(\frac{\partial y}{\partial \nu}\Big) = g \quad \text{in } \Sigma_0 = \Gamma_0 \times (0,\infty), \\
y(0) = y_0, \quad \frac{\partial y}{\partial t}(0) = y_1 \quad \text{in} \quad \Omega,
\end{gathered}
\tag{1.1}
$$

where M is a C^1-function such that $M(\lambda) \geqslant \lambda_0 > 0$ for every $\lambda \geqslant 0$ and $f(y) = |y|^\alpha y$ for $\alpha \geqslant 0$.

Keywords: existence and uniqueness, Galerkin method, nondegenerate wave equation

MSC 2000: 35L70, 35L15, 65M60

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^2-boundary Γ. Let (Γ_0, Γ_1) be a partition of Γ, both parts having positive measure and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \varphi$. Let ν be the unit normal vector pointing toward the exterior of Ω and let $\frac{\partial}{\partial \nu}$ be the normal derivative. Let $M \in C^1([0,\infty);\mathbb{R})$ be a function such that $M(\lambda) \geqslant \lambda_0 > 0$ for every $\lambda \geqslant 0$.

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Our model was inspired by the problem

$$
\begin{aligned}
(1.2)\qquad & y_{tt} - M(\|\nabla y\|^2)\Delta y = f(y) \quad \text{in } Q = \Omega \times (0,T),\\
& y = 0 \quad \text{in } \Gamma_1 \times (0,T),\\
& y(0) = y_0, \quad y_t(0) = y_1 \quad \text{in } \Omega.
\end{aligned}
$$

This problem has its origin in the canonical model of Kirchhoff and Carrier which describes small vibrations of an elastic streched string. More precisely, we have

$$
(1.3)\qquad \varrho h \frac{\partial^2 y}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x \right\} \frac{\partial^2 y}{\partial x^2} + f \quad \text{for} \quad 0 < x < L, \quad t \geqslant 0,
$$

where y is the lateral deflection, x the space coordinate, t the time, E the Young modulus, ϱ the mass density, h the cross section area, L the length, p_0 the initial axial tension and f the external force. Kirchhoff was the first to introduce (1.3) in the study of oscillations of stretched strings and plates, so that (1.3) is called the wave equation of Kirchhoff type after him. Moreover, we call (1.3) a degenerate equation when $p_0 = 0$ and a nondegenerate one when $p_0 > 0$. In this paper, we show the existence of a unique weak and strong solution of problem (1.1). The works related to those kinds of problems treat homogeneous boundary conditions. In order to obtain the existence of solutions the authors employ the Galerkin method and make use of a special basis, that is, the basis formed by the eigenfunctions $(w_j)_{j\in\mathbb{N}}$ which possess the property

$$
(1.4)\qquad -\Delta w_j = \lambda_j w_j.
$$

In this paper, we use Galerkin's approximation and take into account nonhomogeneous boundary conditions but we cannot use the basis (1.4). Hence, we can not pass to the limit using the standard argument of compactness and so we have to find an other argument.

Our paper is organized as follows. In Section 2, we give the notation and main result. In Section 3 we prove the existence and uniqueness of a weak and strong solution of problem (1.1).

2. Notation and main result

In this section we present some notation that will be used throughout the paper and we state the main result. Let $V = \{v \in H^1(\Omega)\colon\ v = 0 \text{ on } \Gamma_1\}$ be endowed with the topology given by the norm $\|\nabla \cdot \|_{L^2(\Omega)}$. Note that V is a Hilbert subspace of $H^1(\Omega)$.

We first prepare the following well known lemma which will be needed later.

Lemma 2.1 (Sobolev-Poincaré [4]). *If either* $1 \leqslant q < +\infty \quad (N = 1, 2)$ *or* $1 \leqslant q \leqslant \frac{2N}{N-2} \quad (N \geqslant 3)$, *then there is a positive constant* C_* *such that*

$$\|u\|_q \leqslant C_* \|\nabla u\|_2 \quad \textit{for} \quad u \in V.$$

We write

$$(u, v) = \int_\Omega u(x)v(x)\,\mathrm{d}x \quad \text{and} \quad (u, v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)\,\mathrm{d}\Gamma.$$

We define the energy and the potential including the nonlinear terms associated with equation (1.1) by

$$E(y) = \|y'\|^2 + J(y), \quad J(y) = \overline{M}(\|\nabla y\|^2) - \frac{2}{\alpha+2}\|y\|_{\alpha+2}^{\alpha+2}, \tag{2.1}$$

where $\overline{M}(s) = \int_0^s M(r)\,\mathrm{d}r$.

We define a modified potential well by

$$\mathcal{W} = \{y \in V \mid I(y) = \overline{M}(\|\nabla y\|^2) - \|y\|_{\alpha+2}^{\alpha+2} > 0\}.$$

Now we are able to state the main result.

Theorem 2.1. *Let* $\{y_0, y_1, g\} \in \mathcal{W} \times L^2(\Omega) \times L^2(0, \infty;\ L^2(\Gamma_0))$. *If* $0 \leqslant \alpha \leqslant \frac{2}{N-4}$ ($0 \leqslant \alpha < \infty$ *if* $N \leqslant 4$) *or* $0 \leqslant \alpha \leqslant \frac{2}{N-2}$ ($0 \leqslant \alpha < \infty$ *if* $N = 1, 2$) *and*

$$\left\{\frac{1}{{\lambda_0}^{\alpha+2}}\right\}^{\frac{1}{2}} C_*^{\alpha+2}\Big[\frac{\alpha+2}{\alpha}E(y_0) + C_0^2\|g\|^2_{L^2(0,T;L^2(\Gamma_0))}\Big]^{\frac{\alpha}{2}} < 1,$$

then there exists $T = T(\|\Delta y_0\|, \|\nabla y_1\|) > 0$ *such that the problem* (1.2) *admits a unique weak solution* y *in the class*

$$C([0,T); V) \cap C^1([0,T); L^2(\Omega)) \cap C^2([0,T); L^2(\Omega)).$$

3. Existence and uniqueness of strong and weak solutions

In order to obtain strong solutions, let us consider $\{y_0, y_1, g\} \in \mathcal{W} \cap H^2(\Omega) \times V \cap H^2(\Omega) \times H^1(0, \infty; L^2(\Gamma_0))$. The variational formulation associated with the problem (1.1) is given by

$$(3.1)\qquad \begin{aligned}&(y''(t), w) + (M(\|\nabla y(t)\|^2)\nabla y(t), \nabla w) + (\nabla y'(t), \nabla w)\\ &\quad = (f(y(t)), w) + (g(t), w)_{\Gamma_0},\end{aligned}$$

where $f(y) = |y|^\alpha y$ for $\alpha \geqslant 0$.

We represent by $(w_j)_{j\in\mathbb{N}}$ a basis in $V \cap H^2(\Omega)$ which is orthonormal in $L^2(\Omega)$, and by V_m the subspace of $V \cap H^2(\Omega)$ generated by the first m vectors $w_1, w_2, \ldots, w_m$. We define $y_m(t) = \sum\limits_{i=1}^{m} g_{im}(t) w_i$, where $y_m(t)$ is the solution of the Cauchy problem

$$(3.2)\qquad \begin{aligned}&(y''_m(t), w_j) + (M(\|\nabla y_m(t)\|^2)\nabla y_m(t), \nabla w_j) + (\nabla y'_m(t), \nabla w_j)\\ &\quad = (f(y_m(t)), w_j) + (g(t), w_j)_{\Gamma_0}\end{aligned}$$

with the initial data

$$\begin{aligned}y_m(0) = y_{0m} &\to y_0 \quad \text{in } V \cap H^2(\Omega),\\ y'_m(0) = y_{1m} &\to y_1 \quad \text{in } V \cap H^2(\Omega).\end{aligned}$$

The approximate system is a system of m ordinary differential equations. It is easy to see that (3.1) has a local solution in $[0, t_m)$. The extension of the solution to the whole interval $[0, T]$ is a consequence of the first estimate we are going to obtain below.

A priori estimates

The first estimate. Multiplying both sides of equation (3.2) by $2g'_{jm}(t)$ and summing over j we have

$$(3.3)\qquad \begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|y'_m(t)\|^2 + \overline{M}(\|\nabla y_m(t)\|^2) - \frac{2}{\alpha+2}\|y_m(t)\|_{\alpha+2}^{\alpha+2}\right) + \|\nabla y'_m(t)\|^2\\ &\quad = (g(t), y'_m(t))_{\Gamma_0},\end{aligned}$$

where $\overline{M}(s) = \int_0^s M(r)\,\mathrm{d}r$.

Let C_0 be a positive constant such that $\|v\|_{\Gamma_0} \leqslant C_0\|\nabla v\|$ for every $v \in V$. Then

$$(3.4)\qquad \begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|y'_m(t)\|^2 + \overline{M}(\|\nabla y_m(t)\|^2) - \frac{2}{\alpha+2}\|y_m(t)\|_{\alpha+2}^{\alpha+2}\right) + \|\nabla y'_m(t)\|^2\\ &\quad \leqslant C_0\|g(t)\|_{\Gamma_0}\|\nabla y'_m(t)\| \leqslant \frac{C_0^2}{2}\|g(t)\|_{\Gamma_0}^2 + \frac{1}{2}\|\nabla y'_m(t)\|^2.\end{aligned}$$

Thus

$$\text{(3.5)}\qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\|y_m'(t)\|^2 + \overline{M}(\|\nabla y_m(t)\|^2) - \frac{2}{\alpha+2}\|y_m(t)\|_{\alpha+2}^{\alpha+2}\right) + \|\nabla y_m'(t)\|^2 \leqslant C_0^2\|g(t)\|_{\Gamma_0}^2.$$

Integrating (3.5) over $(0,t)$, we get

$$\begin{aligned}\|y_m'(t)\|^2 &+ \overline{M}(\|\nabla y_m(t)\|^2) - \frac{2}{\alpha+2}\|y_m(t)\|_{\alpha+2}^{\alpha+2} + \int_0^t \|\nabla y_m'(s)\|^2\,\mathrm{d}s \\ &\leqslant \|y_{1m}\|^2 + \overline{M}(\|\nabla y_{0m}\|^2) - \frac{2}{\alpha+2}\|y_{0m}\|_{\alpha+2}^{\alpha+2} + C_0^2\int_0^t \|g(s)\|_{\Gamma_0}^2\,\mathrm{d}s.\end{aligned}$$

Using (2.1), we obtain

$$\text{(3.6)}\qquad E(y_m(t)) + \int_0^t \|\nabla y_m'(s)\|^2\,\mathrm{d}s \leqslant E(y_0) + C_0^2\|g\|_{L^2(0,T;L^2(\Gamma_0))}^2.$$

To proceed in the estimation, we observe that the following lemma holds:

Lemma 3.1. *If $\alpha \leqslant \frac{4}{N-2}$ ($\alpha < \infty$ for $N = 1,2$) then $\mathcal{W}$ is a neighborhood of 0 in V and it is an open set.*

Proof. Using the theory of imbedding and the assumption on M, we get

$$\text{(3.7)}\qquad \begin{aligned}\|y_m(t)\|_{\alpha+2}^{\alpha+2} &\leqslant C_*^{\alpha+2}\|\nabla y_m(t)\|^{\alpha+2} \\ &= C_*^{\alpha+2}\lambda_0^{-1}\|\nabla y_m(t)\|^{\alpha}\lambda_0\|\nabla y_m(t)\|^2 \\ &\leqslant C_*^{\alpha+2}\lambda_0^{-1}\|\nabla y_m(t)\|^{\alpha}\overline{M}(\|\nabla y_m(t)\|^2).\end{aligned}$$

If we choose sufficiently large λ_0 such that $C_*^{\alpha+2}\|\nabla y_m\|^{\alpha} < \lambda_0$, then

$$\text{(3.8)}\qquad \|y_m(t)\|_{\alpha+2}^{\alpha+2} < \overline{M}(\|\nabla y_m\|^2).$$

Thus $I(y_m) = \overline{M}(\|\nabla y_m\|^2) - \|y_m\|_{\alpha+2}^{\alpha+2} > 0$ if $\|\nabla y_m\|$ is sufficiently small and $y_m \neq 0$. Hence $\mathcal{W}$ is a neighborhood of 0 in V and it is an open set. $\square$

To get an a priori bound on y_m, we shall show that $y_m \in \mathcal{W}$.

Corollary. *We assume that $\alpha \leqslant \frac{4}{N-2}$ ($\alpha < \infty$ if $N = 1,2$), $y_0 \in \mathcal{W} \cap H_0^1(\Omega)$ and $y_1 \in H_0^1(\Omega)$. If*

$$\text{(3.9)}\qquad \left\{\frac{1}{\lambda_0{}^{\alpha+2}}\right\}^{\frac{1}{2}} C_*^{\alpha+2}\left[\frac{\alpha+2}{\alpha}E(y_0) + C_0^2\|g\|_{L^2(0,T;L^2(\Gamma_0))}^2\right]^{\frac{\alpha}{2}} < 1,$$

then the solution $y_m(t)$ *of* (3.1) *is contained in* $\mathcal{W}$*, that is,*

$$I(y_m)=\overline{M}(\|\nabla y_m\|^2)-\|y_m\|_{\alpha+2}^{\alpha+2}>0 \quad \textit{on} \quad [0,+\infty).$$

P r o o f. Since $I(y_0)>0$, it follows from the continuity of $y_m(t)$ that

$$I(y_m(t))\geqslant 0 \quad \text{for some interval near} \quad t=0. \tag{3.10}$$

Let $t_{\max}$ be a maximal time (possibly $t_{\max}=T_m$) when (3.10) holds on $[0,t_{\max})$.

Note that

$$\begin{aligned} J(y_m(t)) &= \overline{M}(\|\nabla y_m(t)\|^2)-\frac{2}{\alpha+2}(\|y_m(t)\|_{\alpha+2}^{\alpha+2}) \\ &= \frac{2}{\alpha+2}I(y_m(t))+\frac{\alpha}{\alpha+2}\overline{M}(\|\nabla y_m(t)\|^2) \\ &\geqslant \frac{\alpha}{\alpha+2}\overline{M}(\|\nabla y_m(t)\|^2) \quad \text{on} \quad [0,t_{\max}). \end{aligned} \tag{3.11}$$

By (3.6), (3.11) and (2.1), we have

$$\begin{aligned} \overline{M}(\|\nabla y_m(t)\|^2) &\leqslant \frac{\alpha+2}{\alpha}J(y_m(t))\leqslant \frac{\alpha+2}{\alpha}E(y_m(t)) \\ &\leqslant \frac{\alpha+2}{\alpha}\{E(y_0)+C_0^2\|g\|^2_{L^2(0,T;L^2(\Gamma_0))}\} \quad \text{on} \quad [0,t_{\max}). \end{aligned} \tag{3.12}$$

It follows from the Sobolev-Poincaré inequality, (3.9) and (3.12) that

$$\begin{aligned} \|y_m(t)\|_{\alpha+2}^{\alpha+2} &\leqslant C_*^{\alpha+2}\|\nabla y_m(t)\|^{\alpha+2} \\ &= \frac{1}{\lambda_0^{\frac{\alpha+2}{2}}}C_*^{\alpha+2}\{\lambda_0\|\nabla y_m(t)\|^2\}^{\frac{\alpha+2}{2}} \\ &\leqslant \frac{1}{\lambda_0^{\frac{\alpha+2}{2}}}C_*^{\alpha+2}\left\{\frac{\alpha+2}{\alpha}E(y_0)+C_0^2\|g\|^2_{L^2(0,T;L^2(\Gamma_0))}\right\}^{\frac{\alpha}{2}}\overline{M}(\|\nabla y_m(t)\|^2) \\ &\leqslant \overline{M}(\|\nabla y_m(t)\|^2) \quad \text{on} \quad [0,t_{\max}). \end{aligned} \tag{3.13}$$

Therefore we get $I(y_m(t))>0$ on $[0,t_{\max})$. This implies that we can take $t_{\max}=T_m$. This completes the proof of Corollary. □

Using Corollary, we can deduce an a priori bound for y_m:

$$\begin{aligned} E(y_m(t)) &= \|y_m'(t)\|_2^2+J(y_m(t)) \\ &= \|y_m'(t)\|_2^2+\frac{2}{\alpha+2}I(y_m(t))+\frac{\alpha}{\alpha+2}\overline{M}(\|\nabla y_m(t)\|_2^2) \\ &\geqslant \|y_m'(t)\|_2^2+\frac{\alpha}{\alpha+2}\overline{M}(\|\nabla y_m(t)\|_2^2). \end{aligned} \tag{3.14}$$

Thus, (3.6) and (3.14) imply

$$\|y_m'(t)\|^2 + \frac{\alpha}{\alpha+2}\overline{M}(\|\nabla y_m(t)\|^2) + \int_0^t \|\nabla y_m'(s)\|^2\,\mathrm{d}s \leqslant E(y_0) + C_0^2\|g\|^2_{L^2(0,T;L^2(\Gamma_0))} \leqslant L_1, \tag{3.15}$$

where L_1 is a positive constant independent of $m \in \mathbb{N}$ and $t \in [0,T]$.

The second estimate. Multiplying both sides of equation (3.1) by $2g_{jm}''(t)$ and summing over j, we have

$$\begin{aligned}\|y_m''(t)\|^2 &+ M(\|\nabla y_m(t)\|^2)(\nabla y_m(t), \nabla y_m''(t)) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla y_m'(t)\|^2 \\ &= (|y_m(t)|^\alpha y_m(t), y_m''(t)) + \frac{\mathrm{d}}{\mathrm{d}t}(g(t), y_m'(t))_{\Gamma_0} - (g'(t), y_m'(t))_{\Gamma_0}.\end{aligned} \tag{3.16}$$

On the other hand, we have

$$\begin{aligned}&M(\|\nabla y_m(t)\|^2)(\nabla y_m(t), \nabla y_m''(t)) \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}\{M(\|\nabla y_m(t)\|^2)(\nabla y_m(t), \nabla y_m'(t))\} \\ &\qquad - 2M'(\|\nabla y_m(t)\|^2)|(\nabla y_m(t), \nabla y_m'(t))|^2 \\ &\qquad - M(\|\nabla y_m(t)\|^2)\|\nabla y_m'(t)\|^2.\end{aligned} \tag{3.17}$$

Thus (3.16) and (3.17) imply

$$\begin{aligned}\|y_m''(t)\|^2 &+ \frac{\mathrm{d}}{\mathrm{d}t}\{M(\|\nabla y_m(t)\|^2)(\nabla y_m(t), \nabla y_m'(t))\} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla y_m'(t)\|^2 \\ &= 2M'(\|\nabla y_m(t)\|^2)|(\nabla y_m(t), \nabla y_m'(t))|^2 \\ &\quad + M(\|\nabla y_m(t)\|^2)\|\nabla y_m'(t)\|^2 + (|y_m(t)|^\alpha y_m(t), y_m''(t)) \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}t}(g(t), y_m'(t))_{\Gamma_0} - (g'(t), y_m'(t))_{\Gamma_0}.\end{aligned} \tag{3.18}$$

Now, since $\alpha < \frac{2}{N-2}$, the Sobolev-Poincaré inequality implies

$$\begin{aligned}(|y_m(t)|^\alpha y_m(t), y_m''(t)) &\leqslant \|y_m(t)\|^{\alpha+1}_{2(\alpha+1)}\|y_m''(t)\| \\ &\leqslant C_*\|\nabla y_m(t)\|^{\alpha+1}\|y_m''(t)\| \\ &\leqslant C_\alpha + \frac{1}{2}\|y_m''(t)\|^2,\end{aligned} \tag{3.19}$$

where C_α is a positive constant.

Also, the first estimate implies

$$
\tag{3.20}
\begin{aligned}
2M'(\|\nabla y_m(t)\|^2)|(\nabla y_m(t), \nabla y_m'(t))|^2 &\leqslant C_1\|\nabla y_m'(t)\|^2 \\
M(\|\nabla y_m(t)\|^2)\|\nabla y_m'(t)\|^2 &\leqslant C_2\|\nabla y_m'(t)\|^2
\end{aligned}
$$

and

$$
\tag{3.21}
\begin{aligned}
|(g'(t), y_m'(t))_{\Gamma_0}| &\leqslant \lambda\|g'(t)\|_{\Gamma_0}\|\nabla y_m'(t)\| \\
&\leqslant C_\lambda(\|g'(t)\|^2_{\Gamma_0} + \|\nabla y_m'(t)\|^2).
\end{aligned}
$$

Thus (3.18)–(3.21) imply

$$
\tag{3.22}
\begin{aligned}
&\frac{1}{2}\|y_m''(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\{M(\|\nabla y_m(t)\|^2)(\nabla y_m(t), \nabla y_m'(t))\} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla y_m'(t)\|^2 \\
&\quad\leqslant C_\alpha + C_\lambda\|g'(t)\|^2_{\Gamma_0} + C_3\|\nabla y_m'(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}(g(t), y_m'(t))_{\Gamma_0}.
\end{aligned}
$$

Integrating the inequality (3.22) over $(0,t)$, we obtain

$$
\tag{3.23}
\begin{aligned}
&\frac{1}{2}\int_0^t \|y_m''(s)\|^2\,\mathrm{d}s + \frac{1}{2}\|\nabla y_m'(t)\|^2 \\
&\quad\leqslant M(\|\nabla y_{0m}\|^2)(\nabla y_{0m}, \nabla y_{1m}) \\
&\qquad + \frac{1}{2}\|\nabla y_{1m}\|^2 + M(\|\nabla y_m(t)\|^2)|(\nabla y_m(t), \nabla y_m'(t))| \\
&\qquad + \int_0^t (C_\alpha + C_\lambda\|g'(s)\|^2_{\Gamma_0} + C_3\|\nabla y_m'(s)\|^2)\,\mathrm{d}s \\
&\qquad + (g(t), y_m'(t))_{\Gamma_0} - (g(0), y_{1m})_{\Gamma_0} \\
&\quad\leqslant C_4 + M(\|\nabla y_{0m}\|^2)(\nabla y_{0m}, \nabla y_{1m}) + \frac{1}{2}\|\nabla y_{1m}\|^2 \\
&\qquad + C_5\int_0^t (\|g'(s)\|^2_{\Gamma_0} + \|\nabla y_m'(s)\|^2)\,\mathrm{d}s + C_6\|g(t)\|^2_{\Gamma_0} \\
&\qquad + \frac{1}{4}\|\nabla y_m'(t)\|^2 + C_7\|g(0)\|_{\Gamma_0}\|\nabla y_{1m}\| \\
&\quad\leqslant C_8 + C_9\int_0^t \|\nabla y_m'(s)\|^2\,\mathrm{d}s + \frac{1}{4}\|\nabla y_m'(t)\|^2.
\end{aligned}
$$

Thus

$$
\tag{3.24}
\frac{1}{2}\int_0^t \|y_m''(s)\|^2\,\mathrm{d}s + \frac{1}{4}\|\nabla y_m'(t)\|^2 \leqslant C_{10}\int_0^t (1 + \|\nabla y_m'(s)\|^2)\,\mathrm{d}s.
$$

Using Gronwall's lemma, we have

$$
\tag{3.25}
\int_0^t \|y_m''(s)\|^2\,\mathrm{d}s + \|\nabla y_m'(t)\|^2 \leqslant L_2,
$$

where L_2 is a positive constant independent of $m \in \mathbb{N}$ and $t \in [0,T]$.

The third estimate. Let $m_2 \geqslant m_1$ be two natural numbers and consider $z_m = y_{m_2} - y_{m_1}$. Then we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z_m'(t)\|^2 + 2\|\nabla z_m'(t)\|^2 = -2M(\|\nabla y_{m_2}(t)\|^2)(\nabla y_{m_2}(t), \nabla z_m'(t)) + 2M(\|\nabla y_{m_1}(t)\|^2)(\nabla y_{m_1}(t), \nabla z_m'(t)) + (f(y_{m_2}(t)) - f(y_{m_1}(t)), z_m'(t)). \tag{3.26}$$

On the other hand, we note that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(M(\|\nabla y_{m_2}(t)\|^2)\|\nabla z_m(t)\|^2) \\ &= 2M'(\|\nabla y_{m_2}(t)\|^2)(\nabla y_{m_2}(t), \nabla y_{m_2}'(t))\|\nabla z_m(t)\|^2 \\ &\quad + 2M(\|\nabla y_{m_2}(t)\|^2)(\nabla z_m(t), \nabla z_m'(t)) \\ &= 2M'(\|\nabla y_{m_2}(t)\|^2)(\nabla y_{m_2}(t), \nabla y_{m_2}'(t))\|\nabla z_m(t)\|^2 \\ &\quad + 2M(\|\nabla y_{m_2}(t)\|^2)(\nabla y_{m_2}(t) - \nabla y_{m_1}(t), \nabla z_m'(t)). \end{aligned} \tag{3.27}$$

Then (3.26) and (3.27) imply

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|z_m'(t)\|^2 + M(\|\nabla y_{m_2}(t)\|^2)\|\nabla z_m(t)\|^2) + 2\|\nabla z_m'(t)\|^2 \\ &= 2\{M(\|\nabla y_{m_1}(t)\|^2) - M(\|\nabla y_{m_2}(t)\|^2)\}(\nabla y_{m_1}(t), \nabla z_m'(t)) \\ &\quad + 2M'(\|\nabla y_{m_2}(t)\|^2)(\nabla y_{m_2}(t), \nabla y_{m_2}'(t))\|\nabla z_m(t)\|^2 \\ &\quad + (f(y_{m_2}(t)) - f(y_{m_1}(t)), z_m'(t)). \end{aligned} \tag{3.28}$$

We note that the first estimate (3.15) yields

$$\begin{aligned} &|M(\|\nabla y_{m_1}(t)\|^2) - M(\|\nabla y_{m_2}(t)\|^2)| \\ &\quad \leqslant \int_{\|\nabla y_{m_1}(t)\|^2}^{\|\nabla y_{m_2}(t)\|^2} |M'(\xi)|\,\mathrm{d}\xi \leqslant C_{11}|\,\|\nabla y_{m_2}(t)\|^2 - \|\nabla y_{m_1}(t)\|^2| \\ &\quad \leqslant C_{11}(\|\nabla y_{m_2}(t)\| + \|\nabla y_{m_1}(t)\|)\|\nabla z_m(t)\| \\ &\quad \leqslant C_{12}\|\nabla z_m(t)\|, \end{aligned} \tag{3.29}$$

where C_{11} and C_{12} are positive constants.

From (3.15) and (3.29) we get

$$\begin{aligned} &2|M(\|\nabla y_{m_1}(t)\|^2) - M(\|\nabla y_{m_2}(t)\|^2)|(\nabla y_{m_1}(t), \nabla z_m'(t)) \\ &\quad \leqslant 2C_{12}\|\nabla z_m(t)\|\|\nabla y_{m_1}(t)\|\|\nabla z_m'(t)\| \\ &\quad \leqslant C_{13}\|\nabla z_m(t)\|^2 + \|\nabla z_m'(t)\|^2, \end{aligned} \tag{3.30}$$

where C_{13} is a positive constant.

Again from (3.15) and (3.25), there exists a positive constant C_{14} such that

$$2M'(\|\nabla y_{m_2}(t)\|^2)(\nabla y_{m_2}(t), \nabla y'_{m_2}(t))\|\nabla z_m(t)\|^2 \leqslant C_{14}\|\nabla z_m(t)\|^2. \tag{3.31}$$

We note that for some constant C_{15} we have

$$\begin{aligned} |f(y_{m_2}(t)) - f(y_{m_1}(t))| &= ||y_{m_2}(t)|^\alpha y_{m_2}(t) - |y_{m_1}(t)|^\alpha y_{m_1}(t)| \\ &\leqslant C_{15}(|y_{m_1}(t)|^\alpha + |y_{m_2}(t)|^\alpha)|z_m(t)|. \end{aligned} \tag{3.32}$$

Thus (3.15) and the Sobolev imbedding imply

$$\begin{aligned} &|(f(y_{m_2}(t)) - f(y_{m_1}(t)), z'_m(t))| \\ &\qquad \leqslant C_{15}\int_\Omega (|y_{m_1}(t)|^\alpha + |y_{m_2}(t)|^\alpha)|z_m(t)|\,|z'_m(t)|\,\mathrm{d}x \\ &\qquad \leqslant C_{15}(\|y_{m_1}(t)\|^\alpha_{\alpha N} + \|y_{m_2}(t)\|^\alpha_{\alpha N})\|z_m(t)\|_{\frac{2N}{N-2}}\|z'_m(t)\| \\ &\qquad \leqslant C_{16}(\|\nabla y_{m_1}(t)\|^\alpha + \|\nabla y_{m_2}(t)\|^\alpha)\|\nabla z_m(t)\|\,\|z'_m(t)\| \\ &\qquad \leqslant C_{17}(\|\nabla z_m(t)\|^2 + \|z'_m(t)\|^2). \end{aligned} \tag{3.33}$$

Combining the inequalities (3.30), (3.31) and (3.33), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|z'_m(t)\|^2 + M(\|\nabla y_{m_2}(t)\|^2)\|\nabla z_m(t)\|^2) + \|\nabla z'_m(t)\|^2 \\ &\qquad \leqslant C_{18}\{\|z'_m(t)\|^2 + \|\nabla z_m(t)\|^2\}. \end{aligned} \tag{3.34}$$

Integrating the inequality (3.34) over $(0,t)$, employing the Gronwall lemma we obtain the third estimate

$$\|z'_m(t)\|^2 + \|\nabla z_m(t)\|^2 + \int_0^t \|\nabla z'_m(s)\|^2\,\mathrm{d}s \leqslant L_3, \tag{3.35}$$

where L_3 is a positive constant.

Due to the estimates (3.15), (3.25) and (3.35), we can extract a subsequence (y_μ) of (y_m) such that

$$\begin{aligned} y_\mu &\to y \quad &&\text{strongly} \quad &&\text{in } C^0([0,T];V), \\ y'_\mu &\to y' \quad &&\text{strongly} \quad &&\text{in } C^0([0,T];L^2(\Omega)), \\ y'_\mu &\to y' \quad &&\text{strongly} \quad &&\text{in } L^2([0,T];V), \\ y'_\mu &\to y' \quad &&\text{weakly star} \quad &&\text{in } L^\infty([0,T];V), \\ y''_\mu &\to y'' \quad &&\text{weakly} \quad &&\text{in } L^2([0,T];L^2(\Omega)). \end{aligned} \tag{3.36}$$

Now (3.36) implies

$$\|\nabla y_\mu\|^2 \to \|\nabla y\|^2 \quad \text{in } C^0([0,T]).$$

From the above result and $M \in C^1([0,\infty);R)$ we obtain

$$M(\|\nabla y_\mu\|^2) \to M(\|\nabla y\|^2) \quad \text{in } C^0([0,T]).$$

Thus

$$M(\|\nabla y_\mu\|^2)y_\mu \to M(\|\nabla y\|^2)y \quad \text{in} \quad C^0([0,T];V).$$

Applying a method similar to (3.33), we get for every $\varphi \in L^2(\Omega)$

$$\begin{aligned}|(|y_\mu(t)|^\alpha y_\mu(t) - |y(t)|^\alpha y(t), \varphi)| \leqslant C_{19}(\|\nabla y_\mu(t)\|^\alpha + \|\nabla y(t)\|^\alpha)& \\ \times \|\nabla y_\mu(t) - \nabla y(t)\|\, \|\varphi(t)\| \to 0.&\end{aligned}$$

Thus the function $y : Q \to \mathbb{R}$ satisfies

$$y_{tt} - M(\|\nabla y\|^2)\Delta y - \Delta y' = |y|^\alpha y \quad \text{in } L^2(0,\infty;L^2(\Omega)).$$

Also, taking into account that $\Delta\{M(\|\nabla y\|^2)y + y'\} \in L^2(\Omega)$ and $M(\|\nabla y\|^2)y + y') \in V$ by the generalized Green's formula, we infer $\frac{\partial}{\partial\nu}(M(|\nabla y|^2)y + y') = g \in L^2(0,\infty;L^2(\Gamma_0))$.

This completes the proof.

Remark. We observe that for a.e. $t \geqslant 0$ the function $y\colon \Omega \to \mathbb{R}$ is the weak solution to the elliptic problem

$$\begin{aligned}-\Delta\{M(\|\nabla y\|^2)y + y'\} &= |y|^\alpha y - y'' \quad \text{in} \quad L^2(\Omega),\\ M(\|\nabla y\|^2)y + y' &= 0 \quad \text{in } \Gamma_1,\\ \frac{\partial}{\partial\nu}(M(\|\nabla y\|^2)y + y') &= g \in L^2(0,\infty;L^2(\Gamma_0)).\end{aligned}$$

Since $\overline{\Gamma_0} \cap \overline{\Gamma_1}$ is empty, the theory of elliptic problems gives $y \in L^2(0,\infty;H^{\frac{3}{2}}(\Omega))$. Now we can consider $g \in H^1(0,\infty;H^{\frac{1}{2}}(\Gamma_0))$, hence one has $y \in L^2(0,\infty;V \cap H^2(\Omega))$.

Uniqueness. Let y and $\bar{y}$ be two solutions of the problem (1.1). Then defining $z = y - \bar{y}$ and repeating the same argument already used in the third estimate we obtain $\|\nabla z\| = \|z'\| = 0$.

Existence of weak solution. We have just the existence of solutions to problem (1.1) when the initial data is smooth. However, when $\{y_0, y_1, g\} \in \mathcal{W} \times$

$L^2(\Omega) \times L^2(0,\infty; L^2(\Gamma_0))$, there exist $\{y_{\mu 0}, y_{\mu 1}, g_\mu\} \in \mathcal{W} \cap H^2(\Omega) \times V \cap L^2(\Omega) \times H^1(0,\infty; L^2(\Gamma_0))$ such that

$$\{y_{\mu 0}, y_{\mu 1}, g_\mu\} \to \{y_0, y_1, g\} \in \mathcal{W} \times L^2(\Omega) \times L^2(0,\infty; L^2(\Gamma_0))$$

and using the density argument and proceeding analogously to the first and second estimates we can find a sequence $\{y_\mu\}$ of solutions to problem (1.1) such that $y_\mu \in C^0([0,T]; V)$, $y'_\mu \in C^0([0,T]; L^2(\Omega))$ and $y''_\mu \in L^2([0,T]; \; L^2(\Omega))$,

$$\begin{aligned} y_\mu &\to y && \text{strongly} && \text{in } C^0([0,T];V), \\ y'_\mu &\to y' && \text{strongly} && \text{in } C^0([0,T];L^2(\Omega)), \\ y'_\mu &\to y' && \text{strongly} && \text{in } L^2([0,T];V), \\ |y_\mu|^\alpha y_\mu &\to |y|^\alpha y && \text{weakly} && \text{in } L^2(Q), \\ y''_\mu &\to y'' && \text{weakly} && \text{in } L^2([0,T];L^2(\Omega)). \end{aligned} \tag{3.37}$$

The above convergences are sufficient for passing to the limit in order to obtain a weak solution of (1.1) which satisfies

$$y'' - M(\|\nabla y\|^2)\Delta y = |y|^\alpha y \quad \text{in } L^2_{\text{loc}}(0,\infty;V').$$

Moreover, we obtain

$$\frac{\partial}{\partial \nu}(M(\|\nabla y\|^2)y + y') = g \quad \text{in } L^2_{\text{loc}}(0,\infty;L^2(\Gamma_0)). \tag{3.38}$$

Indeed, let us consider the problems

$$\begin{aligned} -\Delta p &= |y|^\alpha y && \text{in } \Omega, \\ p &= 0 && \text{on } \Gamma_1, \\ \frac{\partial p}{\partial \nu} &= 0 && \text{on } \Gamma_0, \end{aligned} \tag{3.39}$$

$$\begin{aligned} -\Delta q &= y' && \text{in } \Omega, \\ q &= 0 && \text{on } \Gamma_1 \\ \frac{\partial q}{\partial \nu} &= g && \text{on } \Gamma_0, \end{aligned} \tag{3.40}$$

which admit unique solutions

$$p, q \in L^2_{\text{loc}}(0,\infty,\mathcal{H}), \quad \text{where} \quad \mathcal{H} = \{u \in V; \; \Delta u \in L^2(\Omega)\}. \tag{3.41}$$

On the other hand, we can write

$$-\Delta\{M(\|\nabla y\|^2)y + y'\} = |y|^\alpha y - y'' \quad \text{in } L^2_{\mathrm{loc}}(0,\infty;V')$$

and considering (3.39) and (3.40) we conclude

$$-\Delta\{M(\|\nabla y\|^2)y + y'\} = -\Delta p + \Delta q' \quad \text{in } \mathcal{D}'_{\mathrm{loc}}(0,\infty;V').$$

Then we deduce

$$-\int_0^T \Delta\{M(\|\nabla y\|^2)y + y'\}(t)\theta(t)\,\mathrm{d}t = -\int_0^T \Delta p(t)\theta(t)\,\mathrm{d}t - \int_0^T \Delta q(t)\theta'(t)\,\mathrm{d}t$$

in V' for all $\theta \in D(0,T)$.

Consequently,

$$\int_0^T \{M(\|\nabla y\|^2)y + y'\}(t)\theta(t)\,\mathrm{d}t = \int_0^T p(t)\theta(t)\,\mathrm{d}t + \int_0^T q(t)\theta'(t)\,\mathrm{d}t$$

in V. The last equality combined with (3.41) allows us to conclude that

$$M(\|\nabla y\|^2)y + y' = p - q' \quad \text{in } H^{-1}_{\mathrm{loc}}(0,\infty;\mathcal{H}).$$

In the same way, considering for each $\mu \in \mathbb{N}$

$$\begin{aligned} -\Delta p_\mu &= |y_\mu|^\alpha y_\mu && \text{in } \Omega,\\ p_\mu &= 0 && \text{on } \Gamma_1,\\ \frac{\partial p_\mu}{\partial \nu} &= 0 && \text{on } \Gamma_0, \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} -\Delta q_\mu &= y'_\mu && \text{in } \Omega,\\ q_\mu &= 0 && \text{on } \Gamma_1,\\ \frac{\partial q_\mu}{\partial \nu} &= g_\mu && \text{on } \Gamma_0, \end{aligned} \tag{3.43}$$

we have $p_\mu, q_\mu \in L^2_{\mathrm{loc}}(0,\infty;H)$ and $-\Delta\{M(|\Delta y_\mu|^2)y_\mu + y'_\mu\} = |y_\mu|^\alpha y_\mu - y''_\mu$ in $L^2_{\mathrm{loc}}(0,\infty;V')$.

Next we are going to prove that

$$q_\mu \to q \quad \text{in } L^2_{\mathrm{loc}}(0,\infty;\mathcal{H}). \tag{3.44}$$

Indeed, first taking into account the generalized Green's formula and considering (3.40) and (3.43), we infer

$$\int_\Omega |\nabla(q_\mu - q)|^2 \,\mathrm{d}x = \int_\Omega |(y'_\mu - y')(q_\mu - q)|^2 \,\mathrm{d}x + \int_{\Gamma_0} |(g_\mu - g)(q_\mu - q)|^2 \,\mathrm{d}\Gamma.$$

Integrating it over $[0, T]$, we obtain

$$\int_0^T |\nabla q_\mu(t) - \nabla q(t)|^2 \,\mathrm{d}t \leqslant C \int_0^T |y'_\mu(t) - y'(t)|^2 + |g_\mu(t) - g(t)|^2_{\Gamma_0} \,\mathrm{d}t. \tag{3.45}$$

However, (3.40) and (3.43) yield

$$\begin{aligned} \|q_\mu - q\|^2_{L^2(0,T;\mathcal{H})} &= \int_0^T |\nabla q'_\mu(t) - \nabla q(t)|^2 + |\Delta q_\mu(t) - \Delta q(t)|^2 \,\mathrm{d}t \\ &= \int_0^T |\nabla q'_\mu(t) - \nabla q(t)|^2 + |y'_\mu(t) - y'(t)|^2 \,\mathrm{d}t. \end{aligned} \tag{3.46}$$

Combining (3.37), (3.45) and (3.46), we conclude

$$q_\mu \to q \quad \text{in } L^2_{\mathrm{loc}}(0, \infty; \mathcal{H}).$$

Analogously, we get

$$p_\mu \to p \quad \text{in } L^2_{\mathrm{loc}}(0, \infty; \mathcal{H}). \tag{3.47}$$

Thus from (3.44) and (3.47) we have

$$p_\mu - q'_\mu \to p - q' \quad \text{in } H^{-1}_{\mathrm{loc}}(0, \infty; \mathcal{H}).$$

Therefore

$$g_\mu = \frac{\partial}{\partial \nu}(p_\mu - q'_\mu) \to \frac{\partial}{\partial \nu}(p - q') \quad \text{in } H^{-1}_{\mathrm{loc}}(0, \infty; H^{-\frac{1}{2}}(\Gamma_0)). \tag{3.48}$$

On the other hand

$$g_\mu \to g \quad \text{in } L^2_{\mathrm{loc}}(0, \infty; L^2(\Gamma_0)). \tag{3.49}$$

Then combining (3.48) and (3.49), we deduce the desired result

$$\frac{\partial}{\partial \nu}(M(\|\nabla y\|^2) y + y') = g \quad \text{in } L^2_{\mathrm{loc}}(0, \infty; L^2(\Gamma_0)).$$

Uniqueness. Let y_1 and y_2 be two weak solutions to problem (1.1). Then defining $z = y_1 - y_2$, one has

$$z'' - \Delta\{M(\|\nabla y_1\|^2)y_1 + y_1' - (M(\|\nabla y_2\|^2)y_2 + y_2')\} = |y_1|^\alpha y_1 - |y_2|^\alpha y_2 \quad \text{in } L^2_{\text{loc}}(0,\infty;V'),$$

$$z = 0 \quad \text{in } \Sigma_0,$$

$$\frac{\partial}{\partial\nu}\{M(\|\nabla y_1\|^2)y_1 + y_1' - (M(\|\nabla y_2\|^2)y_2 + y_2')\} = 0 \quad \text{in } L^2_{\text{loc}}(0,\infty;L^2(\Gamma_0)),$$

$$z(0) = 0, \quad z'(0) = 0.$$

Then noting that $z' \in L^2(0,\infty;V)$ we see that the duality $\langle z'', z'\rangle_{V'\times V}$ makes sense. Consequently,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z'(t)\|^2 + \|\nabla z'(t)\|^2 = {} & M(\|\nabla y_2\|^2)(\nabla y_2(t), \nabla z'(t)) \\ & - M(\|\nabla y_1\|^2)(\nabla y_1(t), \nabla z'(t)) \\ & + (|y_1(t)|^\alpha y_1(t) - |y_2(t)|^\alpha y_2(t), z'(t)).\end{aligned}$$

From the above equality and making use arguments analogous to those in the third estimate we deduce that $\|z'(t)\|^2 = \|\nabla z(t)\|^2 = 0$. This completes the proof. □

References

[1] *M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. S. Prates Filho and J. A. Soriano*: Existence and exponential decay for a Kirchhoff-Carrier model with viscosity. J. Math. Anal. Appl. *226* (1998), 40–60.

[2] *R. Ikehata*: On the existence of global solutions for some nonlinear hyperbolic equations with Neumann conditions. TRU Math. *24* (1988), 1–17.

[3] *J. L. Lions*: Quelques méthode de résolution des probléme aux limites nonlinéaire. Dunod Gauthier–Villars, Paris (1969).

[4] *T. Matsuyama and R. Ikehata*: On global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping terms. J. Math. Anal. Appl. *204* (1996), 729–753.

[5] *M. Nakao*: Asymptotic stability of the bounded or almost periodic solutions of the wave equations with nonlinear damping terms. J. Math. Anal. Appl. *58* (1977), 336–343.

[6] *K. Narasimha*: Nonlinear vibration of an elastic string. J. Sound Vibration *8* (1968), 134–146.

[7] *K. Nishihara and Y. Yamada*: On global solutions of some degenerate quasilinear hyperbolic equation with dissipative damping terms. Funkcial. Ekvac. *33* (1990), 151–159.

[8] *K. Ono*: Global existence, decay and blowup of solutions for some mildly degenerate nonlinear Kirchhoff strings. J. Differential Equations *137* (1997), 273–301.

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