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ON MULTIPLICITIES OF SIMPLE SUBQUOTIENTS IN GENERALIZED VERMA MODULES

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Abstract. We reduce the problem on multiplicities of simple subquotients in an α-stratified generalized Verma module to the analogous problem for classical Verma modules.

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1. Introduction

The study of α-stratified modules over a simple complex finite-dimensional Lie algebra was originated in [3] where several basic properties of such modules were obtained. The class of α-stratified modules contains the so-called generalized Verma modules (GVM). These modules are completely different from another family of GVMs introduced and studied in [9]. The α-stratified GVM were investigated in [5, 6, 8] where a BGG-like criterion for the existence of a non-trivial homomorphism between two α-stratified GVMs was established.

One of the most important results about classical Verma modules is the so-called Kazhdan-Lusztig theorem describing the multiplicities of simple subquotients in a Verma module (see for example [1] and references therein). An analogous result for GVM in the sense of [9] was obtained in [2]. It happened that the answer obtained in [2] is different from the classical Kazhdan-Lusztig theorem. The latter means that the multiplicities of simple subquotients in a GVM (in the sense of [9]) cannot be obtained directly from the analogous multiplicities in the corresponding Verma module.

In the present paper we calculate the multiplicities of simple subquotients in an α-stratified GVM. In fact, with an arbitrary α-stratified GVM we associate a certain Verma module and prove that the required multiplicities coincide with the multiplicities of simple subquotients in this Verma module. This analogy with Verma modules provides one more difference between α-stratified GVMs and GVMs in the sense of [9].

We have to note that one related question for α-stratified modules was solved in [7, Theorem 13.4] in a full generality. In fact, for any simple complex finite-dimensional Lie algebra $\mathfrak{G}$ and its "well-embedded" subalgebra $\mathfrak{G}_1$ of type A_n or C_n, the character of the unique simple quotient of GVM induced from a homogeneous G_1-module was calculated. Here homogeneous means that this module is weight, dense and has weight subspaces of the same dimension (see [7] for details). In the case when $\mathfrak{G}_1$ is of type A_1, simple homogeneous means the same as simple α-stratified. Thus, using the above mentioned result one can calculate the character of the unique simple quotient of an α-stratified GVM.

The paper is organized as follows: in Section 2 we collect all necessary preliminaries. In Section 3 we formulate our main result—Theorem 1, which is proved in Section 4.

2. Preliminaries

Let $\mathbb{C}$ denote the complex numbers, $\mathbb{Z}$ the set of integers and $\mathbb{N}$ the set of all positive integers. For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ its universal enveloping algebra.

Let $\mathfrak{G}$ be a simple complex finite-dimensional Lie algebra and $\mathfrak{H}$ its Cartan subalgebra. Denote by Δ the corresponding root system and choose a base π in Δ. This defines a partition of Δ into two sets of positive (Δ^+) and negative (Δ^-) roots. We will write P for the abelian subgroup in $\mathfrak{H}^*$ generated by the elements from Δ. For $\beta \in \Delta$ let $\mathfrak{G}_\beta$ denote the corresponding root subspace in $\mathfrak{G}$. Fix a Weyl-Chevalley basis X_α, $\alpha \in \Delta$, H_α, $\alpha \in \pi$. Set

$$\varrho = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta.$$

Fix $\alpha \in \pi$. Let $\mathfrak{G}^\alpha$ denote the sl(2)-subalgebra of $\mathfrak{G}$ corresponding to the root α. Set $\mathfrak{N}^\alpha_\pm = \sum\limits_{\beta \in \Delta^+ \setminus \{\alpha\}} \mathfrak{G}_{\pm\beta}$, $\mathfrak{H}^\alpha = \{h \in \mathfrak{H} \mid \alpha(h) = 0\}$, $\pi_\alpha = \pi \setminus \{\alpha\}$. Then we have the following decomposition: $\mathfrak{G} = \mathfrak{G}^\alpha \oplus \mathfrak{N}^\alpha_- \oplus \mathfrak{H}^\alpha \oplus \mathfrak{N}^\alpha_+$. For $\mathfrak{H}_\alpha = \mathfrak{G}^\alpha \cap \mathfrak{H}$ one obtains $\mathfrak{G}^\alpha = \mathfrak{G}_\alpha \oplus \mathfrak{H}_\alpha \oplus \mathfrak{G}_{-\alpha}$.

For a $\mathfrak{G}$-module V and $\lambda \in \mathfrak{H}^*$ let V_λ denote the weight space with respect to weight λ. A $\mathfrak{G}$-module V will be called a *weight module* if it decomposes into a direct sum of its weight spaces. A weight $\mathfrak{G}$-module V is called *α-stratified* ([3]) if the actions of X_α and $X_{-\alpha}$ are injective on V. All modules considered in this paper are supposed to be weight modules with finite-dimensional weight spaces.

Consider the quadratic Casimir operator $c = (H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha$ in $U(\mathfrak{G}_\alpha)$. Any pair $a, b \in \mathbb{C}$ defines a unique indecomposable $\mathfrak{G}^\alpha$-module $N(a, b)$ such that $X_{-\alpha}$ acts bijectively on $N(a, b)$, all non-trivial weight spaces of $N(a, b)$ are one-dimensional, a is an eigenvalue of H_α and b is the (unique!) eigenvalue of c. One has $N(a, b) \simeq N(a + 2l, b)$ for any $l \in \mathbb{Z}$.

Since $\mathfrak{H} = \mathfrak{H}_\alpha \oplus \mathfrak{H}^\alpha$ we can rewrite an arbitrary $\lambda \in \mathfrak{H}^*$ as $\lambda = \lambda_\alpha + \lambda^\alpha$ where $\lambda_\alpha \in \mathfrak{H}_\alpha$ and $\lambda^\alpha \in \mathfrak{H}^\alpha$. Let $a, b \in \mathbb{C}$ and let $\lambda \in \mathfrak{H}^*$ be such that $(\lambda - \varrho)(H_\alpha) = (\lambda_\alpha - \varrho)(H_\alpha) = a$. We can define the structure of an $\mathfrak{H}$-module on $N(a, b)$ by setting $hv = (\lambda - \varrho)^\alpha(h)v$ for all $h \in \mathfrak{H}^\alpha$ and all $v \in N(a, b)$. Further, we can consider $N(a, b)$ as $D = \mathfrak{H} + \mathfrak{G}^\alpha \oplus \mathfrak{N}_+^\alpha$-module by setting $\mathfrak{N}_+^\alpha N(a, b) = 0$.

The $\mathfrak{G}$-module

$$M_\alpha(\lambda, b) = U(\mathfrak{G}) \bigotimes_{U(D)} N(a, b)$$

is called the *generalized Verma module* associated with $\mathfrak{G}, \mathfrak{H}, \pi, \alpha, \lambda, b$. One can easily prove that $M_\alpha(\lambda, b)$ is α-stratified if and only if $b \neq (a + 1 + 2l)^2$ for all $l \in \mathbb{Z}$ (see also [3, Theorem 2.1]). We will denote by $L_\alpha(\lambda, b)$ the unique simple quotient of $M_\alpha(\lambda, b)$. It is well-known that $M_\alpha(\lambda, b)$ has a composition series [3, Theorem 2.8 (i)]. For $\lambda \in \mathfrak{H}^*$ we will write $M(\lambda)$ for the Verma module with the highest weight $\lambda - \varrho$ ([4, 7.1.4]) and $L(\lambda)$ for its unique simple quotient.

3. Main theorem

Fix an analytic branch of the square root function satisfying the condition $\sqrt{1} = 1$. For arbitrary $\lambda \in \mathfrak{H}^*$ and $b \in \mathbb{C}$ set

$$f(\lambda, b) = \lambda - \frac{\lambda(H_\alpha) + \sqrt{b}}{\alpha(H_\alpha)}\, \alpha.$$

Theorem 1. *Suppose that $M_\alpha(\lambda, b)$ is α-stratified. Then the multiplicity of $L_\alpha(\mu, d)$ as a simple subquotient in a composition series of $M_\alpha(\lambda, b)$ equals the multiplicity of $L(f(\mu, d))$ as a simple subquotient in a composition series of $M(f(\lambda, b))$.*

For $u \in \mathbb{C}$ consider the $\mathfrak{G}^\alpha$-module

$$T(u) = \bigoplus_{a \in \mathbb{C}/2\mathbb{Z}} N(a, u)$$

and the corresponding induced module

$$M_T(\lambda, u) = U(\mathfrak{G}) \bigotimes_{U(D)} T(u).$$

A weight $\mathfrak{G}$-module V will be called *normal* provided $X_{-\alpha}$ acts bijectively on V. It follows from the definition of $N(a, b)$ that $M_T(\lambda, b)$ is normal.

Lemma 1. *Let V be a normal weight $\mathfrak{G}$-module and W a normal submodule of V. Then the module V/W is normal.*

Proof. Since V is normal it follows that $X_{-\alpha}$ acts surjectively on V/W. Moreover, since W is normal it follows that the pre-image of any element from W is contained in W and thus $X_{-\alpha}$ acts injectively on V/W. Combining these results we obtain that V/W is normal. $\square$

Consider a normal $\mathfrak{G}$-module V. Let $U(\alpha)$ denote the localization of $U(\mathfrak{G})$ with respect to the multiplicative set $\{X_{-a}^n \mid n \in \mathbb{N}\}$. $U(\alpha)$ is well-defined by [7, Lemma 4.2]. Since V is normal, we can define the $U(\alpha)$-module $V(\alpha) = U(\alpha) \otimes_{U(\mathfrak{G})} V$. By [7, Lemma 4.3] there exists a unique polynomial extension $\{\theta_x \mid x \in \mathbb{C}\}$ of the family of automorphisms $\theta_x \colon U(\alpha) \to U(\alpha)$, $x \in \mathbb{Z}$ such that $\theta_x(v) = X_{-\alpha}^x v X_{-\alpha}^{-x}$, $x \in \mathbb{Z}$. For a $U(\alpha)$-module W and $x \in \mathbb{C}$ we will denote by $\theta_x(W)$ the $U(\alpha)$-module which is equal to W as a vector space and $v \cdot w = \theta(v)w$ for all $v \in U(\alpha)$, $w \in W$. Clearly, one can consider any $U(\alpha)$-module as a $U(\mathfrak{G})$-module by restriction.

Set $P_\alpha = \left\{ \sum_{\beta \in \pi_\alpha} z_\beta \beta \mid z_\beta \in \mathbb{C} \right\}$ and $P(\alpha) = P + P_\alpha$. Let V be a weight $\mathfrak{G}$-module and $\lambda \in \mathfrak{H}^*$. We will denote by $V(\lambda)$ the direct summand $\sum_{\mu \in \lambda + P(\alpha)} V_\mu$ of V. For $\lambda_1, \lambda_2 \in \mathfrak{H}^*$ let $x(\lambda_1, \lambda_2)$ denote the unique complex number such that $\lambda_2 - (\lambda_1 + x(\lambda_1, \lambda_2)\alpha)$ belongs to P_α. A weight $\mathfrak{G}$-module V will be called *α-homogeneous* provided $V(\lambda_2) \simeq \theta_{x(\lambda_1, \lambda_2)} V(\lambda_1)$ for all $\lambda_1, \lambda_2 \in \mathfrak{H}^*$. It follows immediately from the definition that $M_T(\lambda, b)$ is α-homogeneous. One can easily see that the quotient of an α-homogeneous module by an α-homogeneous submodule is again α-homogeneous.

Let V be an α-homogeneous $\mathfrak{G}$-module. By a *solid structure* on V we will mean a family of linear maps $\psi(y) = \theta_{x(y\alpha, 0)}^{-1} \circ \varphi(y) \colon V(0) \to V(y\alpha)$, $y \in \mathbb{C}$, where $\varphi(y)$,

$y \in \mathbb{C}$ are isomorphisms of $V(0)$, which can be chosen in an arbitrary way. If a solid structure on V is given, V will be called a *solid* module. We will say that an α-homogeneous submodule W of V is *solid* provided

$$W(\lambda_1) = \psi(x(\lambda_1, 0)) \circ \psi^{-1}(x(\lambda_2, 0))(W(\lambda_2)).$$

It follows immediately from the definition that $M_T(\lambda, b)$ can be viewed as a solid α-homogeneous module (remark that the only automorphisms of the zero part of $M_T(\lambda, b)$ are scalars by [3], hence all $\varphi(y)$ are scalars). One can easily see that the quotient of a solid α-homogeneous module by a solid α-homogeneous submodule (whose solid structure is inherited from the big module) is again solid α-homogeneous.

Lemma 2. *Let V be a solid α-homogeneous module and let W be a normal submodule in V. Then the submodule $\hat{W}$ of V defined by*

$$\hat{W}(\mu) = \sum_{\mu' \in \mathfrak{H}^*} \psi(x(\mu, 0)) \circ \psi^{-1}(x(\mu', 0))(W(\mu')),$$

$\mu \in \mathfrak{H}^*$ is the unique minimal solid normal α-homogeneous submodule containing W.

P r o o f. Clearly, $\hat{W}$ is solid, α-homogeneous and contains W. It is normal by the definition of θ_x. Its minimality follows directly from the construction. The uniqueness follows from the solidness. □

The submodule $\hat{W}$ constructed in Lemma 2 will be called the *α-homogeneous hat* of W. A $\mathfrak{G}$-module V is said to be *simple normal* if there are no non-trivial normal $\mathfrak{G}$-submodules in V.

Lemma 3. *Let V be a solid α-homogeneous $\mathfrak{G}$-module and let W be its simple normal submodule. Let $\hat{W}$ be the α-homogeneous hat of W. Then $\hat{W}(\mu)$ is simple normal for any $\mu \in \mathfrak{H}^*$.*

P r o o f. Since W is simple normal it follows that $W = W(\mu')$ for some $\mu' \in \mathfrak{H}^*$. Thus $W(\mu') = \psi(x(\mu', 0)) \circ \psi^{-1}(x(\mu, 0))(\hat{W}(\mu))$. Suppose that $\hat{W}(\mu)$ is not simple normal and contains a non-trivial normal submodule, say N. Then $\psi(x(\mu', 0)) \circ \psi^{-1}(x(\mu, 0))(N)$ is a non-trivial normal submodule in $W(\mu')$, which contradicts our assumptions. □

Lemma 4. *Let V be solid α-homogeneous and let $W = W(\mu)$, $\mu \in \mathfrak{H}^*$, be a normal submodule in V. Suppose that W has a composition series,*

$$W = W_0 \supset W_1 \supset \ldots \supset W_k = 0,$$

such that all simple quotients $W^i = W_i/W_{i+1}$, $0 \leqslant i \leqslant k$, are normal. Let $\hat{W}$ be the α-homogeneous hat of W. Then $\hat{W}$ has a filtration

$$\hat{W} = \hat{W}_0 \supset \hat{W}_1 \supset \ldots \supset \hat{W}_k = 0,$$

such that each $\hat{W}_i$ is the α-homogeneous hat of W_i for all $0 \leqslant i \leqslant k$. Moreover, $\hat{W}^i = \hat{W}_i/\hat{W}_{i+1}$ is the α-homogeneous hat of W^i in $V/\hat{W}_{i+1}$ and $\hat{W}^i(\xi)$ is simple normal for all $\xi \in \mathfrak{H}^*$.

P r o o f. Follows from Lemma 3 and Lemma 1 by trivial induction in k. □

Lemma 5. *Suppose that W is simple normal. Then W contains the unique subquotient N such that $X_{-\alpha}$ acts injectively on N. Moreover, this subquotient is a submodule of W.*

P r o o f. As any simple subquotient of W on which X_{-a} acts injectively defines some normal subquotient of W, the first statement follows from the assumption that W is simple normal. The second statement follows from the bijectivity of $X_{-\alpha}$. □

Now we are ready to prove our main theorem.

P r o o f of Theorem 1. Consider the module $M_T(\lambda, b)$. Clearly, it is normal and we can view it as a solid α-homogeneous module with respect to an arbitrary solid structure. Consider its normal submodule $M_\alpha(\lambda, b)$. One can see that $M_T(\lambda, b)$ is the α-homogeneous hat of $M_\alpha(\lambda, b)$. Let $N = (M_T(\lambda, b))(f(\lambda, b))$. By Lemma 4 any composition series of $M_\alpha(\lambda, b)$ leads to a filtration of N with simple normal subquotients. By Lemma 5 each simple normal subquotient of N has a unique simple submodule on which $X_{-\alpha}$ acts injectively. Clearly, this correspondence is a bijection between the set of all simple subquotients of $M_\alpha(\lambda, b)$ and all simple subquotients of $M(f(\lambda, b))$ on which $X_{-\alpha}$ acts injectively. The rest follows from the trivial observation that the module corresponding to $L_\alpha(\mu, d)$ is exactly $L(f(\mu, d))$. Theorem 1 is proved. □

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