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SEMIREGULARITY OF CONGRUENCES IMPLIES CONGRUENCE MODULARITY AT 0

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Abstract. We introduce a weakened form of regularity, the so called semiregularity, and we show that if every diagonal subalgebra of $\mathscr{A} \times \mathscr{A}$ is semiregular then $\mathscr{A}$ is congruence modular at 0.

Keywords: regularity, modularity, semiregularity, modularity at 0

MSC 2000: 08A30, 08B10

Recall that an algebra $\mathscr{A}$ is *regular* if for every two congruences $\Theta, \Phi \in \operatorname{Con} \mathscr{A}$ the following holds

$$\text{if} \quad [a]_\Theta = [a]_\Phi \quad \text{for some } a \in A \quad \text{then} \quad \Theta = \Phi.$$

Note that this condition can be rewritten in the form:

$$\text{if} \quad [a]_\Theta = [a]_\Phi \quad \text{for some } a \in A \quad \text{then} \quad [b]_\Theta = [b]_\Phi \quad \text{for each } b \in A.$$

This formulation was used in [2] for introducing local regularity. At first we say that an algebra $\mathscr{A}$ has 0 if 0 is a nullary (term) operation of $\mathscr{A}$. An algebra $\mathscr{A}$ with 0 is *locally regular* if for each $\Theta, \Phi \in \operatorname{Con} \mathscr{A}$ the following holds:

$$\text{if} \quad [a]_\Theta = [a]_\Phi \quad \text{for some} \quad a \in \mathscr{A} \quad \text{then} \quad [0]_\Theta = [0]_\Phi.$$

The paper [2] contains examples of locally regular algebras and two characterizations of varieties of these algebras.

It was shown in [1] that if every subalgebra of the direct power $\mathscr{A} \times \mathscr{A}$ is regular then $\mathscr{A}$ is congruence modular, i.e. the congruence lattice $\operatorname{Con} \mathscr{A}$ is modular.

The concept of congruence modularity was weakened in [3] as follows:

An algebra $\mathscr{A}$ with 0 is *congruence modular at 0* if for each $\Theta, \Phi, \Psi \in \operatorname{Con}\mathscr{A}$ with $\Psi \subseteq \Phi$ the following holds

$$[0]_{\Phi\cap(\Theta\vee\Psi)} = [0]_{(\Phi\cap\Theta)\vee\Psi}.$$

Let $\mathscr{A} = (A, F)$ be an algebra. Denote by $\omega_A = \{\langle a, a\rangle; a \in \mathscr{A}\}$ the so called *diagonal* of $\mathscr{A}$, i.e. the least congruence on $\mathscr{A}$. A subalgebra $\mathscr{B}$ of the direct square $\mathscr{A} \times \mathscr{A}$ is called a *diagonal subalgebra* whenever $\omega_A \subseteq \mathscr{B}$.

Let us consider the conditions

(i) Every diagonal subalgebra of $\mathscr{A} \times \mathscr{A}$ is regular.

(ii) Every diagonal subalgebra of $\mathscr{A} \times \mathscr{A}$ is locally regular (with respect to the term $(0, 0)$).

We can ask whether there exists an intermediate property between the conditions (i) and (ii) which ensures the congruence modularity at 0 for $\mathscr{A}$.

Definition. Let $\mathscr{A}$ be an algebra with 0. We say that a diagonal subalgebra $\mathscr{B}$ of $\mathscr{A} \times \mathscr{A}$ is *semiregular* if for every $\alpha, \beta \in \operatorname{Con}\mathscr{B}$ the following holds: if $[(a, a)]_\alpha = [(a, a)]_\beta$ for some $a \in A$ then $[(0, a)]_\alpha = [(0, a)]_\beta$ whenever $(0, a) \in \mathscr{B}$.

Now let us deal with the condition

(iii) Every diagonal subalgebra of $\mathscr{A} \times \mathscr{A}$ is semiregular.

Then we have

$$\text{(i)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)}$$

(since the element $(0, 0)$ is contained in each diagonal subalgebra).

Applying an approach similar to that of [1] for regularity, we will show the connection between semiregularity and modularity at 0. For this, we need the following

Lemma. *Let every diagonal subalgebra of $\mathscr{A} \times \mathscr{A}$ be semiregular, let $\Psi, \Phi \in \operatorname{Con}\mathscr{A}$ and R be a reflexive and compatible relation on $\mathscr{A}$. If $\Psi \subseteq \Phi$ and $\Phi \cap R \subseteq \Psi$ then*

$$[\langle x_1, x_2\rangle \in R, \langle 0, y_2\rangle \in R, \langle x_1, 0\rangle \in \Phi, \langle x_2, y_2\rangle \in \Psi] \Rightarrow \langle x_1, 0\rangle \in \Psi.$$

P r o o f. Let every diagonal subalgebra of $\mathscr{A} \times \mathscr{A}$ be semiregular and Θ, Ψ, R satisfy the assumptions. Of course, R is a diagonal subalgebra of $\mathscr{A} \times \mathscr{A}$. Introduce the following two congruences α, β on R:

$$\langle (x_1, x_2), (y_1, y_2)\rangle \in \alpha \quad \text{if} \quad \langle x_1, y_1\rangle \in \Theta \quad \text{and} \quad \langle x_2, y_2\rangle \in \Psi,$$
$$\langle (x_1, x_2), (y_1, y_2)\rangle \in \beta \quad \text{if} \quad \langle x_1, y_1\rangle \in \Psi \quad \text{and} \quad \langle x_2, y_2\rangle \in \Psi.$$

Since $\Psi \subseteq \Phi$, we have $\beta \subseteq \alpha$. First we prove $[(y_2, y_2)]_\alpha = [(y_2, y_2)]_\beta$. Suppose $(z_1, z_2) \in [(y_2, y_2)]_\beta$ for some $\langle z_1, z_2\rangle \in R$. Then $\langle y_2, z_1\rangle \in \Phi$, $\langle y_2, z_2\rangle \in \Psi \subseteq \Phi$ thus also $\langle z_1, z_2\rangle \in \Phi$, i.e. $\langle z_1, z_2\rangle \in \Phi \cap R \subseteq \Psi$.

Together with $\langle y_2, z_2\rangle \in \Psi$ this gives $\langle y_2, z_1\rangle \in \Psi$ thus $\langle (y_2, y_2), (z_1, z_1)\rangle \in \alpha$ proving our equality. Since R is semiregular, this implies

$$[(0, y_2)]_\alpha = [(0, y_2)]_\beta.$$

By the assumption, $\langle 0, x_1\rangle \in \Phi$, $\langle y_2, x_2\rangle \in \Psi$, i.e. $\langle x_1, x_2\rangle \in [(0, y_2)]_\alpha = [(0, y_2)]_\beta$ thus also $\langle x_1, 0\rangle \in \Psi$. □

Theorem. *If every diagonal subalgebra of $\mathscr{A} \times \mathscr{A}$ is semiregular then $\mathscr{A}$ is congruence modular at 0.*

Proof. Let every diagonal subalgebra of $\mathscr{A} \times \mathscr{A}$ be semiregular, $\Theta, \Phi, \Psi \in \operatorname{Con} \mathscr{A}$ and $\Psi \subseteq \Phi$. To prove congruence modularity at 0 we need only to show that

$$[0]_{\Phi \cap (\Psi \vee \Theta)} \subseteq [0]_{\Psi \vee (\Phi \cap \Theta)}.$$

Denote by $R_0 = \Theta$ and define inductively

$$R_{k+1} = R_k \cdot \Psi \cdot \Theta \qquad \text{for } k = 0, 1, 2, \ldots$$

Hence, we need to prove

$$(*) \qquad [0]_{\Phi \cap R_k} \subseteq [0]_{\Psi \vee (\Phi \cap \Theta)}$$

for every $k = 0, 1, 2, \ldots$.

For $k = 0$ this holds trivially. Suppose that $(*)$ holds for some $k \geqslant 0$ and let us prove it for $k + 1$. Let $a \in [0]_{\Phi \cap R_{k+1}}$. Then there exist $b, c \in A$ such that

$$\langle a, 0\rangle \in \Phi, \quad \langle a, b\rangle \in R_k, \quad \langle b, c\rangle \in \Psi, \quad \langle c, 0\rangle \in \Theta.$$

However, $\Theta \subseteq R$ gives $\langle 0, c\rangle \in R_k$.

Set $\Psi^* = \Psi \vee (\Phi \cap \Theta)$. Then $\Psi^* \subseteq \Phi$ and $\Phi \cap R_k \subseteq \Psi^*$. Evidently, R_k is a diagonal subalgebra of $\mathscr{A} \times \mathscr{A}$. In account of the Lemma, we obtain $\langle a, 0\rangle \in \Psi^*$ which proves $(*)$ for $k + 1$. By induction, we have shown that $\mathscr{A}$ is congruence modular at 0. □

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