P. M. G. Manchón
Hypersurfaces in $\mathbb{R}^n$ and critical points in their external region

Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 1, 1–9

Persistent URL: `http://dml.cz/dmlcz/127696`

Czechoslovak Mathematical Journal, 52 (127) (2002), 1–9

HYPERSURFACES IN $\mathbb{R}^n$ AND CRITICAL POINTS IN THEIR EXTERNAL REGION

P. M. G. Manchón, Madrid

(Received July 7, 1997)

Abstract. In this paper we study the hypersurfaces M^n given as connected compact regular fibers of a differentiable map $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, in the cases in which f has finitely many nondegenerate critical points in the unbounded component of $\mathbb{R}^{n+1} - M^n$.

Keywords: hypersurface in $\mathbb{R}^n$, nondegenerate critical point, noncompact Morse Theory, h-cobordism, Palais-Smale condition

MSC 2000: 57R80

1. Introduction

Let $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a differentiable map (throughout this paper differentiable means differentiable of class C^∞) and let M^n be a connected compact regular fiber of f. By the Jordan-Brouwer Separation Theorem ([3]) M^n bounds a compact cobordism and if f is a Morse function, we can study the hypersurface M^n through the critical points in this cobordism. But what happens if f has degenerate critical points in the bounded cobordism, and f has finitely many critical points in the external region of $\mathbb{R}^{n+1} - M^n$ and all of them are nondegenerate? (This possibility is not odd: see Example in Section 2.) In this case, Classical Morse Deformation Theory does not work in the external region: this cobordism is not compact and it does not have a known first level.

The purpose of this paper is to solve this problem. To explain the answer, suppose f (or a good enough modification of f) extends to $S^{n+1} = \mathbb{R}^{n+1} \cup \{\infty\}$ through a maximal point ∞. Then, the nonempty inverse image of a regular value greater than

The results in this paper are contained in the author's Ph. D. Thesis, written under the direction of Professor E. Outerelo.

any critical value (of the map f defined on $\mathbb{R}^{n+1}$) would be the sphere S^n and we would have a Morse function on a compact cobordism in the external region with S^n as a first level. Although this is the solution, our approach is necessarily different because, in general, we can not extend the function f to S^{n+1}. We prove:

Theorem 2. *Let M^n be a connected compact hypersurface in $\mathbb{R}^{n+1}$ $(n \geqslant 2)$, and let $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a differentiable map with $M^n = f^{-1}(0)$ and 0 a regular value of f. Suppose f satisfies the condition of Palais-Smale or has compact fibers. Then, if f does not have critical points in the unbounded connected component of $\mathbb{R}^{n+1} - M^n$, we have that:*

i) *M^n is a homotopy 3-sphere if $n = 3$.*
ii) *M^n is homeomorphic to S^4 if $n = 4$.*
iii) *M^n is diffeomorphic to S^n if $n \geqslant 5$ or $n = 2$.*

We recall that a differentiable map $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ satisfies the condition of Palais-Smale if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^{n+1}$ such that $\{\|Df(x_n)\|\}_{n\in\mathbb{N}}$ tends to 0 and $\{f(x_n) \mid n \in \mathbb{N}\}$ is a bounded set, we have that $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence (see [10]).

Lemma 2. *Let $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a differentiable map with compact fibers $(n \in \mathbb{N})$. Then, if $\lambda \in \mathbb{R}$ is a regular value of f which is greater than any critical value of f, we have that $f^{-1}(\lambda)$ is connected (perhaps empty).*

Lemma 2 is a generalization of Theorem 3 in [2]. We can use this lemma as well for proving the connectedness of the fibers of the model function $g(x, y) = \frac{x^4}{4} + \frac{x^3}{3} - x^2 + y^2$ used in the characterization of the disk (see [4], p. 195).

Let now $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a proper differentiable function, and let $M^n = f^{-1}(0)$ be a regular level hypersurface. Suppose f has only finitely many critical points in the external region, and all of them are nondegenerate. As f is proper, f is unbounded. Assume for example that f is not bounded from above. Choose a regular value λ which is an upper bound of the set of the critical values. It is clear that $\left(f^{-1}([0,\lambda]); M^n, f^{-1}(\lambda)\right)$ is a compact cobordism and $f|_{f^{-1}([0,\lambda])}\colon f^{-1}([0,\lambda]) \to [0,\lambda]$ is a Morse function on this cobordism. Now, by Lemma 2 the fiber $f^{-1}(\lambda)$ is connected, and by Theorem 2 it is the sphere (differentiable if $n = 2$ or $n \geqslant 5$). Thus we know the first level of the cobordism and hence we can study M^n by means of the Classical Morse Deformation Theory. For example, we deduce:

Corollary 1. *Let S be a connected surface in $\mathbb{R}^3$ obtained as the zero set of a proper and differentiable function $f\colon \mathbb{R}^3 \to \mathbb{R}$ where 0 is a regular value of f. Then, if f has only one critical point in the unbounded connected component of $\mathbb{R}^3 - S$, and it is nondegenerate, S is diffeomorphic to the torus.*

Corollary 2. *Let M^n be a compact and connected hypersurface in $\mathbb{R}^{n+1}$ with $n \geqslant 5$. Then the following statements are equivalent:*

i) *M^n is diffeomorphic to S^n.*

ii) *There exists a differentiable map $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ such that 0 is a regular value of f, $M^n = f^{-1}(0)$, f does not have critical points in the unbounded connected component of $\mathbb{R}^{n+1} - M^n$, and f satisfies the condition of Palais-Smale.*

Moreover, the result also holds true if in ii) *we substitute "f satisfies the condition of Palais-Smale" by either the condition "f has compact fibers"* (statement iii)) *or the condition "f is a proper map"* (statement iv)).

First of all, we prove that every compact hypersurface (not necessarily connected) is a regular level set of a function with the nice properties of Theorem 2:

Theorem 1. *Let M^n be a compact hypersurface in $\mathbb{R}^{n+1}$. Then there exists a proper Morse function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ with finitely many critical points, and such that 0 is a regular value of f and M^n is its inverse image.*

In the proofs of the theorems and of Lemma 2 we will need the following topological result about proper maps:

Lemma 1. *Let $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a continuous map with $n \geqslant 1$. Suppose f has compact fibers and is not bounded from above. Then f is bounded from below and proper.*

2. Proofs

We begin with the proof of Lemma 1 for differentiable maps (recall that differentiable means differentiable of class C^∞), since this is the case we will need and it illustrates better the situation of the fibers of the map. The proof of Lemma 1 can be found in [6].

Proposition 1. *Let $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a differentiable map with $n \geqslant 1$. Suppose f has compact fibers and is not bounded from above. Then f is bounded from below and proper.*

P r o o f. As the map f is of class C^∞, the set of the regular values of f is dense in $\mathbb{R}$ by the Second Theorem of Sard ([5], 8.3.10). In particular there exists a sequence $\{t_m\}_{m\in\mathbb{N}} \subset \mathbb{R}$ of regular values of f which converges to $+\infty$, and we can suppose that the fibers of these points $f^{-1}(t_m)$ are nonempty because f is not bounded from above. As f has also compact fibers, $f^{-1}(t_m)$ is a nonempty compact differentiable submanifold (without boundary) of $\mathbb{R}^{n+1}$, and by the Jordan-Brouwer Separation Theorem, $\mathbb{R}^{n+1} - f^{-1}(t_m)$ has finitely many connected components, all bounded except one which is unbounded, and all are open (here we need the hypothesis $n \in \mathbb{N}$). Let $\mathcal{A}_m$ be the unbounded connected component of $\mathbb{R}^{n+1} - f^{-1}(t_m)$. Then $f(\mathcal{A}_m) \subset (t_m, +\infty)$ since f is not bounded from above and $\mathbb{R}^{n+1} - \mathcal{A}_m$ is compact. That f is bounded from below follows then from the facts that $f(\mathcal{A}_m)$ is (by t_m) and that $\mathbb{R}^{n+1} - \mathcal{A}_m$ is compact. (Note that we have only used one regular value of f with compact and nonempty fiber.)

We see now that f is proper, proving that for every $\mu \in \mathbb{R}$, $\mu > 0$, there exists a natural number N such that if $\|x\| > N$, then $|f(x)| > \mu$. As the sequence $\{t_m\}_{m\in\mathbb{N}}$ converges to $+\infty$, there exists $m_0 \in \mathbb{N}$ such that $t_{m_0} > \mu$, and since $\mathbb{R}^{n+1} - \mathcal{A}_{m_0}$ is compact, there exists $N > 0$ with $\mathbb{R}^{n+1} - \mathcal{A}_{m_0} \subset B[0, N] = \{x \in \mathbb{R}^{n+1} \mid \|x\| \leqslant N\}$. Then, if $x \in \mathbb{R}^{n+1}$ is such that $\|x\| > N$, $x \notin B[0, N]$, so $x \notin \mathbb{R}^{n+1} - \mathcal{A}_{m_0}$, that is $x \in \mathcal{A}_{m_0}$, and hence $|f(x)| \geqslant f(x) > t_{m_0} > \mu$. This finishes the proof of Proposition 1. □

P r o o f of Theorem 1. Suppose first that the hypersurface is connected: we can assume that the hypersurface in $\mathbb{R}^{n+1}$ is a hypersurface in S^{n+1}, the Alexandroff compactification of the Euclidean Space, and that it divides the sphere into two connected components with the hypersurface as common boundary. The closure of each component is a cobordism, and admits a Morse function (with the hypersurface as the zero level set of both Morse functions). By the compactness of the cobordisms, the Morse functions have finitely many critical points.

Now we can paste the Morse functions to obtain a Morse function on the sphere, keeping exactly the critical points of the original functions and with the hypersurface as the inverse image of zero. The details of this construction are along the line of Theorem 3.4 of [7], using strongly the uniqueness in Theorem 1.4 of the same reference.

By the Morse Lemma, we can obtain the above function with only an absolute maximal point and an absolute minimal point, and by the Thom Isotopy Extension Theorem ([4]) we can assume that the maximal point is the one used for the Alexandroff compactification of $\mathbb{R}^{n+1}$. In this way we get a Morse function on $\mathbb{R}^{n+1}$ with finitely many critical points such that 0 is a regular value, the hypersurface is its inverse image, and all the fibers of the map are compact.

However, the function we have obtained is not necessarily proper. In any case the supremum $b > 0$ of the function is not in the image and we then compose the function with an appropriate diffeomorphism of $(-\infty, b)$ onto $(-\infty, +\infty)$. By Proposition 1 the composite map is then a solution to the theorem in the connected case.

Although the proof of the disconnected case is similar, some remarks are necessary: first of all, by a generalization of the Jordan-Brouwer Separation Theorem the hypersurface divides the sphere into one more connected component than those of the hypersurface, and each component of the hypersurface will be a component of the boundary of exactly two of these components. Secondly, if A is a connected component of the hypersurface M^n and X and Y are the two cobordisms with $\partial X = A = \partial Y$ (adjacent cobordisms), the chosen Morse function for X will be negative if for Y it is positive. Note that this is possible even though the cobordisms may have many connected components of M^n in the boundary. The remainder of the proof runs basically as in the connected case. $\square$

We now prove the main result:

P r o o f of Theorem 2. Let G_0 and G_1 be the connected components of $\mathbb{R}^{n+1} - M^n$, so M^n is the boundary of both G_0 and G_1 and one of them, for example G_0, is bounded. Moreover G_0 and G_1 are open sets of $\mathbb{R}^{n+1}$ and $\overline{G}_i = G_i \cup M^n$ are differentiable submanifolds of $\mathbb{R}^{n+1}$ with $\partial(\overline{G}_i) = M^n$ for $i = 0, 1$.

As M^n is compact, there exists $r > 0$ with $M^n \subset B(0, r) = \{x \in \mathbb{R}^{n+1} \mid \|x\| < N\}$, hence $\overline{G}_0 \subset B(0, r)$. Then $X = \overline{G}_1 \cap B[0, r]$ is a differentiable submanifold of $\mathbb{R}^{n+1}$ with $\partial X = M^n + S^n_r$ the disjoint union of M^n and S^n_r, where S^n_r is the sphere of radius r centered at the origin of $\mathbb{R}^{n+1}$.

It suffices to prove that X is an h-cobordism between M^n and S^n_r if $n \geqslant 2$, this is, the maps $M^n \hookrightarrow X$ and $S^n_r \hookrightarrow X$ are homotopy equivalences. Of course, in that case, the theorem follows from the h-cobordism Theorem if $n \geqslant 5$ ([7], Theorem 9.1) and from [1], Corollary 7.1B in the case $n = 4$.

By an already well known argument (see for example [7], Theorem 9.1) it suffices to prove that M^n and X are simply-connected and the inclusion map $M^n \hookrightarrow X$ is a homotopy equivalence. As G_i are connected, $M^n = f^{-1}(0)$ and 0 is a regular value of f, we can assume that $f(G_0) \subset (-\infty, 0)$ and $f(G_1) \subset (0, +\infty)$.

Suppose first that f satisfies the condition of Palais-Smale. Let $C(f)$ be the set of critical points of f. Then, if $C(f) \cap G_1 = \emptyset$, there exists a diffeomorphism $\psi\colon M^n \times [0, +\infty) \to \overline{G}_1$ with $\psi(x, 0) = x$ for every $x \in M^n$. To see this, note first that there is an $\varepsilon > 0$ with $f(C(f)) \cap [-\varepsilon, 0] = \emptyset$. It follows that (see [9], Theorem 5.9) there exists a diffeomorphism $h\colon f^{-1}(0) \times (-\varepsilon, +\infty) \to f^{-1}(-\varepsilon, +\infty)$ with $h(x, 0) = x$ for every $x \in f^{-1}(0)$ and $h(f^{-1}(0) \times \{d\}) = f^{-1}(d)$ for every $d \in (-\varepsilon, +\infty)$, and so the map $\psi\colon M^n \times [0, +\infty) \to \overline{G}_1$ defined by $\psi(x, t) = h(x, t)$ is a diffeomorphism and $\psi(x, 0) = x$ for every $x \in M^n$ (note that f has then compact fibers).

Then the diagram

$$\begin{array}{ccc} M^n & \hookrightarrow & \overline{G}_1 \\ & {\scriptstyle j_0}\searrow \quad \nearrow{\scriptstyle \psi} & \\ & M^n \times [0, +\infty) & \end{array}$$

where $j_0(x) = (x, 0)$, is obviously commutative, and the inclusion map $M^n \hookrightarrow \overline{G}_1$ is a homotopy equivalence. On the other hand, the inclusion map $X \hookrightarrow \overline{G}_1$ is also a homotopy equivalence since there exists a strong deformation retract of $\overline{G}_1$ on X. As $M^n \subset X \subset \overline{G}_1$, we can conclude that the inclusion map $M^n \hookrightarrow X$ is a homotopy equivalence.

Let us see now that $\overline{G}_1$ is simply-connected. Choose $d > 0$ such that the hypersurface $f^{-1}(d) = \psi(M^n \times \{d\})$ is disjoint from $B[0, r]$. Then, if $i\colon\ f^{-1}(d) \hookrightarrow \overline{G}_1$ is the inclusion map, the induced homomorphism $i_*\colon\ \pi_1(f^{-1}(d)) \to \pi_1(\overline{G}_1)$ is the zero map because $f^{-1}(d) \subset \mathbb{R}^{n+1} - B[0, r] \subset \overline{G}_1$ and $n \geqslant 2$. Now, as the diagram

$$\begin{array}{ccc} M^n \times \{d\} & \longrightarrow & f^{-1}(d) \\ \big\downarrow & & \big\downarrow {\scriptstyle i} \\ M^n \times [0, +\infty) & \xrightarrow{\ \psi\ } & \overline{G}_1 \end{array}$$

is commutative, i_* is an isomorphism and hence $\pi_1(\overline{G}_1) = 0$.

In this way the proof of the theorem is finished when f satisfies the condition of Palais-Smale. Suppose now that f has compact fibers: if f is not bounded from above, f is proper by Proposition 1 and hence it satisfies Palais-Smale and we may apply the part of the theorem already proved. If f is bounded from above, $f(G_1) = (0, b)$ with $b > 0$ since $C(f) \cap G_1 = \emptyset$. We then consider a strictly increasing diffeomorphism $\beta\colon\ (-\infty, b) \to \mathbb{R}$ such that $\beta(t) = t$ for every $t \leqslant b/2$. Then the composite map $g = \beta f$ is proper by Proposition 1, hence it fulfills the condition of Palais-Smale, and of course $M^n = g^{-1}(0)$ and 0 is a regular value of g. As before, we then apply the part of the theorem already proved. This completes the proof. □

Note that under the hypothesis of Theorem 2, the map f has compact fibers in any case. We now prove a result about the connectedness of regular level sets:

P r o o f of Lemma 2. We first see that if there is a regular value λ of f which is an upper bound of $f(C(f))$ and its inverse image is nonempty, then $C(f)$ is bounded. Of course, the inverse image of such a regular value is a compact hypersurface, and by the Jordan-Brouwer Separation Theorem, its complement has finitely many connected components, all bounded except one which we call $\mathcal{M}$. Moreover every bounded component has critical points (at least extreme points) of f, hence f takes values less than λ in these components. Then, as $\mathcal{M}$ is connected and λ is a regular value of f, necessarily $f(\mathcal{M}) \subset (\lambda, +\infty)$ and so there are no critical points in $\mathcal{M}$ because λ is an upper bound of $f(C(f))$. So we have that $C(f)$ is included in the union of the bounded connected components of $\mathbb{R}^{n+1} - f^{-1}(\lambda)$ and hence it is bounded.

Note also that we can assume that f is not bounded from above: let $b \in \mathbb{R} \cup \{+\infty\}$ be the supremum of $\mathrm{im}(f)$. If f is bounded from above, $b \in \mathbb{R}$. Now, if $b \in \mathrm{im}(f)$, $b \in f(C(f))$ and if λ is a regular value of f which is an upper bound of $f(C(f))$, of course $f^{-1}(\lambda)$ is empty. The other possibility is that $b \notin \mathrm{im}(f)$: if a is the infimum of $\mathrm{im}(f)$, then $a < b$ and there exists $c \in \mathbb{R}$, $a < c < b$. We consider then a diffeomorphism $\alpha\colon (-\infty, b) \to \mathbb{R}$ satisfying the condition $\alpha(t) = t$ for every $t \leqslant c$, and the differentiable map $g = \alpha \circ f$. This function has compact fibers too, and it is unbounded. Now, if λ is a regular value of f which is an upper bound of $f(C(f))$, then $\alpha(\lambda)$ is a regular value of g which is an upper bound of $g(C(g))$, and of course, $f^{-1}(\lambda) = g^{-1}(\alpha(\lambda))$.

Now suppose that f is not bounded from above, so f is proper by Proposition 1. We have then that $f^{-1}(\lambda_1)$ and $f^{-1}(\lambda_2)$ are diffeomorphic if $\lambda_1 < \lambda_2$ are two regular values of f which are upper bounds of $f(C(f))$: in fact, as f is proper, $\big(f^{-1}([\lambda_1, \lambda_2]); f^{-1}(\lambda_1), f^{-1}(\lambda_2)\big)$ is a compact cobordism, and

$$f|_{f^{-1}([\lambda_1,\lambda_2])}\colon\ f^{-1}([\lambda_1, \lambda_2]) \to [\lambda_1, \lambda_2]$$

is a Morse function on this cobordism without critical points. So, by the First Theorem of Morse Deformation Theory, the fibers $f^{-1}(\lambda_1)$ and $f^{-1}(\lambda_2)$ are diffeomorphic manifolds.

From the above discussion, it suffices to prove that there is a regular value of f which is an upper bound of $f(C(f))$ such that its fiber is connected when f is a proper function, not bounded from above, and $f(C(f))$ is bounded from above. Then, we already know that $C(f) \subset B(0, s)$ for some $s > 0$. Besides, if μ is a regular value of f with $f^{-1}(\mu) \cap B(0, s) = \emptyset$, there exists a connected component A of $\mathbb{R}^{n+1} - f^{-1}(\mu)$ with $C(f) \subset A$. By the Jordan-Brouwer Separation Theorem and the existence of relative extreme points, if μ is a regular value of f and $\mathbb{R}^{n+1} - f^{-1}(\mu)$ has more than two connected components, at least two of these components have critical points of f, hence there is no connected component of $\mathbb{R}^{n+1} - f^{-1}(\mu)$ containing $C(f)$. Now, Lemma 2 follows from the fact that $f(B[0, s]))$ is bounded. □

Remark. In spite of Lemma 2, Theorem 2 requires the hypothesis of the connectedness of the hypersurface. Why?

Let now $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a proper differentiable function, and let $M^n = f^{-1}(0)$ be a regular level hypersurface. Suppose f has only finitely many critical points in the external region, and all of them are nondegenerate. As f is proper, f is unbounded. Assume for example that f is not bounded from above. Choose a regular value λ which is an upper bound of the set of the critical values. It is clear that $\big(f^{-1}([0,\lambda]); M^n, f^{-1}(\lambda)\big)$ is a compact cobordism and $f|_{f^{-1}([0,\lambda])}\colon f^{-1}([0,\lambda]) \to [0,\lambda]$ is a Morse function on this cobordism. Now, by Lemma 2 the fiber $f^{-1}(\lambda)$ is connected, and by Theorem 2 it is the sphere (differentiable if $n = 2$ or $n \geqslant 5$). So, we know the first level of the cobordism and hence we can study M^n by means of the Classical Morse Deformation Theory. For example, using the relation between Morse functions and surgery ([8]) we can prove Corollary 1 stated in the Introduction.

Example. From the last result, we infer easily that if $g\colon \mathbb{R}^3 \to \mathbb{R}$ is the polynomial map defined by

$$g(x_1, x_2, x_3) = x_1^4 + x_2^4 + x_3^4 + 2x_1^2x_2^2 + 2x_1^2x_3^2 + 2x_2^2x_3^2 + 6x_1^2 - 10x_2^2 - 10x_3^2 + 9,$$

then $S = g^{-1}(0)$ is a torus.

That S is connected follows from the Jordan-Brouwer Separation Theorem and from the next points:

i) $C(g) = \{(0,0,0)\} \cup \{(0, x_2, x_3) \in \mathbb{R}^3 \mid x_2^2 + x_3^2 = 5\}$ has exactly two connected components.

ii) $\{(0,0,0)\} \subset A$, where A is the only unbounded connected component of $\mathbb{R}^3 - S$ ($g(t,0,0) = t^4 + 6t^2 + 9 > 0$ for every $t \geqslant 0$).

iii) Every bounded connected component of $\mathbb{R}^3 - S$ has necessarily critical points (relative extreme points).

On the other hand g has only one critical point in the unbounded connected component of $\mathbb{R}^3 - S$: this is $\{p \in \mathbb{R}^3 \mid g(p) > 0\}$ and $C(g) \cap \{p \in \mathbb{R}^3 \mid g(p) > 0\} = \{(0,0,0)\}$. It is obvious that g fulfills all the other hypothesis of Corollary 1, hence S is a torus.

Making use of the Differentiable Schoenflies Theorem in dimensions greater than four we can prove (we need again Proposition 1):

Proposition 2. *Let M^n be a hypersurface in $\mathbb{R}^{n+1}$, with $n \geqslant 4$. Then, if M^n is diffeomorphic to S^n, there exists a proper differentiable map $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ such that 0 is a regular value of f, $M^n = f^{-1}(0)$, f has only one critical point and it is nondegenerate (hence f is a Morse function).*

P r o o f of Corollary 2. Of course ii) and iii) follows from iv). By Theorem 2, i) follows from ii) or iii). Finally i) implies iv) by Proposition 2. □

References

[1] *M. H. Freedman and F. S. Quinn*: Topology of 4-Manifolds. Princeton Univ. Press, Princeton, NJ., 1990.

[2] *J. M. Gamboa and C. Ueno*: Proper poynomial maps: the real case. Lectures Notes in Mathematics, 1524. Real Algebraic Geometry. Proceedings. Rennes, 1991, pp. 240–256.

[3] *V. Guillemin and A. Pollack*: Differential Topology. Prentice-Hall, Englewood Cliffs, NJ., 1974.

[4] *M. W. Hirsch*: Differential Topology. Springer-Verlag, Berlin, 1976.

[5] *J. Margalef and E. Outerelo*: Differential Topology. North Holland, Amsterdam, 1992.

[6] *P. M. G. Manchón*: Ph. D. Thesis. Universidad Complutense de Madrid.

[7] *J. Milnor*: Lectures on the h-cobordism Theorem. Princeton Univ. Press, Princeton, NJ., 1965.

[8] *J. Milnor*: A procedure for killing the homotopy groups of differentiable manifolds. Proc. Sympos. Pure Math, Vol. 3 Amer. Math. Soc. Providence, R.I. 1961, pp. 39–55.

[9] *R. S. Palais*: Lusternik-Schnirelman theory on Banach manifolds. Topology *5* (1966), 115–132.

[10] *R. S. Palais and S. Smale*: A generalized Morse theory. Bull. Amer. Math. Soc. *70* (1964), 165–172.

Author's address: Universidad Complutense, Facultad de Matemáticas, Dpto. de Geometría y Topología, Ciudad Universitaria s/n, 28040 Madrid, Spain, e-mail: `pmanchon@terra.es`; `pmanchon@ma.upm.es`.