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John R. Graef; C. Qian

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GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM OF A NONLINEAR DIFFERENCE EQUATION

J. R. Graef, Chattanooga, and C. Qian, Mississippi State

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Abstract. The authors consider the nonlinear difference equation

$$x_{n+1} = \alpha x_n + x_{n-k} f(x_{n-k}), \quad n = 0, 1, \ldots \tag{0.1}$$

where

$$\alpha \in (0,1), \ \ k \in \{0, 1, \ldots\} \ \text{ and } \ f \in C^1[[0,\infty),[0,\infty)]$$

with $f'(x) < 0$.

They give sufficient conditions for the unique positive equilibrium of (0.1) to be a global attractor of all positive solutions. The results here are somewhat easier to apply than those of other authors. An application to a model of blood cell production is given.

Keywords: nonlinear difference equation, global attractivity, oscillation

MSC 2000: 39A10, 92D25

1. Introduction

Our aim in this paper is to study the global attractivity of the nonlinear difference equation

$$x_{n+1} = \alpha x_n + x_{n-k} f(x_{n-k}), \quad n = 0, 1, \ldots \tag{1.1}$$

where

$$\alpha \in (0,1), \ \ k \in \{0, 1, \ldots\} \ \text{ and } \ f \in C^1[[0,\infty),[0,\infty)] \ \text{ with } f'(x) < 0. \tag{1.2}$$

Clearly, $\bar{x} = f^{-1}(1-\alpha)$ is the unique positive equilibrium of (1.1). If we let

$$x_{-k}, x_{-k+1}, \ldots, x_0 \tag{1.3}$$

be $k+1$ given nonnegative numbers with $x_0 > 0$, then (1.1) has a unique positive solution with initial condition (1.3). Results on the global attractivity of the positive equilibrium of equations of the form (1.1) have been obtained by Ivanov [2] and Karakostas, Philos and Sficas [3]. However, their results involve some implicit conditions which can make them difficult to apply. In the next section, we establish a criteria ensuring that the positive equilibrium $\bar{x}$ is a global attractor of all positive solutions of (1.1). This is accomplished under different conditions than those imposed in [2]–[3] and, moreover, our hypotheses will be much easier to verify.

Our motivation for studying (1.1) comes from the fact that some special cases of (1.1) arise as discrete models of various biological phenomena. For example, the equation

$$x_{n+1} = \alpha x_n + \frac{\beta x_{n-k}}{1 + x_{n-k}^r}, \tag{1.4}$$

where $\alpha \in (0,1)$ and $\beta, r \in (0,\infty)$, is a discrete version of a model of haematopoiesis (blood cell production). The global attractivity of (1.4) is studied in [2] and [3]. By applying our result for (1.1), we establish some new global attractivity results for (1.4); we will discuss this in Section 3.

In a recent paper [1], the global stability of the nonlinear difference equation

$$x_{n+1} = \alpha x_n + f(x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.5}$$

where

$$\alpha \in (0,1), \;\; k \in \{0,1,\ldots\}, \;\text{ and }\; f \in C^1[[0,\infty),[0,\infty)] \;\text{ with }\; f'(x) < 0, \tag{1.6}$$

is studied by using Liapunov's method. The asymptotic behavior of positive solutions of (1.1) is quite different from the global behavior of positive solutions of (1.5) since the nonlinear term in (1.5) is a decreasing function, while the nonlinear term in (1.1) is a "tent" function. For example, if a positive solution of (1.5) does not oscillate about the positive equilibrium of the equation, this solution must be monotonic (see [1]), but this is not the case for (1.1). Hence, in this paper we need to take a different approach in analyzing the behavior of the solutions.

2. Attractivity of the equilibrium

In this section, we give sufficient conditions under which the positive equilibrium $\bar{x}$ of (1.1) is a global attractor of all positive solutions. First, we introduce some lemmas that are needed to establish our main result.

Lemma 1. *If $\{x_n\}$ is a positive solution of (1.1) that is eventually less than or equal to $\bar{x}$, then it is persistent. Furthermore, if the function $xf(x)$ is bounded, then every positive solution $\{x_n\}$ of (1.1) is bounded.*

Proof. Let $\{x_n\}$ be a positive solution of (1.1) that satisfies

$$x_n \leqslant \bar{x} \quad \text{for } n \geqslant n_0 \tag{2.1}$$

where n_0 is a positive integer. We claim that $\{x_n\}$ is persistent. Otherwise,

$$\liminf_{n\to\infty} x_n = 0. \tag{2.2}$$

Let $\varepsilon = \min\{x_n\colon\ n_0 \leqslant n \leqslant n_0 + k\}$; then $\varepsilon > 0$. We claim that

$$x_n \geqslant \frac{\varepsilon}{2} \quad \text{for } n \geqslant n_0 + k.$$

If not, then there exists a positive integer $n_1 > n_0 + k$ such that

$$x_{n_1} < \frac{\varepsilon}{2} \quad \text{and} \quad x_n \geqslant \frac{\varepsilon}{2} \quad \text{for } n_0 + k \leqslant n < n_1. \tag{2.3}$$

Observe that from (1.1) we have

$$\alpha(x_{n_1} - x_{n_1-1}) = -(1-\alpha)x_{n_1} + x_{n_1-k-1}f(x_{n_1-k-1})$$

which, in view of (2.3), implies that

$$0 > -(1-\alpha)\frac{\varepsilon}{2} + \frac{\varepsilon}{2}f(x_{n_1-k-1}).$$

Hence, it follows that $f(x_{n_1-k-1}) < 1-\alpha$. Then, by noting that $f(\bar{x}) = 1-\alpha$ and the strict decreasing property of f, we see that $x_{n_1-k-1} > \bar{x}$, which contradicts (2.1). Hence, (2.2) can not hold, and so $\{x_n\}$ is persistent.

Next, assume that the function $xf(x)$ in (1.1) is bounded and $\{x_n\}$ is a positive solution of (1.1). Then there is a positive number B such that

$$|xf(x)| \leqslant B \quad \text{for } x \geqslant 0,$$

and so it follows from (1.1) that

$$x_{n+1} \leqslant \alpha x_n + B, \quad n = 0, 1, 2, \dots$$

By an easy induction, we see that

$$x_n \leqslant x_0 \alpha^n + \frac{B}{1-\alpha}(1-\alpha^n), \quad n = 0, 1, \dots,$$

which clearly implies that $\{x_n\}$ is bounded. This completes the proof of the lemma. □

We will say that a sequence $\{x_n\}$ is *oscillatory* if it has arbitrarily large zeros, and it is *nonoscillatory* otherwise. An oscillatory sequence $\{x_n\}$ is *strictly oscillatory* if it actually changes signs. (An oscillatory sequence that is not strictly oscillatory, i.e., it has arbitrarily large zeros but is ultimately nonnegative or nonpositive, has been referred to as a Z-type sequence in the literature.) A sequence $\{x_n\}$ is said to *oscillate about* K if $\{x_n - K\}$ is oscillatory.

Lemma 2. *Every positive solution of* (1.1) *that is not strictly oscillatory about* $\bar{x}$ *converges to* $\bar{x}$.

Proof. First, assume that $\{x_n\}$ is a solution of (1.1) that is eventually greater than or equal to $\bar{x}$. We will show that

$$\mu = \limsup_{n\to\infty} x_n = \bar{x}. \tag{2.4}$$

If (2.4) fails to hold, then $\mu > \bar{x}$ and there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$n_i \geqslant n_0, \quad \lim_{i\to\infty} x_{n_i} = \mu, \quad \text{and} \quad x_{n_i} - x_{n_i-1} \geqslant 0. \tag{2.5}$$

Now, (1.1) can be written in the form

$$\alpha(x_{n+1} - x_n) + (1-\alpha)x_{n+1} = x_{n-k} f(x_{n-k}), \tag{2.6}$$

so from (2.5), it follows that

$$(1-\alpha)x_{n_i} \leqslant x_{n_i-k-1} f(x_{n_i-k-1}) \leqslant x_{n_i-k-1} f(\bar{x}) = (1-\alpha)x_{n_i-k-1}. \tag{2.7}$$

Clearly, this implies that $x_{n_i} \leqslant x_{n_i-k-1}$, and so $\lim\limits_{i\to\infty} x_{n_i-k-1} = \mu$. Then, taking limits of both sides of (2.7), we find that $(1-\alpha)\mu \leqslant \mu f(\mu)$, and so $f(\mu) \geqslant 1-\alpha$, which is a contradiction. Hence, $\mu = \bar{x}$, and so (2.4) holds, which clearly implies that $\{x_n\}$ converges to $\bar{x}$.

Next, assume that $\{x_n\}$ is a positive solution of (1.1) that is eventually less than or equal to $\bar{x}$. We claim that

$$\eta = \liminf_{n\to\infty} x_n = \bar{x}. \tag{2.8}$$

Otherwise, $\eta < \bar{x}$ and there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$n_i \geqslant n_0, \quad \lim_{i\to\infty} x_{n_i} = \eta, \quad \text{and} \quad x_{n_i} - x_{n_i-1} \leqslant 0.$$

Then, from this and (2.6), we obtain

$$(1-\alpha)x_{n_i} \geqslant x_{n_i-k-1} f(x_{n_i-k-1}) \geqslant x_{n_i-k-1} f(\bar{x}) = (1-\alpha)x_{n_i-k-1}. \tag{2.9}$$

Clearly, this implies that $x_{n_i} \geqslant x_{n_i-k-1}$, and so $\lim\limits_{i\to\infty} x_{n_i-k-1} = \eta$. Then taking the limit on both sides of (2.9), we find that $(1-\alpha)\eta \geqslant \eta f(\eta)$. From Lemma 1, we see that $\eta \neq 0$. Hence, it follows that $f(\eta) \leqslant 1-\alpha$ and so $\eta \geqslant \bar{x}$, which is a contradiction. Therefore, (2.8) holds, and this implies that $\{x_n\}$ converges to $\bar{x}$. The proof of the lemma is now complete. $\square$

Now, we are ready to give our main result.

Theorem 1. *Assume that* (1.2) *holds, the function* $xf(x)$ *is bounded, and*

$$\frac{(\alpha^{-(k+1)}-1)^2 c_1^2 c_2^2 \bar{x}^4}{(1-\alpha)^2((1-\alpha)-c_1\bar{x})((1-\alpha)-c_2\bar{x})} < 1, \tag{2.10}$$

where c_1 *and* c_2 *are two negative constants such that*

$$f'(x) \geqslant c_1 \ \text{for } x \in (0,\bar{x}) \quad \text{and} \quad f'(x) \geqslant c_2 \ \text{for } x \in (\bar{x},\infty).$$

Then $\bar{x}$ *is a global attractor of all positive solutions of* (1.1).

Proof. From Lemma 2, we see that every positive solution of (1.1) that is not strictly oscillatory about $\bar{x}$ converges to $\bar{x}$. Hence, we only need to show that every positive solution that is strictly oscillating about $\bar{x}$ also tends to $\bar{x}$.

Suppose that $\{x_n\}$ is a positive solution that is strictly oscillatory about $\bar{x}$. Let

$$L = \limsup_{n\to\infty} x_n \quad \text{and} \quad l = \liminf_{n\to\infty} x_n. \tag{2.11}$$

Then, by Lemma 1,

$$0 \leqslant l \leqslant \bar{x} \leqslant L < \infty. \tag{2.12}$$

To complete the proof, it suffices to show that $l = \bar{x} = L$. Suppose, for the sake of contradiction, that this is not the case. Then, there are three possibilities:

$$\text{(i)}\ l < \bar{x} < L; \qquad \text{(ii)}\ l = \bar{x} < L; \qquad \text{(iii)}\ l < \bar{x} = L.$$

First, assume that (i) holds. Since $\{x_n\}$ strictly oscillates about $\bar{x}$, there are two interlacing sequences $\{n_i'\}$ and $\{n_i''\}$ of positive integers such that

$$\begin{aligned} n_i' < n_i'' &< n_{i+1}', \quad && i = 1, 2, \ldots, \\ x_{n_i'} &> \bar{x}, && i = 1, 2, \ldots, \quad \lim_{i \to \infty} x_{n_i'} = L, \end{aligned}$$

and

$$x_{n_i''} < \bar{x}, \qquad i = 1, 2, \ldots, \quad \lim_{i \to \infty} x_{n_i''} = l.$$

Now, choose a sequence $\{n_i\}$ of positive integers with

$$n_i'' \leqslant n_i < n_{i+1}', \quad x_{n_i} < \bar{x}, \quad \text{and} \quad x_{n_i+1} > \bar{x}, \quad i = 1, 2, \ldots$$

For each $i = 1, 2, \ldots$, let M_i and m_i be integers in $(n_i, n_{i+1}]$ such that

$$x_{M_i} = \max\{x_j \colon n_i < j \leqslant n_{i+1}\} \quad \text{and} \quad x_{m_i} = \min\{x_j \colon n_i < j \leqslant n_{i+1}\}.$$

Clearly, for each $i = 1, 2, \ldots$

$$x_{M_i} > \bar{x} \quad \text{and} \quad x_{M_i} - x_{M_i - 1} \geqslant 0 \tag{2.13}$$

and

$$x_{m_i} < \bar{x} \quad \text{and} \quad x_{m_i} - x_{m_i - 1} \leqslant 0. \tag{2.14}$$

Since $n_{i+1}', n_{i+1}'' \in (n_i, n_{i+1})$,

$$x_{M_i} \geqslant x_{n_{i+1}'} \quad \text{and} \quad x_{m_i} \leqslant x_{n_{i+1}''}.$$

Hence, it follows that

$$\lim_{i \to \infty} x_{M_i} = L \quad \text{and} \quad \lim_{i \to \infty} x_{m_i} = l. \tag{2.15}$$

From (1.1) we see that

$$x_{M_i} - x_{M_i - 1} + (1 - \alpha) x_{M_i - 1} = x_{M_i - 1 - k} f(x_{M_i - 1 - k})$$

which, in view of (2.13), implies that

$$x_{M_i-1} \leqslant \frac{1}{1-\alpha}\, x_{M_i-1-k} f(x_{M_i-1-k}).$$

Combining this inequality and the equality

$$x_{M_i} = \alpha x_{M_i-1} + x_{M_i-1-k} f(x_{M_i-1-k}),$$

we obtain

$$x_{M_i} \leqslant \frac{1}{1-\alpha}\, x_{M_i-1-k} f(x_{M_i-1-k}). \tag{2.16}$$

Now, we claim that there exists a positive integer I such that

$$x_{M_i-1-k} < \bar{x} \quad \text{for } i \geqslant I. \tag{2.17}$$

Otherwise, there is a subsequence $\{M_{i_j}\}$ of $\{M_i\}$ such that

$$x_{M_{i_j}-1-k} \geqslant \bar{x}, \quad j = 1, 2, \ldots$$

Since $f(x_{M_{i_j}-1-k}) \leqslant 1-\alpha$, (2.16) implies

$$x_{M_{i_j}-1-k} \geqslant x_{M_{i_j}},$$

and so

$$\lim_{j\to\infty} x_{M_{i_j}-1-k} = L.$$

Hence, (2.16) yields

$$L \leqslant \frac{1}{1-\alpha}\, L f(L),$$

which implies that $L = \bar{x}$. This contradicts (i), and so (2.17) must hold. From (2.16) and (2.17), we have

$$x_{M_i} \leqslant \frac{1}{1-\alpha}\, \bar{x} f(x_{M_i-1-k}) \quad \text{for } i \geqslant I,$$

which, in view of the monotonicity of f, yields

$$x_{M_i-1-k} \leqslant f^{-1}\Big(\frac{1-\alpha}{\bar{x}}\, x_{M_i}\Big) \quad \text{for } i \geqslant I. \tag{2.18}$$

By (2.11), given an $\varepsilon > 0$, there exists a positive integer $n_0 \geqslant M_I$ such that

$$l - \varepsilon < x_n < L + \varepsilon \quad \text{for } n > n_0 + k, \tag{2.19}$$

and so

$$l - \overline{x} - \varepsilon < x_n - \overline{x} < L - \overline{x} + \varepsilon \quad \text{for } n > n_0 + k.$$

Now, if $x_{n-k} > \overline{x}$, then

$$(x_{n-k} - \overline{x}) f(x_{n-k}) \leqslant (L - \overline{x} + \varepsilon)(1 - \alpha) \quad \text{for } n > n_0 + k, \tag{2.20}$$

while if $x_{n-k} \leqslant \overline{x}$, then (2.20) holds since the left hand side is nonpositive. Observe that (1.1) can be written in the form

$$x_{n+1} - \alpha x_n = \overline{x} f(x_{n-k}) + (x_{n-k} - \overline{x}) f(x_{n-k}). \tag{2.21}$$

Multiplying (2.21) by $\alpha^{-(n+1)}$, and summing from $n = M_i - 1 - k$ to $n = M_i - 1$, we obtain

$$\begin{aligned} \alpha^{-M_i} x_{M_i} = \alpha^{-(M_i - 1 - k)} x_{M_i - 1 - k} &+ \overline{x} \sum_{j = M_i - 1 - k}^{M_i - 1} \alpha^{-(j+1)} f(x_{j-k}) \\ &+ \sum_{j = M_i - 1 - k}^{M_i - 1} \alpha^{-(j+1)} (x_{j-k} - \overline{x}) f(x_{j-k}). \end{aligned}$$

Applying (2.18)–(2.20), for sufficiently large i, we have

$$\begin{aligned} \alpha^{-M_i} x_{M_i} \leqslant \alpha^{-(M_i - 1 - k)} & f^{-1}\Big(\frac{1-\alpha}{\overline{x}} x_{M_i}\Big) \\ &+ \left[\frac{\overline{x} f(l - \varepsilon)}{1 - \alpha} + L - \overline{x} + \varepsilon\right] [\alpha^{-M_i} - \alpha^{-(M_i - 1 - k)}], \end{aligned}$$

and so it follows that

$$x_{M_i} \leqslant \alpha^{k+1} f^{-1}\Big(\frac{1-\alpha}{\overline{x}}\, x_{M_i}\Big) + \left[\frac{\overline{x} f(l - \varepsilon)}{1 - \alpha} + L - \overline{x} + \varepsilon\right] [1 - \alpha^{k+1}].$$

Letting $i \to \infty$ and noting that ε is arbitrary, we obtain

$$L \leqslant \alpha^{k+1} f^{-1}\Big(\frac{1-\alpha}{\overline{x}}\, L\Big) + \left[\frac{\overline{x} f(l)}{1 - \alpha} + L - \overline{x}\right] (1 - \alpha^{k+1}),$$

which yields

$$L - f^{-1}\Big(\frac{1-\alpha}{\bar{x}}\,L\Big) \leqslant \frac{(\alpha^{-(k+1)}-1)\bar{x}}{1-\alpha}\,[f(l)-(1-\alpha)]. \tag{2.22}$$

By a similar argument, we can establish that

$$l - f^{-1}\Big(\frac{1-\alpha}{\bar{x}}\,l\Big) \geqslant \big(\alpha^{-(k+1)}-1\big)\bar{x}[f(L)-(1-\alpha)]. \tag{2.23}$$

From the Mean Value Theorem,

$$L - f^{-1}\Big(\frac{1-\alpha}{\bar{x}}L\Big) = \Big[1 - \Big(f^{-1}\Big(\frac{1-\alpha}{\bar{x}}\xi\Big)\Big)'\frac{1-\alpha}{\bar{x}}\Big](L-\bar{x}),$$

where $\xi \in (\bar{x}, L)$. Since

$$\Big(f^{-1}\Big(\frac{1-\alpha}{\bar{x}}\,\xi\Big)\Big)' = \frac{1}{f'(\lambda)},$$

where $\lambda \in (0,\bar{x})$ satisfies $f(\lambda) = (1-\alpha)\bar{x}^{-1}\xi$, we have

$$L - f^{-1}\Big(\frac{1-\alpha}{\bar{x}}\,L\Big) = \Big(1 - \frac{1}{f'(\lambda)}\frac{1-\alpha}{\bar{x}}\Big)(L-\bar{x}).$$

Hence, (2.22) can be written in the form

$$\Big(1 - \frac{1}{f'(\lambda)}\frac{1-\alpha}{\bar{x}}\Big)(L-\bar{x}) \leqslant \frac{(\alpha^{-(k+1)}-1)\bar{x}}{1-\alpha}\,[f(l)-(1-\alpha)],$$

and so it follows that

$$\Big(1 - \frac{1-\alpha}{c_1\bar{x}}\Big)(L-\bar{x}) \leqslant \frac{(\alpha^{-(k+1)}-1)\bar{x}}{1-\alpha}\,[f(l)-(1-\alpha)], \tag{2.24}$$

where c_1 is a constant satisfying $f'(x) \geqslant c_1$ for $x \in (0,\bar{x})$. By a similar argument and the fact that $l-\bar{x}<0$, (2.23) yields

$$\Big(1 - \frac{1-\alpha}{c_2\bar{x}}\Big)(l-\bar{x}) \geqslant \frac{(\alpha^{-(k+1)}-1)\bar{x}}{1-\alpha}\,[f(L)-(1-\alpha)], \tag{2.25}$$

where c_2 is a constant satisfying $f'(x) \geqslant c_2$ for $c_2 \in (\bar{x},\infty)$. Now let

$$U = L - \bar{x} \quad \text{and} \quad u = l - \bar{x}.$$

Then, $0 < U < \infty$, $-\bar{x} < u < 0$, and (2.24) and (2.25) can be written in the form

$$\begin{aligned} U &\leqslant A_1[f(u+\bar{x})-(1-\alpha)], \\ u &\geqslant A_2[f(U+\bar{x})-(1-\alpha)] \end{aligned} \tag{2.26}$$

where

$$A_1 = \frac{(\alpha^{-(k+1)} - 1)c_1\bar{x}^2}{(1-\alpha)(c_1\bar{x} - (1-\alpha))} \quad \text{and} \quad A_2 = \frac{(\alpha^{-(k+1)} - 1)c_2\bar{x}^2}{(1-\alpha)(c_2\bar{x} - (1-\alpha))}.$$

Let

$$g(x) = f(x+\bar{x}) - (1-\alpha), \quad x \geqslant -\bar{x}.$$

Since f is decreasing, it follows from (2.26) that

$$U \leqslant A_1 g(A_2 g(U)). \tag{2.27}$$

Now, consider the function

$$h(x) = x - A_1 g(A_2 g(x)), \quad x \geqslant 0.$$

Observe that $h(0) = 0$ and

$$\begin{aligned} h'(x) &= 1 - A_1 A_2 g'(A_2 g(x)) g'(x) \\ &= 1 - A_1 A_2 f'\big(A_2(f(x+\bar{x}) - (1-\alpha)) + \bar{x}\big) f'(x+\bar{x}) \\ &\geqslant 1 - A_1 A_2 c_1 c_2 > 0. \end{aligned}$$

Thus, $h(x) > 0$ for $x > 0$, that is,

$$x > A_1 g(A_2 g(x)) \quad \text{for } x > 0.$$

Clearly, this contradicts (2.27). Hence, (i) can not hold. Now, assume that (ii) holds. Then, from the above argument, we see that L satisfies (2.24). Since $l = \bar{x}$, (2.24) clearly implies that $L = \bar{x}$, which contradicts (ii). Finally, since (2.25) implies $l = \bar{x}$ if $L = \bar{x}$, we see that (iii) can not hold as well. Hence, we must have $L = l = \bar{x}$, and this completes the proof of the theorem.

The following result is a consequence of Theorem 1. While it does not give as sharp a result as Theorem 1, it easier to apply.

Corollary 1. *Assume that* (1.2) *holds, the function* $xf(x)$ *is bounded, and*

$$-d\bar{x} < \frac{(1-\alpha)\Big(1 + \sqrt{1 + 4(\alpha^{-(k+1)} - 1)}\Big)}{2(\alpha^{-(k+1)} - 1)} \tag{2.28}$$

where d *is a negative constant such that*

$$f'(x) \geqslant d \quad \textit{for } x \in (0, \infty).$$

Then $\bar{x}$ *is a global attractor of all positive solutions of* (1.1).

P r o o f. By the quadratic formula, (2.28) implies

$$(\alpha^{-(k+1)}-1)(-d\bar{x})^2-(1-\alpha)(-d\bar{x})-(1-\alpha)^2<0. \tag{2.29}$$

Clearly, (2.29) is equivalent to

$$\frac{(\alpha^{-(k+1)}-1)(-d\bar{x})^2}{(1-\alpha)((1-\alpha)-d\bar{x})}<1. \tag{2.30}$$

Since we can choose c_1 and c_2 in Theorem 1 such that $d \leqslant \min\{c_1,c_2\}<0$, we see that (2.10) holds, and so $\bar{x}$ is a global attractor of all positive solutions. This completes the proof. □

3. Applications

In this section, we apply our main result to an equation that is derived from mathematical biology. Consider the difference equation

$$x_{n+1}=\alpha x_n+\frac{\beta x_{n-k}}{1+x_{n-k}^r},\quad n=0,1,\ldots \tag{3.1}$$

with

$$\alpha\in(0,1),\ \ \beta\in(0,\infty),\ \ \alpha+\beta>1,\ \ r\in(0,\infty),\ \text{ and }\ k\in\{0,1,\ldots\}, \tag{3.2}$$

and where the initial conditions $x_{-k},\ldots,x_0$ are nonnegative. Equation (3.1) is a discrete analogue of the delay differential equation

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t}=\frac{\beta_0\theta^nP(t-\tau)}{\theta^n+P^n(t-\tau)}-\gamma P(t),\quad t\geqslant 0, \tag{3.3}$$

which has been proposed by Mackey and Glass [5] (also see Kocic and Ladas [4]) as a model of haematopoiesis, i.e., blood cell production. Here, β_0, θ, γ, τ and n are positive constants and $P(t)$ denotes the density of mature cells in blood circulation.

Equation (3.1) has a positive equilibrium at $\bar{x}=((\alpha+\beta-1)/(1-\alpha))^{1/r}$. The following theorem gives a sufficient condition for $\bar{x}$ to be a global attractor of all positive solutions.

Theorem 2. *Assume that* (3.2) *holds. If* $r>1$ *and*

$$\frac{(\alpha^{-(k+1)}-1)^2c_1^2c_2^2\bar{x}^4}{(1-\alpha)^2((1-\alpha)-c_1\bar{x})((1-\alpha)-c_2\bar{x})}<1,$$

where

$$c_1 = -\frac{\beta}{4r}(r-1)^{1-1/r}(1+r)^{1+1/r} \quad \textit{and} \quad c_2 = -\frac{r}{\beta}(\alpha+\beta-1)^{1-1/r}(1-\alpha)^{1+1/r},$$

and, in particular, if

$$\begin{aligned}(3.4)\qquad &\frac{\beta}{4r}(r-1)^{1-1/r}(r+1)^{1+1/r}\Big(\frac{\alpha+\beta-1}{1-\alpha}\Big)^{1/r} \\ &< \frac{(1-\alpha)\Big(1+\sqrt{1+4(\alpha^{-(k+1)}-1)}\Big)}{2(\alpha^{-(k+1)}-1)},\end{aligned}$$

then $\bar{x}$ is a global attractor of all positive solutions of (3.1).

Proof. Equation (3.1) is in the form of (1.1) with $f(x) = \beta/(1+x^r)$, $x > 0$. Clearly, the function $xf(x)$ is bounded for $x \geqslant 0$. Observe that

$$f'(x) = \frac{-\beta r x^{r-1}}{(1+x^r)^2}, \quad x > 0$$

and

$$f''(x) = \frac{-\beta r x^{r-2}((r-1)-(r+1)x^r)}{(1+x^r)^3}, \quad x > 0.$$

Clearly, $f'(x)$ has minimum at $x^* = ((r-1)/(r+1))^{1/r}$ and

$$(3.5)\qquad f'(x^*) = -\frac{\beta}{4r}(r-1)^{1-1/r}(1+r)^{1+1/r}.$$

Since $f'(x)$ is decreasing for $x < x^*$ and increasing for $x > x^*$, we may either have $c_1 = f'(x^*)$ and $c_2 = f'(\bar{x})$, or $c_1 = f'(\bar{x})$ and $c_2 = f'(x^*)$. In either case, by (3.5) and the fact that

$$f'(\bar{x}) = -\frac{r}{\beta}(\alpha+\beta-1)^{1-1/r}(1-\alpha)^{1+1/r},$$

the hypotheses of Theorem 1 are satisfied. In particular, (3.4) is (2.28) with $d = f'(x^*)$ in Corollary 1. This completes the proof of the theorem. □

Remark 1. The global attractivity of (3.1) has been studied in [2] and [3]. For the case that $0 < r \leqslant 1$, Ivanov [2] showed that $\bar{x}$ is a global attractor. If $r > 1$, Ivanov [2] and Karakostas et al. [3] showed that

$$(3.6)\qquad \beta \leqslant (1-\alpha)\frac{4r}{(r-1)^2}$$

and

$$(3.7)\qquad \beta \leqslant (1-\alpha)\frac{r}{r-1}$$

are sufficient conditions for $\bar{x}$ to be a global attractor of all positive solutions of (3.1), respectively. Clearly, the "delay k" does not play any role in these two conditions. Our conditions in Theorem 2 are different from these two conditions, and in particular, the "delay k" plays an essential role in our conditions.

Example 1. Consider equation (3.1) with $\alpha = 0.99$, $r = 2$, $k = 1$ and

$$\beta = 5(1-\alpha)\frac{r}{r-1} = 0.1.$$

Clearly, neither (3.6) nor (3.7) is satisfied. However, since

$$\frac{\beta}{4r}(r-1)^{1-1/r}(r+1)^{1+1/r}\Big(\frac{\alpha+\beta-1}{1-\alpha}\Big)^{1/r} < 0.2$$

and

$$\frac{(1-\alpha)\big(1+\sqrt{1+4(\alpha^{-(k+1)}-1)}\,\big)}{2(\alpha^{-(k+1)}-1)} > 0.4,$$

we see that (3.4) is satisfied, and so by Theorem 2, $\bar{x}$ is a global attractor of all positive solutions of (3.1).

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Authors' addresses: J. R. Graef, Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, U.S.A., e-mail: `john-graef@utc.edu`; C. Qian, Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762, U.S.A., e-mail: `qian@math.msstate.edu`.