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THE INVERSE CARRIER PROBLEM

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Abstract. The problem was motivated by Borůvka's definitions of the carrier and the associated carrier. The inverse carrier problem is precisely defined and partially solved. Examples are given.

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1. Inverse carrier problem

The coefficient $q(t)$ of the second order differential equation $y'' = q(t)y$ is called its *carrier*. In Borůvka [1], formulas are given for the *associated carrier* $Q(t)$: the transformation of variables $Y = y'/\sqrt{|q(t)|}$ takes $y'' = q(t)y$ into $Y'' = Q(t)Y$.

The *inverse carrier problem* has been isolated from the discussion of carriers in Borůvka [1], p. 8. It can be stated as follows.

Given $Q(t)$ continuous on $0 \leqslant t < T$ ($T = \infty$ allowed), solve the differential equation

$$Q(t) = \sqrt{|q(t)|}\bigl(1/\sqrt{|q(t)|}\bigr)'' + q(t)$$

for a function $q(t) \neq 0$ on $0 \leqslant t < T$ which is twice continuously differentiable.

Borůvka's theory implies $Q(t)$ is exactly the *associated carrier* of $q(t)$ on $0 \leqslant t < T$.

To clarify the nature of the inverse carrier problem, the following new definition is introduced.

Definition. The *inverse carrier system* for carrier $Q(t)$ continuous on $0 \leqslant t < T$ is the two-dimensional nonautonomous system of differential equations

$$\begin{cases} u' = q + u^2 - Q(t), \\ q' = 2qu, \\ u(0) = u_0, \quad q(0) = q_0. \end{cases}$$

The inverse carrier system satisfies the hypotheses of Picard's existence-uniqueness theorem for a nonlinear vector system

$$\begin{cases} X'(t) = F(t, X(t)), \\ X(0) = X_0. \end{cases}$$

Picard's theorem implies the system has a unique solution $u(t)$, $q(t)$ defined on a maximal interval of existence $0 \leqslant t < H$, where $H = \infty$ is allowed. First-order differential equation methods apply to write

$$q(t) = q_0 \mathrm{e}^{2\int_0^t u(x)\,\mathrm{d}x}.$$

The exponential representation implies $q(t)$ is never zero, if $q_0 \neq 0$, it is defined everywhere where $u(t)$ is defined, and $q(t)$ is twice continuously differentiable. The connection between the *inverse carrier system* and the *inverse carrier problem* of Borůvka is as follows.

Theorem 1. *If the inverse carrier system has a solution pair $u(t)$, $q(t)$ defined on $0 \leqslant t < T$ and $q_0 \neq 0$, then $q(t)$ has an associated first carrier $Q(t)$ on $0 \leqslant t < T$.*

P r o o f. We shall establish the identity

$$Q(t) = q(t) + \sqrt{|q(t)|}\big(1/\sqrt{|q(t)|}\big)''.$$

The exponential representation of $q(t)$ implies $q(t) \neq 0$ and $q(t)$ is of class C^2. Use the first differential equation and from the second equation write $2u = q'/q$ to obtain

$$\begin{aligned}
Q(t) &= q(t) + u^2(t) - u'(t) \\
&= q(t) + u^2(t) - \frac{1}{2}\left(\frac{q''(t)}{q(t)} - 4\left(\frac{q'(t)}{2q(t)}\right)^2\right) \\
&= q(t) + u^2(t) - \frac{1}{2}\frac{q''(t)}{q(t)} + 2u^2(t) \\
&= q(t) + 3\left(\frac{q'(t)}{2q(t)}\right)^2 - \frac{1}{2}\frac{q''(t)}{q(t)} = q(t) + \frac{3}{4}\left(\frac{q'(t)}{q(t)}\right)^2 - \frac{1}{2}\frac{q''(t)}{q(t)} \\
&= q(t) + \sqrt{|q(t)|}\big(1/\sqrt{|q(t)|}\big)''.
\end{aligned}$$

This completes the proof of the theorem. □

The inverse carrier problem does not have a unique solution in general. This is because different initial values u_0, q_0 result in different carriers $q(t)$ with the same associated carrier $Q(t)$, by virtue of Picard's theorem. It is not possible by Picard's theorem alone to infer that $u(t)$ exists on $0 \leqslant t < T$. Extension of the solution is the critical issue.

It is *always true* that $Q(t)$ is the associated carrier of some $q(t)$ on an interval $0 \leqslant t < H$. This is perhaps less interesting, since the domain $0 \leqslant t < T$ of the original function $Q(t)$ has been changed. In particular, even if $Q(t)$ is studied on a closed interval, then Picard's interval $0 \leqslant t < H$ may be only a subinterval.

It remains to give examples of nontrivial functions $Q(t)$ for which the inverse carrier problem has at least one solution. It will turn out that such examples have an infinite number of solutions $q(t)$.

3. Inverse carriers

Some insight into the problem of maximal extension of the solution $u(t)$ in the inverse carrier system can be obtained by viewing the differential equation for $u(t)$ as a Riccati differential equation

$$u' = r(t) + u^2.$$

It is well-known that the transformation $u = -y'/y$ changes the Riccati equation into a second order differential equation $y'' + r(t)y = 0$. In the case of the inverse carrier equations, $r(t) = q(t) - Q(t)$ depends on u itself, precisely,

$$r(t) = q_0 \mathrm{e}^{2\int_0^t u(x)\,\mathrm{d}x} - Q(t).$$

If $r(t) \leqslant 0$, then the second order differential equation has a solution $y(t)$ such that y and y' are positive on $0 \leqslant t < \infty$. In this case $u(t)$ is negative and consequently $q(t)$ has a negative exponent. If this is the case and $q_0 > 0$, then $q(t) \leqslant q_0$ and $r(t) \leqslant q_0 - Q(t)$. This motivates the assumptions below.

Theorem 2. *Let $Q(t)$ be continuous, $u_0 < 0$, $q_0 > 0$ and $Q(t) \geqslant q_0$ for $0 \leqslant t < T$ ($T = \infty$ allowed). Then $u(t)$, $q(t)$ in the inverse carrier system exist for $0 \leqslant t < T$.*

The hypotheses $Q(t) \geqslant q_0$, $u_0 < 0$, $q_0 > 0$ can be discarded, provided $u_0 q_0 < 0$ and the two initial value problems

$$\begin{aligned} V'' &= Q(t)V, & V(0) &= 1, & V'(0) &= -u_0, \\ W'' &= (Q(t) - q_0)W, & W(0) &= 1, & W'(0) &= -u_0 \end{aligned}$$

have positive solutions with $u_0 V'(t) < 0$, $u_0 W'(t) < 0$ on $0 \leqslant t < T$.

Proof. Given $r(t) \geqslant 0$ and $y'' = r(t)y$, $y(0) = y_0 > 0$, $y'(0) = y_1 > 0$, then y and y' increase, by sign analysis of derivatives. Therefore, the first case is a special case of the more general second statement.

Assume $Q(t) \geqslant q_0$, $u_0 < 0$ and $q_0 > 0$ for definiteness.

The idea of the proof is to apply the comparison theorem in Hartman [2] to three differential equations

$$\begin{cases} v'(t) = -Q(t) + v^2(t), & v(0) = u_0, \\ u'(t) = q(t) - Q(t) + u^2(t), & u(0) = u_0, \\ w'(t) = q_0 - Q(t) + w^2(t), & w(0) = u_0. \end{cases}$$

According to the comparison theorem, $u(t)$ is trapped between the solutions $v(t)$ and $w(t)$ on an interval $0 \leqslant t < H$. Extension theory for ordinary differential equations and the trapping inequality imply $u(t)$ exists on $0 \leqslant t < T$, provided v and w exist on $0 \leqslant t < T$.

The technical hypothesis to satisfy for the comparison theorem is

$$-Q(t) + x^2 \leqslant q(t) - Q(t) + x^2 \leqslant q_0 - Q(t) + x^2$$

for $0 \leqslant t < T$, $-\infty < x < \infty$ (inequalities are reversed for $u_0 > 0$). Since $q(t) > 0$ it suffices to show $q(t) \leqslant q_0$. This will be done by showing under the given assumptions that $u(t) < 0$ on its maximal interval of existence.

The comparison equations are known to be related via substitutions $v = -V'/V$ and $w = -W'/W$ to the second order equations $V'' - Q(t)V = 0$ and $W'' + (q_0 - Q(t))W = 0$, with initial data $V(0) = W(0) = 1$, $V'(0) = W'(0) = -u_0$.

Due to the hypotheses, these initial value problems have positive solutions V and W defined on $0 \leqslant t < T$. Further, $u_0 V' < 0$ and $u_0 W' < 0$. Therefore, v, u and w have the same sign as u_0 on their common interval of existence.

This completes the proof of the theorem for the case $u_0 < 0$, $q_0 > 0$. If $u_0 > 0$ and $q_0 < 0$, then $u(t) > 0$ is proved instead, which follows because u is trapped between positive functions. The other details given above remain unchanged. $\square$

Example 1. The inverse carrier problem for $y'' = (1+t)y$ on $0 \leqslant t < \infty$ has a solution $q(t)$ on $0 \leqslant t < \infty$ such that for certain combinations V and W of Airy's wave functions

$$\left(\frac{1}{V(t)}\right)^2 \leqslant q(t) \leqslant \left(\frac{1}{W(t)}\right)^2.$$

To *verify* these statements, let $q_0 = -u_0 = 1$. It suffices to write the solutions of the comparison equations $v' = -(1+t) + v^2$, $w' = -t + w^2$ as fractions $v = -V'/V$, $w = -W'/W$ where $V'' = (1+t)V$, $W'' = tW$, $V(0) = W(0) = 1$, $V'(0) = W'(0) = -u_0 = 1$. As argued in the proof of the theorem above, $v(t) \leqslant u(t) \leqslant w(t)$. Therefore,

$$-2 \ln V(t) \leqslant 2 \int_0^t u(x)\,\mathrm{d}x \leqslant -2 \ln W(t).$$

The representation

$$q(t) = q_0 \mathrm{e}^{2\int_0^t u(x)\,\mathrm{d}x}$$

implies the inequality

$$\left(\frac{1}{V(t)}\right)^2 \leqslant q(t) \leqslant \left(\frac{1}{W(t)}\right)^2.$$

The graphs of the three comparison functions appear below. The name *Airy's equation* is attached to $V'' = tV$. The Airy functions A_i, B_i are a special basis of this equation. By a change of variables, $A_i(t+1)$, $B_i(t+1)$ are a basis for $W'' = (1+t)W$. This completes the verification of the statement in the example.

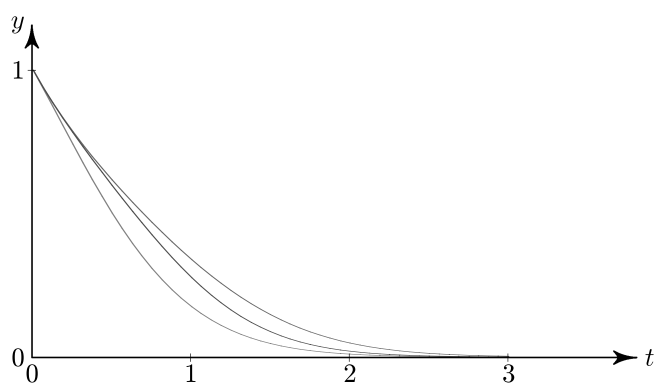

Figure 1. $\left(1/V(t)\right)^2 \leqslant q(t) \leqslant \left(1/W(t)\right)^2$.

Example 2. The inverse carrier problem for $y'' = (1+\sin^2 t)y$ on $0 \leqslant t < \infty$ has a solution $q(t)$ on $0 \leqslant t < \infty$ such that for certain positive functions V and W,

$$\left(\frac{1}{V(t)}\right)^2 \leqslant q(t) \leqslant \left(\frac{1}{W(t)}\right)^2.$$

The verification is similar to Example 1, except that the second order equations $V'' = \sin^2 tV$ and $W'' = (1+\sin^2 t)W$ do not have closed-form solutions. Also similar is the graphic.

The possibility of obtaining an analytic solution for the inverse carrier has diminished, in view of the examples. Nevertheless, some progress is possible, using the methods of first order differential equations.

Theorem 3. *Let $F(u)$ be continuous for $u \geqslant 1$ and let c be a constant. Define*

$$\begin{aligned} K(u) &= 2\int_1^u F(x)\,\mathrm{d}x + c^2, \\ g(y) &= \int_1^y K(u)^{-1/2}\,\mathrm{d}u, \\ G(t) &= g^{-1}(t) \ \text{ provided it exists.} \end{aligned}$$

Assume c and F are such that $K(u) > 0$ for $u \geqslant 1$. Let for some t_0

$$Q(t) = \frac{F(G(t-t_0))}{G(t-t_0)} + \left(\frac{1}{G(t-t_0)}\right)^2.$$

Then an inverse carrier for $Q(t)$ on $t \geqslant t_0$ is

$$q(t) = \left(\frac{1}{G(t-t_0)}\right)^2.$$

The theorem remains true if F, Q, q are replaced by $-F$, $-Q$, $-q$.

Proof. It has to be shown that $Q = \sqrt{q}(1/\sqrt{q})'' + q$. Let $y(t) = 1/\sqrt{q(t)} = G(t-t_0)$. Then $t - t_0 = g(y(t))$ and differentiation gives

$$1 = g'(y)y' = K(y)^{-1/2}y'.$$

Hence

$$(y')^2 = K(y).$$

Differentiate this equation to obtain

$$y'y'' = F(y)y'.$$

If $K(y) > 0$, then $y' > 0$ and by division

$$y'' = F(y).$$

This implies

$$\begin{aligned} Q(t) &= y^{-1}F(y) + y^{-2} = y^{-1}y'' + y^{-2} \\ &= \sqrt{q}(1/\sqrt{q})'' + q. \end{aligned}$$

This completes the proof of the theorem, except for the remark about negatives. In the latter case, the details of the proof remain unchanged except at the end, where $y'' = -F(y)$, $-q = y^{-2}$ and then

$$\begin{aligned} -Q(t) &= y^{-1}F(y) + y^{-2} = -y^{-1}y'' + y^{-2} \\ &= -\sqrt{q}(1/\sqrt{q})'' - q. \end{aligned}$$

$\square$

Example 3. Let A and B be real constants, $A > 0$, and let $t_0 = 1/A - B/A$. The inverse carrier problem for $Q(t) = (At + B)^{-2}$ has a solution $q(t) = (At + B)^{-2}$ on $t_0 \leqslant t < \infty$.

This example appears in Laitoch [3]. Direct verification is possible by showing that $Q = \sqrt{q}(1/\sqrt{q})'' + q$.

To obtain q from Q via Theorem 3 let $F(u) = 0$ and $c = A$. Then $K(u) = A^2$, $g(y) = (y - 1)/A$, $G(t) = At + 1$,

$$\begin{aligned} Q(t) &= \left(\frac{1}{A(t - t_0) + 1}\right)^2, \\ q(t) &= \left(\frac{1}{A(t - t_0) + 1}\right)^2. \end{aligned}$$

Computing $A(t - t_0) + 1 = At + B$ completes the verification.

Similarly, the example $Q(t) = -(At + B)^{-2}$ with a solution $q(t) = -(At + B)^{-2}$ can be derived from Theorem 3 with $F(u) = 0$, using the statement about negatives.

The case $A = 0$ is uninteresting, because then Q is constant, in which case $Q = \sqrt{q}(1/\sqrt{q})'' + q$ has a solution $q = Q$.

Example 4. The inverse carrier problem for $Q(t) = 1 + \operatorname{csch}^2 t$ on $1 \leqslant t < \infty$ has a solution $q(t) = \operatorname{csch}^2(t)$.

Direct verification is possible by showing that $Q = \sqrt{q}(1/\sqrt{q})'' + q$. Instead, it will be shown how to obtain q from Q via Theorem 3.

Let $F(u) = u$. Then F is continuous for $u \geqslant 1$. Define $c^2 = 2$ and $t_0 = \operatorname{arcsinh}(1)$. By direct integration,

$$K(u) = 2\int_1^u F(x)\,\mathrm{d}x + c^2 = u^2 + 1,$$
$$g(y) = \int_1^y K(u)^{-1/2}\,\mathrm{d}u = \operatorname{arcsinh}(y) - t_0.$$

Then

$$G(t) = \sinh(t + t_0).$$

Finally,

$$Q(t) = \frac{F(\sinh(t))}{\sinh(t)} + \frac{1}{\sinh^2(t)} = 1 + \operatorname{csch}^2(t),$$
$$q(t) = \frac{1}{\sinh^2(t)} = \operatorname{csch}^2(t).$$

Since $\operatorname{arcsinh}(1) = 0.88$, the pair of carriers exist on $1 \leqslant t < \infty$.

References

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