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Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 2, 429–437

Persistent URL: `http://dml.cz/dmlcz/127730`

Czechoslovak Mathematical Journal, 52 (127) (2002), 429–437

THE MONOTONE CONVERGENCE THEOREM FOR MULTIDIMENSIONAL ABSTRACT KURZWEIL VECTOR INTEGRALS

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(Received May 27, 1999)

Abstract. We prove two versions of the Monotone Convergence Theorem for the vector integral of Kurzweil, $\int_R \mathrm{d}\alpha(t)f(t)$, where R is a compact interval of $\mathbb{R}^n$, α and f are functions with values on $L(Z,W)$ and Z respectively, and Z and W are monotone ordered normed spaces. Analogous results can be obtained for the Kurzweil vector integral, $\int_R \alpha(t)\,\mathrm{d}f(t)$, as well as to unbounded intervals R.

Keywords: Monotone Convergence Theorem, Kurzweil vector integral, ordered normed spaces

MSC 2000: 26A39, 26A42

1. Ordered normed spaces

As basic references on ordered normed spaces we mention [2] and also [6].

Definition 1. An ordered normed space is a pair $(Z,\leqslant)$ where Z is a normed vector space and $\leqslant$ is an ordering relation in Z such that

i) $z\leqslant w$ implies $z+x\leqslant w+x$ for $z,w,x\in Z$;
ii) $z\leqslant w$ implies $\lambda z\leqslant \lambda w$ for $z,w\in Z$ and λ real and non-negative;
iii) the "positive cone" $C=\{z\in Z;\ z\geqslant 0\}$ is closed in Z.

Remark 1. Let $z,w\in Z$, where $(Z,\leqslant)$ is an ordered normed space. Then

i) $z\leqslant w$ is equivalent to $w-z\geqslant 0$;
ii) $z\leqslant 0$ if and only if $0\leqslant -z$;
iii) the cone C is convex, $C+C\subset C$ and $C\cap(-C)=\{0\}$;
iv) every subspace of an ordered normed space is also an ordered normed space with the induced order;

v) every closed subspace of an ordered Banach space is also an ordered Banach space with the induced order;
vi) if $(Z, \leqslant_Z)$ and $(W, \leqslant_W)$ are ordered normed spaces, then $Z \times W$ with the ordering "$(z_1, z_2) \leqslant_{Z\times W} (w_1, w_2)$ if and only if $z_1 \leqslant_{Z\times W} w_1$ and $z_2 \leqslant_{Z\times W} w_2$" is an ordered normed space.

Example 2. If (Ω, μ) is a measure space and X is an ordered Banach space, then $L_p(\Omega, X)$, $1 \leqslant p \leqslant \infty$, with the ordering "$f \leqslant g$ if and only if $f(t) \leqslant g(t)$ μ-almost everywhere" is an ordered Banach space. If K is a complete metric space and X is an ordered Banach space, then the space of all continuous functions from K to X, $C(K, X)$, is an ordered Banach space with the ordering "m-almost everywhere", m standing for the Lebesgue measure.

Definition 3. An ordered normed space $(Z, \leqslant)$ is called *monotone* if and only if

$$0 < z \leqslant w \text{ implies } \|z\| \leqslant \|w\|.$$

Example 4. Let X be a Banach space and $[a, b]$ a compact interval of $\mathbb{R}$. By $BV_a([a, b], X)$ we mean the space of all functions $f\colon [a, b] \to X$ of bounded variation (i.e., $V(f) < \infty$) such that $f(a) = 0$. Then $BV_a([a, b], X)$ is a Banach space when equipped with the norm $\|f\| = V(f)$. It is possible to have $f, g \in BV_a([a, b], X)$ such that $f \leqslant g$ m-almost everywhere, but $V(g) < V(f)$ and therefore $\|g\| < \|f\|$. Hence, the ordering of $BV_a([a, b], X)$ is not monotone.

Definition 5. Let $(Z, \leqslant_Z)$ and $(W, \leqslant_W)$ be ordered normed spaces and consider a mapping $T\colon Z \to W$. Then T is called *increasing* if and only if $z_1 \leqslant_Z z_2$ implies $T(z_1) \leqslant_W T(z_2)$ and T is called *positive* if and only if $z \geqslant 0$ implies $T(z) \geqslant 0$.

Remark 2. In Definition 5, if T is linear, then T is increasing if and only if T is positive.

Definition 6. Let $(Z, \leqslant)$ be a monotone ordered normed space and $(z_n)_{n\in\mathbb{N}}$ a non-decreasing sequence in Z. We write $z_n\!\uparrow$ in Z. Then
i) $0 \leqslant z_n\!\uparrow$ in Z is bounded if and only if $\sup\limits_n \|z_n\| < \infty$;
ii) $0 \leqslant z_n\!\uparrow$ in Z is upper bounded if and only if there exists $z \in Z$ such that $z \geqslant z_n$ for every $n \in \mathbb{N}$;
iii) $z = \lim\limits_n z_n$ if and only if $\|z - z_n\| \to 0$ as $n \to \infty$;
iv) $z = \sup\limits_n z_n$ if and only if $z \geqslant z_n$ for every $n \in \mathbb{N}$ and $w \in Z$ with $w \geqslant z_n$ for every $n \in \mathbb{N}$, implies $w \geqslant z$.

Let $(Z, \leqslant)$ be a monotone ordered normed space. If $z_n \uparrow$ in Z, then $z = \lim_n z_n$ implies $z = \sup_n z_n$. Consider, however, the following properties which the space $(Z, \leqslant)$ may or may not possess:

(A) if $z_n \uparrow$ in Z and there is $z = \sup_n z_n$, then $z = \lim_n z_n$;

(B) if $z_n \uparrow$ in Z is a Cauchy sequence, then it is convergent;

(C) if $z_n \uparrow$ in Z is bounded from above, then there is $z = \sup_n z_n$;

(D) if $z_n \uparrow$ in Z is bounded from above, then there is $z = \lim_n z_n$;

(E) if $z_n \uparrow$ in Z is bounded, then there is $z = \lim_n z_n = \sup_n z_n$.

Then (E) implies (B) and (D); (D) holds if and only if both (A) and (C) hold.

2. Kurzweil and Henstock vector integrals

Let X and Y be normed spaces and $L(X, Y)$ the normed space of all linear continuous functions from X to Y. Throughout this section, we consider functions $\alpha\colon R \to L(X, Y)$ and $f\colon R \to X$, where R is a compact interval of $\mathbb{R}^n$ with sides parallel to the coordinate axes. However, for simplicity of notation and proofs, we work on the two-dimensional case. Thus, when $t = (t_1, t_2)$ and $s = (s_1, s_2)$ are points of $\mathbb{R}^2$ such that $t \leqslant s$ (i.e., $t_i \leqslant s_i$, $i = 1, 2$), we denote by $[t, s]$ the corresponding closed interval and we write $|[t, s]| = m([t, s])$, where m denotes the Lebesgue measure.

Definition 7. Let $\mathcal{I}(R)$ be the set of all closed intervals contained in R. A *division* d of R is a finite set of nonoverlapping closed intervals of R whose union is R and we write $d = (J_i)$ or simply (J_i), where $J_i \in \mathcal{I}(R)$. By nonoverlapping intervals we mean that their interiors are pairwise disjoint. We say that $d = (\xi_i, J_i)$ is a *tagged division* of R if (J_i) is a division of R and $\xi_i \in J_i$ for every i. We denote by TD_R the set of all tagged divisions of R. A *tagged partial division* of R is any subset of a tagged division of R and, in this case, we write $d \in \mathrm{TPD}_R$. A *gauge* of a subset S of R is any function $\delta\colon S \to]0, \infty[$. Given a gauge δ of R, we say that $d = (\xi_i, J_i) \in \mathrm{TPD}_R$ is δ-fine if $J_i \subset \{t \in R;\ |t - \xi_i| < \delta(\xi_i)\}$ for every i. Let $c \in \mathbb{R}$, $0 < c \leqslant 1$, and $J \in \mathcal{I}(R)$ with sides of the lengths h and k such that $h \leqslant k$. Then J is called *c-regular* if and only if $h/k \geqslant c$; $d = (\xi_i, J_i) \in \mathrm{TPD}_R$ is called c-regular if and only if J_i is c-regular for every i.

Let $\mathcal{A}(\mathcal{I}(R), X)$ be the set of all functions $F\colon \mathcal{I}(R) \to X$ which are additive on intervals, that is, $F(J) = F(J_1) + F(J_2)$ for every $J, J_1, J_2 \in \mathcal{I}(R)$ such that $J = J_1 \cup J_2$ and J_1 and J_2 are nonoverlapping.

Definition 8. Let $\alpha \in \mathcal{A}(\mathcal{I}(R), L(X,Y))$. Then $f\colon R \to X$ is *regularly α-Kurzweil integrable* (we write $f \in {}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,X)$) with vector integral $I \in Y$ (we write $I = {}^{\mathrm{rK}}\!\int_R \mathrm{d}\alpha f = {}^{\mathrm{rK}}\!\int_R \mathrm{d}\alpha(t) f(t)$) if for every $\varepsilon > 0$ and every $0 < c < 1$ there exists a gauge δ of R such that

$$\Big\| \sum_i \alpha(J_i) f(\xi_i) - {}^{\mathrm{rK}}\!\!\int_R \mathrm{d}\alpha f \Big\| < \varepsilon$$

for every c-regular δ-fine $d = (\xi_i, J_i) \in \mathrm{TD}_R$.

We say that $f\colon R \to X$ is *regularly α-Henstock integrable* (we write $f \in {}^{\mathrm{r}}\mathrm{H}^{\alpha}(R,X)$) if there exists $F^{\alpha} \in \mathcal{A}(\mathcal{I}(R), Y)$ (called the associated function of f) such that given $\varepsilon > 0$ and $0 < c < 1$, there exists a gauge δ of R such that

$$\sum_i \|\alpha(J_i) f(\xi_i) - F^{\alpha}(J_i)\| < \varepsilon$$

for every c-regular δ-fine $d = (\xi_i, J_i) \in \mathrm{TD}_R$.

In what follows we put $R = [a,b] = [a_1,b_1] \times [a_2,b_2]$, where $a = (a_1,a_2)$ and $b = (b_1,b_2)$ belong to $\mathbb{R}^2$.

Remark 3. Given $\alpha \in \mathcal{A}(\mathcal{I}(R), L(X,Y))$, let ${}^{\mathrm{r}}I^{\alpha}(R,X)$ denote one of the spaces ${}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,X)$ or ${}^{\mathrm{r}}\mathrm{H}^{\alpha}(R,X)$.

i) If $f \in {}^{\mathrm{r}}I^{\alpha}(R,X)$ and $J \in \mathcal{I}(R)$, then $f \in {}^{\mathrm{r}}I^{\alpha}(J,X)$; we define the indefinite integral $F^{\alpha}\colon \mathcal{I}(R) \to Y$ of $f \in {}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,X)$ by $F^{\alpha}(J) = {}^{\mathrm{rK}}\!\int_J \mathrm{d}\alpha f$, for every $J \in \mathcal{I}(R)$.

ii) ${}^{\mathrm{r}}\mathrm{H}^{\alpha}(R,X) \subset {}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,X)$ and if $f \in {}^{\mathrm{r}}\mathrm{H}^{\alpha}(R,X)$, then the associated function F^{α} is its indefinite integral.

iii) If $J_1, J_2 \in \mathcal{I}(R)$ are such that J_1 and J_2 are nonoverlapping and $J = J_1 \cup J_2$ is an interval, then $f \in {}^{\mathrm{r}}I^{\alpha}(J,X)$ whenever $f \in {}^{\mathrm{r}}I^{\alpha}(J_i,X)$ for $i = 1,2$.

Remark 4. Given $\alpha \in \mathcal{A}(\mathcal{I}(R), L(X,Y))$ and $f \in {}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,X)$, we define a function $\tilde{f}^{\alpha}\colon R \to Y$ by $\tilde{f}^{\alpha}(t) = {}^{\mathrm{rK}}\!\int_{[a,t]} \mathrm{d}\alpha f$ for every $t \in R = [a,b]$. We may associate $\tilde{f}^{\alpha}$ with a function of intervals which we still denote by $\tilde{f}\colon \mathcal{I}(R) \to Y$. In this case,

$$\tilde{f}([t,s]) = \tilde{f}(s) - \tilde{f}(t_1,s_2) - \tilde{f}(s_1,t_2) + \tilde{f}(t),$$

where $[t,s] = [t_1,s_1] \times [t_2,s_2]$. Reciprocally, it is possible to associate a function $F^{\alpha} \in \mathcal{A}(\mathcal{I}(R), X)$ with a function from R to X by the relation

$$F^{\alpha}(t) = F^{\alpha}([a,t]) - F^{\alpha}([a,(a_1,t_2)]) - F^{\alpha}([a,(t_1,a_2)]) + F^{\alpha}([a,a]).$$

When $\alpha(t)=t$, we write ${}^{\mathrm{r}}\mathrm{K}(R,X)$, ${}^{\mathrm{r}}\mathrm{H}(R,X)$ and $\tilde f$ instead of ${}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,X)$, ${}^{\mathrm{r}}\mathrm{H}^{\alpha}(R,X)$ and $\tilde f^{\alpha}$, respectively. If moreover $R=[a,b]\subset\mathbb{R}$, then $\alpha(J)=m(J)=|J|$ for $J\in\mathcal{I}(R)$, and the regularity does not play any role. In this case, we denote by $K([a,b],X)$ and by $\mathrm{H}([a,b],X)$ respectively the spaces of Kurzweil and of Henstock integrable functions.

Definition 9. Given $\alpha\colon R\to L(X,Y)$, let ${}^{\mathrm{r}}I^{\alpha}(R,X)$ denote one of the spaces ${}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,X)$ or ${}^{\mathrm{r}}\mathrm{H}^{\alpha}(R,X)$. Two functions $g,f\in{}^{\mathrm{r}}I^{\alpha}(R,X)$ are equivalent if $\tilde g^{\alpha}=\tilde f^{\alpha}$. Then the space of all equivalence classes of functions of ${}^{\mathrm{r}}I^{\alpha}(R,X)$ endowed with the Alexiewicz norm

$$f\in{}^{\mathrm{r}}I^{\alpha}(R,X)\mapsto\|f\|_A=\|\widetilde{f}\|_{\infty}=\sup\Big\{{}^{\mathrm{rK}}\!\!\int_{[a,t]}\mathrm{d}\alpha f\,;\ t\in R=[a,b]\Big\}$$

is denoted by ${}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,X)_A$ or by ${}^{\mathrm{r}}\mathrm{H}^{\alpha}(R,X)_A$ depending on the choice of ${}^{\mathrm{r}}I^{\alpha}(R,X)$. We write $f\in{}^{\mathrm{r}}I^{\alpha}(R,X)_A$ to indicate that $f=f_{\Phi}\in\Phi$, where $\Phi\in{}^{\mathrm{r}}I^{\alpha}(R,X)_A$.

Example 10. With the notation of Definition 9 and the ordering "$f\leqslant g$ m-almost everywhere", ${}^{\mathrm{r}}I^{\alpha}(R,X)_A$ is an ordered normed space. Besides, if $\alpha\colon R\to L(X,Y)$ is such that $\alpha(J)$ is positive for every $J\in\mathcal{I}(R)$, then ${}^{\mathrm{r}}I^{\alpha}(R,X)_A$ is monotone.

Definition 11. Let $0<c<1$. The c-variation $\mathrm{V}^c(F)$ of a function $F\in\mathcal{A}(\mathcal{I}(R),E)$ is the smallest $p\in[0,\infty]$ such that for every c-regular division (J_i) of R,

$$\sum_i\|F(J_i)\|<p.$$

If $\mathrm{V}^c(F)<\infty$, then F is of *c-bounded variation*. If moreover

$${}^{\mathrm{r}}\mathrm{V}(F)=\sup\{\mathrm{V}^c(F);\ 0<c<1\}<\infty,$$

then F is of *regular bounded variation* and we write $F\in{}^{\mathrm{r}}\mathrm{BV}(\mathcal{I}(R),X)$.

Definition 12. Let $0<c<1$. We say that a function $F\in\mathcal{A}(\mathcal{I}(R),X)$ is *c-differentiable* at $\xi\in R$ and $f(\xi)$ is its *c-derivative* (we write $\mathrm{D}^cF(\xi)=f(\xi)$) if for every $\varepsilon>0$ there is a neighbourhood V of ξ such that for every c-regular $J\in\mathcal{I}(R)$ with $\xi\in J\subset V$,

$$\|F(J)-f(\xi)|J|\|<\varepsilon|J|.$$

If $\mathrm{D}^cF(\xi)=f(\xi)$ for every c, then F is *regularly differentiable* at $\xi\in R$ and $f(\xi)$ is its *regular derivative* (we write ${}^{\mathrm{r}}\mathrm{D}F(\xi)=f(\xi)$). If F is c-differentiable at ξ for every $\xi\in R$, then F is c-differentiable at R. If F is regularly differentiable at ξ for every $\xi\in R$, then F is regularly differentiable at R.

Throughout this section we assume that W and Z are monotone ordered normed spaces and Z satisfies property (B), that is, Z is also a Banach space. Other properties of W and Z will be considered later.

Lemma 13 (Saks-Henstock Lemma). *Let $\alpha \in \mathcal{A}(\mathcal{I}(R), L(Z,W))$, $f \in {}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,Z)$, $\varepsilon > 0$, $0 < c < 1$ and let δ be the gauge of R from the definition of $f \in {}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,Z)$. Then, for every c-regular δ-fine $d = (\xi_i, J_i) \in \mathrm{TPD}_R$,*

$$\left\| \sum_i [\alpha(J_i) f(\xi_i) - {}^{\mathrm{rK}}\!\!\int_{J_i} \mathrm{d}\alpha f] \right\| < \varepsilon.$$

Proof. It is sufficient to adapt the one-dimensional proof—see [7], Proposition 16—for the multidimensional case. $\square$

The proof of the next result follows the ideas of [4], Theorem 6. We use the notation $0 \leqslant \alpha$ for $\alpha\colon R \to L(Z,W)$ such that $\alpha(J)$ is positive, for every $J \in \mathcal{I}(R)$.

Theorem 14 [Monotone Convergence Theorem—version 1]. *Let $0 \leqslant \alpha \in {}^{\mathrm{r}}\mathrm{BV}(\mathcal{I}(R), L(Z,W))$ and $0 \leqslant f_n\!\uparrow$ in ${}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,Z)_A$. Suppose Z and W satisfy property* A *and there exists $\sup\limits_n f_n(t) = f(t)$ for every $t \in R$. Let $I_n = {}^{\mathrm{rK}}\!\int_R \mathrm{d}\alpha f$ for every $n \in \mathbb{N}$, and suppose there exists $\sup\limits_n I_n = I$. Then $f \in {}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,Z)_A$ and*

$$\lim_n {}^{\mathrm{rK}}\!\!\int_R \mathrm{d}\alpha f_n = I = {}^{\mathrm{rK}}\!\!\int_R \mathrm{d}\alpha f.$$

Proof. 1) If $0 \leqslant \alpha$, then $0 \leqslant {}^{\mathrm{rK}}\!\int_R \mathrm{d}\alpha f$, for every $0 \leqslant f$ in ${}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,Z)$. Hence, the regular vector integral of Kurzweil is positive and, therefore, $0 \leqslant f_n\!\uparrow$ and $0 \leqslant \alpha$ imply $I_n\!\uparrow$ in W. By the hypothesis, there exists $\sup\limits_n I_n = I$ and W satisfies (A). Thus $I = \lim\limits_n I_n$ and then, for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \geqslant n_\varepsilon$,

$$\|I - I_n\| < \varepsilon.$$

2) $f_n \in {}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,Z)$, for every $n \in \mathbb{N}$. Then for each $\varepsilon \cdot 2^{-n}$ and each c, $0 < c < 1$, there is a gauge δ_n of R such that for every c-regular δ-fine $d_n = (\xi_{i_n}, J_{i_n}) \in \mathrm{TD}_R$,

$$\left\| \sum_{i_n} \alpha(J_{i_n}) f_n(\xi_{i_n}) - I_n \right\| < \frac{\varepsilon}{2^n}.$$

3) From the facts that $f_n\uparrow$ in ${}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,Z)_A$, there is $\sup\limits_n f_n(t)=f(t)$ for every $t\in R$, and Z satisfies (A) it follows that $f(t)=\lim\limits_n f_n(t)$ for every $t\in R$. Hence, given $t\in R$, there exists $n(t)\geqslant n_\varepsilon$ such that

$$\|f(t)-f_{n(t)}(t)\|<\varepsilon.$$

4) Let $0<c<1$ and δ be a gauge of R defined by $\delta(t)=\delta_{n(t)}(t)$ for every $t\in R$. Then, for every c-regular δ-fine $d=(\xi_i,J_i)\in\mathrm{TD}_R$,

$$\begin{aligned}\Big\|\sum_i \alpha(J_i)f_n(\xi_i)-I\Big\| &\leqslant \Big\|\sum_i \alpha(J_i)f_n(\xi_i)-\sum_{i_n}\alpha(J_{i_n})f_{n(\xi_i)}(\xi_{i_n})\Big\|\\ &\quad+\Big\|\sum_{i_n}\alpha(J_{i_n})f_{n(\xi_i)}(\xi_{i_n})-\sum_i {}^{\mathrm{rK}}\!\!\int_{J_i}\mathrm{d}\alpha f\Big\|\\ &\quad+\Big\|\sum_i {}^{\mathrm{rK}}\!\!\int_{J_i}\mathrm{d}\alpha f-I\Big\|=(*).\end{aligned}$$

Since $\alpha\in{}^{\mathrm{r}}\mathrm{BV}(\mathcal{I}(R),L(Z,W))$, then $\sum\limits_i\|\alpha(J_i)\|\leqslant{}^{r}\mathrm{V}(\alpha)$. Then it follows by 3) that the first summand of $(*)$ is smaller than

$$\sum_i\|\alpha(J_i)\|\,\|f(\xi_i)-f_{n(\xi_i)}(\xi_i)\|<\varepsilon\cdot{}^{r}\mathrm{V}(\alpha).$$

We may rewrite the second summand of $(*)$ as

$$\Big\|\sum_n\sum_{n(\xi_i)=n}\Big[\alpha(J_i)f_n(\xi_i)-{}^{\mathrm{rK}}\!\!\int_{J_i}\mathrm{d}\alpha f\Big]\Big\|,$$

which is smaller than $\sum\limits_n\varepsilon/2^n$ by 2) and by the Saks-Henstock Lemma (Lemma 13) applied to each $\sum\limits_{n(\xi_i)=n}$.

5) Let $\nu=\inf\limits_i\{n(\xi_i)\}$ and $\mu=\sup\limits_i\{n(\xi_i)\}$. Then $I_\nu\leqslant I_\mu$. By 3), $n(\xi_i)\geqslant n_\varepsilon$ for every i. Hence $\nu\geqslant n_\varepsilon$ and therefore it follows by 1) and from the fact that $I_n\uparrow$ in W that $I-\varepsilon<I_\nu$. Besides, $I_\mu\leqslant I$, since $\sup\limits_n I_n=I$. Thus $I-\varepsilon<I_\nu\leqslant I_\mu\leqslant I$ and the third summand of $(*)$ in 4) is smaller than ε. This completes the proof. $\square$

With the notation of Theorem 14 we now consider $0\leqslant f_n\uparrow$ in ${}^{\mathrm{r}}\mathrm{K}^{\alpha}(R,Z)_A$ upper bounded, $0\leqslant\alpha\in{}^{\mathrm{r}}\mathrm{BV}(\mathcal{I}(R),L(Z,W))$ and Z and W satisfying property (D) instead of property (A). Then,

1′) if I_n is upper bounded, then there exists $I=\lim\limits_n I_n$ (since W satisfies (D)) and we have 1);

2′) $f_n \in {}^{\mathrm{r}}\mathrm{K}^{\alpha}(R, Z)_A$ for every $n \in \mathbb{N}$, and we have 2);

3′) if $f_n\uparrow$ is upper bounded, then given $t \in R$, there exists $f(t) = \lim\limits_n f_n(t)$ (since Z satisfies (D)) and we have 3);

4′) $\alpha \in {}^{\mathrm{r}}\mathrm{BV}(\mathcal{I}(R), L(Z, W))$ and we have 4).

Thus, we proved the following

Theorem 15 (Monotone Convergence Theorem—version 2)**.** *Let $0 \leqslant \alpha \in {}^{\mathrm{r}}\mathrm{BV}(\mathcal{I}(R), L(Z, W))$ and let $0 \leqslant f_n\uparrow$ in ${}^{\mathrm{r}}\mathrm{K}^{\alpha}(R, Z)_A$ be upper bounded. Suppose Z and W possess property* (D)*. Let $I_n = {}^{\mathrm{rK}}\!\int_R \mathrm{d}\alpha f$ for every $n \in \mathbb{N}$, and suppose the sequence $(I_n)_{n\in\mathbb{N}}$ is upper bounded. Then there exists $f \in {}^{\mathrm{r}}\mathrm{K}^{\alpha}(R, Z)_A$ with $f(t) = \lim\limits_n f_n(t)$ for every $t \in R$, and*

$$\lim_n {}^{\mathrm{rK}}\!\!\int_R \mathrm{d}\alpha f_n = {}^{\mathrm{rK}}\!\!\int_R \mathrm{d}\alpha f.$$

An example by Birkhoff (see [1]) shows that even in the one-dimensional case, if Z possesses property (E) (which implies properties (A), (B), (C) and (D)), then the Monotone Convergence Theorem need not be valid for the Henstock integral and, therefore, for the Henstock vector integrals.

Proposition 16. *If $f \in {}^{\mathrm{r}}\mathrm{H}(R, Z)$, then there exists ${}^{\mathrm{r}}\mathrm{D}\tilde{f}\ (t) = f(t)$ for m-almost every $t \in R$.*

For a proof of Proposition 16, see [5], Theorem 2.2, or alternatively see [3] for the analogue of this result concerning vector integrals.

Example 17. Let

$$Z = l_2(IN) = \Big\{ z = (z_n)_{n\in\mathbb{N}},\ \ z_n \in \mathbb{R};\ \sum_n |z_n|^2 < \infty \Big\}$$

be equipped with the norm $\|z\| = \Big(\sum\limits_n |z_n|^2\Big)^{1/2}$. Then Z possesses property (E). By e_{ij} we mean the doubly infinite set of orthonormal vectors of Z. We define a function $f\colon\ [0,1] \subset \mathbb{R} \to Z$ by

$$f = \sum_i f_i,$$

where $f_i(t) = 2^i e_{ij}$ for $2^{-i} \leqslant t < j \cdot 2^{-i} + 1 \cdot 2^{-2i}$ and $j = 0, 1, 2, \ldots, 2^i - 1$. Given $n \in \mathbb{N}$, let $f_n = f_1 + f_2 + \ldots + f_n$. Then each $f_n \in \mathrm{H}([0,1], Z)$. However, we assert that $f \notin \mathrm{H}([0,1], Z)$. Indeed, in [1], Birkhoff showed that $\tilde{f}$ is nowhere differentiable. Thus, by Proposition [16], the result follows.

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