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## SECOND CENTRALIZERS OF PARTIAL TRANSFORMATIONS

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*Abstract.* Second centralizers of partial transformations on a finite set are determined. In particular, it is shown that the second centralizer of any partial transformation  $\alpha$  consists of partial transformations that are locally powers of  $\alpha$ .

*Keywords:* partial transformation, second centralizer

*MSC 2000:* 20M20

## 1. INTRODUCTION

The semigroup  $PT_n$  of partial transformations on the set  $X = \{1, \dots, n\}$  consists of the functions whose domain and range are included in  $X$ , with composition as the semigroup operation. For  $\alpha \in PT_n$ , the sets

$$C(\alpha) = \{\gamma \in PT_n : \alpha \circ \gamma = \gamma \circ \alpha\} \text{ and} \\ C^2(\alpha) = \{\beta \in PT_n : \gamma \circ \beta = \beta \circ \gamma \text{ for each } \gamma \in C(\alpha)\}$$

are subsemigroups of  $PT_n$ , called the (first) *centralizer* of  $\alpha$  and the *second centralizer* of  $\alpha$ , respectively. Note that  $C^2(\alpha) \subseteq C(\alpha)$ .

The purpose of this paper is to determine the second centralizers in  $PT_n$ . The second centralizers in the semigroup  $T_n$  of full transformations on the set  $X$  are described in [7].

Obviously, every power  $\alpha^t$  ( $t \geq 0$ ) of  $\alpha \in PT_n$  is an element of  $C^2(\alpha)$ . If  $\alpha$  is not a nilpotent, then  $\{\alpha^t : t \geq 0\}$  is a proper subset of  $C^2(\alpha)$  since the zero (empty) transformation is in  $C^2(\alpha) \setminus \{\alpha^t : t \geq 0\}$ . Thus, in general,  $C^2(\alpha)$  does not consist of just the powers of  $\alpha$ . We show, however, that the elements of  $C^2(\alpha)$  are locally powers of  $\alpha$ .

More specifically, every  $\alpha \in PT_n$  induces a partition  $\{N, A_1, \dots, A_m\}$  of the set  $X = \{1, \dots, n\}$ . ( $A_1, \dots, A_m$  correspond to the weakly connected components containing a cycle in the digraph representation of  $\alpha$ ;  $N$  corresponds to the subgraph of the digraph representation obtained by removing all such components.)

Suppose that  $\beta \in C^2(\alpha)$ . We show that  $\beta$  restricted to  $N$  is equal to  $\alpha^t$  restricted to  $N$  for some  $t \geq 0$ . Similarly,  $\beta$  restricted to  $A_i$  ( $i = 1, \dots, m$ ) is either 0 or is equal to  $\alpha^{t_i}$  restricted to  $A_i$  for some  $t_i \geq 0$ . These necessary conditions are not sufficient for  $\beta$  to be in  $C^2(\alpha)$ . In addition, the exponents  $t, t_1, \dots, t_m$  must be related in a certain way. We prove that the “local powers” requirement together with these relations completely determine  $C^2(\alpha)$ .

## 2. FIRST CENTRALIZERS

This section introduces the terminology used throughout the paper and describes the first centralizers of partial transformations. Centralizers in  $PT_n$  have been studied in [3], [4], [5] and [6].

Let  $\alpha \in PT_n$ . The domain and range of  $\alpha$  will be denoted by  $\text{dom } \alpha$  and  $\text{ran } \alpha$ , respectively. If  $\beta \in PT_n$  is such that  $x\alpha = x\beta$  whenever  $x \in \text{dom } \alpha \cap \text{dom } \beta$ , we define the *join*  $\alpha\beta$  of  $\alpha$  and  $\beta$  as the partial transformation with  $\text{dom}(\alpha\beta) = \text{dom } \alpha \cup \text{dom } \beta$  that coincides with  $\alpha$  on  $\text{dom } \alpha$  and with  $\beta$  on  $\text{dom } \beta$ . Note that the join  $\alpha\beta$  (which, if defined, is simply the union of  $\alpha$  and  $\beta$ ) is distinct from the product (composition)  $\alpha \circ \beta$ .

For  $k \geq 1$ , let  $i_1, i_2, \dots, i_k$  be distinct elements of  $X$  such that  $i_1\alpha = i_2, i_2\alpha = i_3, \dots, i_{k-1}\alpha = i_k$ . Then  $\alpha$  restricted to the set  $\{i_1, \dots, i_{k-1}\}$  is called a *chain* in  $\alpha$  of length  $k$  (or a  $k$ -chain in  $\alpha$ ) and denoted  $(i_1 i_2 \dots i_k)$ . (Note that if  $k = 1$ , then  $(i_1)$  is the zero transformation.) If, in addition,  $i_k\alpha = i_1$  then  $\alpha$  restricted to the set  $\{i_1, i_2, \dots, i_k\}$  is called a *circuit* in  $\alpha$  of length  $k$  (or a  $k$ -circuit in  $\alpha$ ) and denoted  $(i_1 i_2 \dots i_k)$ .

Let  $\eta = (i_1 \dots i_k)$  be a chain in  $\alpha$ . The set  $\{i_1, \dots, i_k\}$  is called the *span* of  $\eta$  and denoted  $\text{span } \eta$ . If  $i_1 \notin \text{ran } \alpha$  and  $i_k \notin \text{dom } \alpha$ , we say that  $\eta$  is a *maximal* chain in  $\alpha$ . Note that  $(i_1)$  is a maximal chain in  $\alpha$  if and only if  $i_1 \notin \text{dom } \alpha \cup \text{ran } \alpha$ .

If  $\eta = (i_1 \dots i_u x_r)$  is a chain in  $\alpha$  and  $\varrho = (x_0 \dots x_{k-1})$  is a circuit in  $\alpha$  ( $u, k \geq 1$ ) such that  $i_1 \notin \text{ran } \alpha$  and  $\{i_1, \dots, i_u, x_r\} \cap \{x_0, \dots, x_{k-1}\} = \{x_r\}$ , we say that  $\eta$  is a *cilium* attached to  $\varrho$  at  $x_r$ . To distinguish cilia from maximal chains, we will use the right angle “ $\angle$ ” for the former and the right bracket “ $\rfloor$ ” for the latter. If  $\eta_1, \dots, \eta_s$  are the cilia in  $\alpha$  attached to  $\varrho$ , then the join  $\lambda = \eta_1 \dots \eta_s \varrho$  is called a *cell* in  $\alpha$ . Note that an isolated circuit (with no cilia) also forms a cell.

Every partial transformation  $\alpha \in PT_n$  is a join

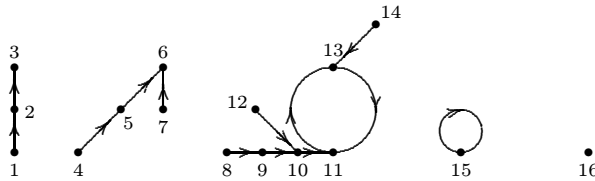
$$(1) \quad \eta_1 \dots \eta_k \lambda_1 \dots \lambda_m$$

of its maximal chains  $\eta_1, \dots, \eta_k$  and its cells  $\lambda_1, \dots, \lambda_m$ . Join (1) is called the *chain-cell decomposition* of  $\alpha$ .

If  $G$  is the digraph representation of  $\alpha$ , then the maximal chains in  $\alpha$  correspond to the simple maximal paths in  $G$ , and the cells in  $\alpha$  correspond to the weakly connected components of  $G$  containing a cycle. For example, the transformation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 2 & 3 & - & 5 & 6 & - & 6 & 9 & 10 & 11 & 13 & 10 & 11 & 13 & 15 & - \end{pmatrix} \in PT_{16}$$

has the digraph representation



and the chain-cell decomposition

$$\alpha = \underbrace{(1 \ 2 \ 3)}_{\eta_1} \underbrace{(4 \ 5 \ 6)}_{\eta_2} \underbrace{(7 \ 6)}_{\eta_3} \underbrace{(16)}_{\eta_4} \underbrace{(8 \ 9 \ 10 \ 11)(12 \ 10 \ 11)(14 \ 13)(11 \ 13)}_{\lambda_1} \underbrace{(15)}_{\lambda_2}.$$

If  $\alpha$  is a full transformation on  $X$ , then there are no maximal chains in  $\alpha$  and so  $\alpha = \lambda_1 \dots \lambda_m$  is a join of its cells. (For applications of the digraph representation of full transformations on  $X$ , see [2] and [1, 6.2].) If  $\alpha$  is a permutation on  $X$ , then  $\alpha$  is a join of its circuits.

Let  $\alpha, \gamma \in PT_n$ . Suppose that  $\eta = (i_1 \dots i_u]$  is a chain in  $\alpha$  and  $\varrho = (x_0 \dots x_{k-1})$  is a circuit in  $\alpha$ . If  $\text{dom } \gamma \cap \text{span } \eta \neq \emptyset$ , we say that  $\gamma$  *meets*  $\eta$ . Similarly, if  $\text{dom } \gamma \cap \text{dom } \varrho \neq \emptyset$ , we say that  $\gamma$  *meets*  $\varrho$ . If  $\xi = (j_1 \dots j_u]$  is a chain in  $\alpha$  such that  $i_1 \gamma = j_1, \dots, i_u \gamma = j_u$ , we say that  $\gamma$  *maps*  $\eta$  *onto*  $\xi$ .

The first centralizers in  $PT_n$  are characterized in [4, Theorem 4] (also see [5, 58.8]).

**Theorem 1.** *Let  $\alpha, \gamma \in PT_n$ . Then  $\gamma \in C(\alpha)$  if and only if for every maximal chain  $\eta = (i_1 \dots i_u]$  in  $\alpha$ , every circuit  $\varrho = (x_0 \dots x_{k-1})$  in  $\alpha$ , and every cilium  $\xi = (j_1 \dots j_v x_r)$  in  $\alpha$  attached to  $\varrho$ , the following conditions are satisfied:*

- (1) *If  $\gamma$  meets  $\eta$ , then there is a maximal chain  $\tau = (k_1 \dots k_w]$  in  $\alpha$  such that  $\gamma$  maps an initial segment  $(i_1 \dots i_p]$  of  $\eta$  ( $p \leq u$ ) onto a terminal segment  $(k_{w-p+1} \dots k_w]$  of  $\tau$  and  $\gamma$  does not meet  $(i_{p+1} \dots i_u]$ ;*
- (2) *If  $\gamma$  meets  $\varrho$ , then there is a circuit  $\delta = (y_0 \dots y_{m-1})$  in  $\alpha$  such that  $m$  divides  $k$ ,  $\gamma$  maps the points  $x_0, x_1, \dots, x_{k-1}$  of  $\text{dom } \varrho$  to  $y_s, y_s \alpha, \dots, y_s \alpha^{k-1}$ , and  $\gamma$  maps*

the points  $j_1, j_2, \dots, j_v, x_r$  of  $\text{span } \xi$  to  $z, z\alpha, \dots, z\alpha^{v-1}, z\alpha^v$ , where  $z$  is on  $\delta$  or some cilium attached to  $\delta$ ;

- (3) If  $\gamma$  does not meet  $\varrho$  but it meets  $\xi$ , then there is a maximal chain  $\tau = (k_1 \dots k_w]$  in  $\alpha$  such that  $\gamma$  maps an initial segment  $(j_1 \dots j_p]$  of  $\xi$  ( $p \leq v$ ) onto a terminal segment  $(k_{w-p+1} \dots k_w]$  of  $\tau$  and  $\gamma$  does not meet  $(j_{p+1} \dots j_v]$ .

### 3. SECOND CENTRALIZERS

Let  $\alpha \in PT_n$  and let  $\lambda$  be a cell in  $\alpha$ . We define the *radius* of  $\lambda$ , written  $r(\lambda)$ , as the largest integer  $u$  such that  $(i_1 \dots i_u x)$  is a cilium in  $\lambda$ . If  $\lambda$  has no cilia, we define  $r(\lambda)$  to be 0. Let  $\eta_1 \dots \eta_k$  be the join of all maximal chains in  $\alpha$  and let  $N = \text{span } \eta_1 \cup \dots \cup \text{span } \eta_k$ . We define the *diameter* of  $N$ , written  $d(N)$ , as the largest integer  $u$  such that  $(i_1 \dots i_u]$  is a maximal chain in  $\alpha$ . If  $N = \emptyset$  (that is, if  $\alpha$  has no maximal chains), we define  $d(N)$  to be 0.

For example, for  $\alpha = \underbrace{(1 \ 2]}_{\eta_1} \underbrace{(3]}_{\eta_2} \underbrace{(6 \ 7 \ 4)(8 \ 4)(4 \ 5)}_{\lambda_1} \underbrace{(10 \ 9)(9)}_{\lambda_2} \underbrace{(11 \ 12)}_{\lambda_3}$ , we have  $N = \{1, 2, 3\}$ ,  $d(N) = 2$ ,  $r(\lambda_1) = 2$ ,  $r(\lambda_2) = 1$ , and  $r(\lambda_3) = 0$ .

Let  $\mathbb{N} = \{0, 1, 2, \dots\}$  denote the set of nonnegative integers. We introduce an element  $-\infty \notin \mathbb{N}$  and agree that for every  $a \in \mathbb{N}$ ,  $-\infty < a$ , and that for every  $\beta \in PT_n$ ,  $\beta^{-\infty} = 0$ , where 0 is the zero (empty) transformation. For  $\beta \in PT_n$  and a subset  $A$  of  $X$ ,  $\beta|A$  will denote the restriction of  $\beta$  to  $A$ . Finally, the length of a circuit  $\varrho$  will be denoted by  $\ell(\varrho)$ .

The following theorem determines the second centralizers of partial transformations.

**Theorem 2.** *Let  $\alpha, \beta \in PT_n$ , let  $\alpha = \eta_1 \dots \eta_k \lambda_1 \dots \lambda_m$  be the chain-cell decomposition of  $\alpha$ , let  $N = \text{span } \eta_1 \cup \dots \cup \text{span } \eta_k$ , and let  $\varrho_i$  be the circuit in the cell  $\lambda_i$ . Then  $\beta \in C^2(\alpha)$  if and only if there are  $t \in \mathbb{N}$  and  $t_1, \dots, t_m \in \mathbb{N} \cup \{-\infty\}$  such that for all  $i, j \in \{1, \dots, m\}$ :*

- (1)  $\beta|N = \alpha^t|N$ ;
- (2)  $\beta| \text{dom } \lambda_i = \alpha^{t_i}| \text{dom } \lambda_i$ ;
- (3) If either  $t < \min\{d(N), r(\lambda_i)\}$  or  $0 \leq t_i < \min\{d(N), r(\lambda_i)\}$ , then  $t_i = t$ ;
- (4) If  $\ell(\varrho_i)$  divides  $\ell(\varrho_j)$ , then:
  - a) If  $t_i \geq 0$  and  $t_j \geq 0$ , then  $t_i \equiv t_j \pmod{\ell(\varrho_i)}$ ;
  - b) If either  $t_i$  or  $t_j$  is less than  $\min\{r(\lambda_i), r(\lambda_j)\}$ , then  $t_i = t_j$ .

Note that (4b) and the convention  $-\infty < a$  for every  $a \in \mathbb{N}$  imply that if  $\ell(\varrho_i)$  divides  $\ell(\varrho_j)$ , then  $t_i = -\infty \iff t_j = -\infty$ . To illustrate Theorem 2, we consider

the following transformations in  $PT_{12}$ :

$$\begin{aligned}\alpha &= \eta_1 \lambda_1 \lambda_2 = (1\ 2\ 3](6\ 7\ 4)(4\ 5)(12\ 8)(8\ 9\ 10\ 11), \\ \beta_1 &= (6\ 2](7\ 3], \\ \beta_2 &= (1\ 2\ 3], \\ \beta_3 &= (6\ 4)(4)(7\ 5)(5)(12\ 8)(8\ 9\ 10\ 11), \\ \beta_4 &= (1\ 2\ 3](6\ 7\ 4)(4\ 5), \\ \beta_5 &= (6\ 4)(4)(7\ 5)(5)(8)(9)(10)(11)(12), \\ \beta_6 &= (1\ 3](6\ 7\ 4)(4\ 5)(12\ 10)(8\ 11\ 10\ 9), \\ \beta_7 &= (1\ 3](6\ 5)(7\ 4)(4\ 5)(12\ 8)(8\ 9\ 10\ 11).\end{aligned}$$

By Theorem 1, each  $\beta_i$  is in  $C(\alpha)$ . Note that  $N = \{1, 2, 3\}$ ,  $d(N) = 3$ ,  $\ell(\varrho_1) = 2$ ,  $\ell(\varrho_2) = 4$ ,  $r(\lambda_1) = 2$ , and  $r(\lambda_2) = 1$ . We apply Theorem 2 to each  $\beta_i$ .

- (1)  $\beta_1 \notin C^2(\alpha)$  since  $\beta_1$  restricted to  $\text{dom } \lambda_1$  is not equal to any power of  $\alpha$  restricted to  $\text{dom } \lambda_1$ .
- (2)  $\beta_2|N = \alpha^1|N$ ,  $\beta_2| \text{dom } \lambda_1 = \alpha^{-\infty}| \text{dom } \lambda_1$ , and  $\beta_2| \text{dom } \lambda_2 = \alpha^{-\infty}| \text{dom } \lambda_2$ , but  $\beta_2 \notin C^2(\alpha)$  since  $1 < \min\{d(N), r(\lambda_1)\}$  and  $1 \neq -\infty$ .
- (3)  $\beta_3|N = \alpha^3|N$ ,  $\beta_3| \text{dom } \lambda_1 = \alpha^2| \text{dom } \lambda_1$ , and  $\beta_3| \text{dom } \lambda_2 = \alpha^1| \text{dom } \lambda_2$ , but  $\beta_3 \notin C^2(\alpha)$  since  $2 \not\equiv 1 \pmod{\ell(\varrho_1)}$ .
- (4)  $\beta_4|N = \alpha^1|N$ ,  $\beta_4| \text{dom } \lambda_1 = \alpha^1| \text{dom } \lambda_1$ , and  $\beta_4| \text{dom } \lambda_2 = \alpha^{-\infty}| \text{dom } \lambda_2$ , but  $\beta_4 \notin C^2(\alpha)$  since  $-\infty < \min\{r(\lambda_1), r(\lambda_2)\}$  and  $1 \neq -\infty$ .
- (5)  $\beta_5|N = \alpha^3|N$ ,  $\beta_5| \text{dom } \lambda_1 = \alpha^2| \text{dom } \lambda_1$ ,  $\beta_5| \text{dom } \lambda_2 = \alpha^0| \text{dom } \lambda_2$ , and  $2 \equiv 0 \pmod{\ell(\varrho_1)}$ , but  $\beta_5 \notin C^2(\alpha)$  since  $0 < \min\{r(\lambda_1), r(\lambda_2)\}$  and  $2 \neq 0$ .
- (6)  $\beta_6|N = \alpha^2|N$ ,  $\beta_6| \text{dom } \lambda_1 = \alpha^1| \text{dom } \lambda_1$ ,  $\beta_6| \text{dom } \lambda_2 = \alpha^3| \text{dom } \lambda_2$ , and  $3 \equiv 1 \pmod{\ell(\varrho_1)}$ , but  $\beta_6 \notin C^2(\alpha)$  since  $1 < \min\{d(N), r(\lambda_1)\}$  and  $1 \neq 3$ .
- (7)  $\beta_7|N = \alpha^2|N$ ,  $\beta_7| \text{dom } \lambda_1 = \alpha^3| \text{dom } \lambda_1$ ,  $\beta_7| \text{dom } \lambda_2 = \alpha^1| \text{dom } \lambda_2$ , and (3) and (4) of Theorem 2 are satisfied, so  $\beta_7 \in C^2(\alpha)$ .

The remainder of the paper will be devoted to proving Theorem 2. It is convenient to lay out the proof of the “only if” part of the theorem as a series of lemmas.

The following two lemmas show that for  $\alpha \in PT_n$  and  $\beta \in C^2(\alpha)$ ,  $\beta$  restricted to  $N$  is equal to some power of  $\alpha$  restricted to  $N$ . (In other words, such a  $\beta$  satisfies (1) of Theorem 2.)

**Lemma 3.** *Let  $\alpha, \beta \in PT_n$  be such that  $\beta \in C^2(\alpha)$ , and let  $\eta = (i_1 \dots i_u]$  be a maximal chain in  $\alpha$ . Then there is  $t \in \{0, \dots, u\}$  such that  $\beta| \text{span } \eta = \alpha^t| \text{span } \eta$ .*

**P r o o f.** If  $\text{dom } \beta \cap \text{span } \eta = \emptyset$ , then  $\beta| \text{span } \eta = \alpha^u| \text{span } \eta$ . Otherwise, by Theorem 1,  $i_1 \in \text{dom } \beta$  and one of the following two cases holds.

*Case 1.*  $i_1 \beta = i_p$  for some  $p \in \{1, \dots, u\}$ .

Then  $\beta|_{\text{span } \eta} = \alpha^{p-1}|_{\text{span } \eta}$  by Theorem 1.

*Case 2.* There is a maximal chain  $\xi = (j_1 \dots j_u]$  in  $\alpha$  such that for some  $p \in \{1, \dots, v\}$ ,  $j_p \notin \{i_1, \dots, i_u\}$  and  $i_1\beta = j_p$ .

We will construct  $\gamma \in C(\alpha)$  such that  $i_1 \notin \text{dom } \gamma$  and  $j_p \in \text{dom } \gamma$ . Set  $\text{dom } \gamma = \{x \in \text{dom } \alpha : x\alpha^q = j_p \text{ for some } q \geq 0\}$ . Define the values of  $\gamma$  so that for every maximal chain  $\mu = (m_1 \dots m_d j_p \dots j_v]$  ( $d \geq 0$ ) in  $\alpha$  whose span contains  $j_p$ ,  $\gamma$  maps the initial segment  $(m_1 \dots m_d j_p]$  of  $\mu$  onto a terminal segment of  $\mu$ . By Theorem 1 and the construction of  $\gamma$ , we have  $\gamma \in C(\alpha)$ ,  $i_1 \notin \text{dom } \gamma$  (since  $j_p \notin \{i_1, \dots, i_u\}$ ), and  $j_p \in \text{dom } \gamma$ . Thus  $i_1(\beta \circ \gamma) = j_p\gamma$  is defined and  $i_1(\gamma \circ \beta)$  is undefined. It follows that  $\gamma \notin C(\beta)$ , which is a contradiction.  $\square$

Recall that for a circuit  $\varrho$ ,  $\ell(\varrho)$  denotes the length of  $\varrho$ . Similarly, for a chain  $\eta$ ,  $\ell(\eta)$  will denote the length of  $\eta$ .

**Lemma 4.** *Let  $\alpha, \beta \in PT_n$  be such that  $\beta \in C^2(\alpha)$ , and let  $\eta = (i_1 \dots i_u]$  and  $\xi = (j_1 \dots j_v]$  be maximal chains in  $\alpha$ . Suppose that  $t \in \{0, \dots, u\}$  and  $w \in \{0, \dots, v\}$  are integers such that  $\beta|_{\text{span } \eta} = \alpha^t|_{\text{span } \eta}$  and  $\beta|_{\text{span } \xi} = \alpha^w|_{\text{span } \xi}$ . If  $t > w$ , then  $\beta|_{\text{span } \xi} = \alpha^t|_{\text{span } \xi}$ .*

*P r o o f.* Suppose  $t > w$ . Proceeding by induction on  $\ell(\eta) + \ell(\xi)$ , we assume that the lemma is true for all maximal chains  $\eta'$  and  $\xi'$  in  $\alpha$  with  $\ell(\eta') + \ell(\xi') > \ell(\eta) + \ell(\xi)$ . We consider three cases.

*Case 1.*  $w = 0$ .

Then  $i_u\beta = i_u\alpha^t$  is undefined (since  $t > w = 0$ ) and  $j_1\beta = j_1\alpha^0 = j_1$ . Define  $\gamma \in PT_n$  by:  $\text{dom } \gamma = \{j_1\}$  and  $j_1\gamma = i_u$ . By Theorem 1,  $\gamma \in C(\alpha)$ . Since  $j_1(\beta \circ \gamma) = j_1\gamma = i_u$  and  $j_1(\gamma \circ \beta) = i_u\beta$  is undefined,  $\gamma \notin C(\beta)$ , which is a contradiction.

*Case 2.*  $w = v$ .

Then  $\beta|_{\text{span } \xi} = 0|_{\text{span } \xi} = \alpha^t|_{\text{span } \xi}$  (since  $t > w$ ).

*Case 3.*  $1 \leq w < v$ .

Then  $j_1\beta = j_{w+1}$ . Let  $m = \min\{u, v\}$ . Since  $w < v$  and  $w < t \leq u$ ,  $w + 1 \leq m$ . Let  $\tau = (k_1 \dots k_b j_m \dots j_v]$  ( $b \geq 0$ ) be a longest maximal chain in  $\alpha$  whose span contains  $j_m$ . We consider two cases.

*Case 3.1.*  $b \leq u - 1$ .

Then we can construct  $\gamma \in C(\alpha)$  that maps the initial segment  $(j_1 \dots j_m]$  of  $\xi$  onto a terminal segment of  $\eta$ . Set  $\text{dom } \gamma = \{x \in \text{dom } \alpha : x\alpha^q = j_m \text{ for some } q \geq 0\}$ . Let  $\mu = (m_1 \dots m_d j_m \dots j_v]$  ( $d \geq 0$ ) be any maximal chain in  $\alpha$  whose span contains  $j_m$ . Since  $\ell(\tau) \geq \ell(\mu)$  and  $b \leq u - 1$ , we have  $u - d \geq u - b \geq 1$  and so  $u \geq d + 1$ . Thus we can define  $\gamma$  so that it maps  $(m_1 \dots m_d j_m]$  onto a terminal segment of  $\eta$ . In particular,  $\gamma$  maps the initial segment  $(j_1 \dots j_m]$  of  $\xi$  onto a terminal segment of  $\eta$ , say  $(i_r \dots i_u]$ .

By Theorem 1 and the construction of  $\gamma$ , we have  $\gamma \in C(\alpha)$ ,  $j_1\gamma = i_r$ , and  $j_{w+1}\gamma = i_{r+w}$ . Since  $i_r\beta = i_r\alpha^t$ , either  $i_r\beta$  is undefined or  $i_r\beta = i_{r+t}$ . Thus  $j_1(\beta \circ \gamma) = j_{w+1}\gamma = i_{r+w}$  and either  $j_1(\gamma \circ \beta) = i_r\beta$  is undefined or  $j_1(\gamma \circ \beta) = i_r\beta = i_{r+t}$ . In either case, since  $t > w$ , it follows that  $\gamma \notin C(\beta)$ , which is a contradiction.

*Case 3.2.*  $b \geq u$ .

Then  $\ell(\tau) = b + v - m + 1 \geq u + v - m + 1 \geq m + v - m + 1 = v + 1 > \ell(\xi)$  and  $\ell(\tau) = b + v - m + 1 \geq u + v - m + 1 \geq u + m - m + 1 = u + 1 > \ell(\eta)$ . By Lemma 3,  $\beta | \text{span } \tau = \alpha^p | \text{span } \tau$  for some  $p \in \{0, \dots, \ell(\tau)\}$ . Suppose  $p > w$ . Then, by the inductive hypothesis applied to  $\tau$  and  $\xi$ ,  $\beta | \text{span } \xi = \alpha^p | \text{span } \xi$ . It follows that  $j_1\beta = j_1\alpha^p \neq j_{w+1} = j_1\beta$ , which is a contradiction. Thus  $p \leq w$ . Then  $t > p$  and so, by the inductive hypothesis applied to  $\eta$  and  $\tau$ ,  $\beta | \text{span } \tau = \alpha^t | \text{span } \tau$ . Note that, since  $\ell(\tau) > \ell(\eta)$  and  $t \in \{0, \dots, \ell(\eta)\}$ , we also have  $t \in \{0, \dots, \ell(\tau)\}$ . Now repeat the argument used in the case  $p > w$  above (with  $p$  replaced by  $t$ ) to obtain a contradiction. This concludes the proof.  $\square$

The next three lemmas show that for  $\alpha \in PT_n$  and  $\beta \in C^2(\alpha)$ ,  $\beta$  restricted to the domain of a cell  $\lambda_i$  is equal to some power (possibly  $-\infty$ ) of  $\alpha$  restricted to that domain. (In other words, such a  $\beta$  satisfies (2) of Theorem 2.)

**Lemma 5.** *Let  $\alpha, \beta \in PT_n$  be such that  $\beta \in C^2(\alpha)$ , and let  $\varrho = (x_0 \dots x_{k-1})$  be a circuit in  $\alpha$ . If  $\text{dom } \beta \cap \text{dom } \varrho \neq \emptyset$ , then there is  $t \in \{0, \dots, k-1\}$  such that  $\beta | \text{dom } \varrho = \alpha^t | \text{dom } \varrho$ .*

*Proof.* By Theorem 1,  $x_0 \in \text{dom } \beta$  and one of the following two cases holds.

*Case 1.*  $x_0\beta = x_t$  for some  $t \in \{0, \dots, k-1\}$ .

Then  $\beta | \text{dom } \varrho = \alpha^t | \text{dom } \varrho$  by Theorem 1.

*Case 2.* There is a circuit  $\delta = (y_0 \dots y_{m-1})$  in  $\alpha$  such that  $\delta \neq \varrho$ ,  $m$  divides  $k$ , and  $x_0\beta = y_p$  for some  $p \in \{0, \dots, m-1\}$ .

Let  $\lambda$  be the cell in  $\alpha$  that has  $\delta$  as the circuit. Define  $\gamma \in PT_n$  by  $\text{dom } \gamma = \text{dom } \lambda$  and  $y\gamma = y$  for every  $y \in \text{dom } \lambda$ . By Theorem 1 and the construction of  $\gamma$ , we have  $\gamma \in C(\alpha)$ ,  $x_0 \notin \text{dom } \gamma$ , and  $y_p\gamma = y_p$ . Since  $x_0(\beta \circ \gamma) = y_p\gamma = y_p$  and  $x_0(\gamma \circ \beta)$  is undefined,  $\gamma \notin C(\beta)$ , which is a contradiction.  $\square$

**Lemma 6.** *Let  $\alpha, \beta \in PT_n$  be such that  $\beta \in C^2(\alpha)$ , and let  $\eta = (i_1 \dots i_u x_0)$  be a cilium in  $\alpha$  attached to a circuit  $\varrho = (x_0 \dots x_{k-1})$ . If  $\text{dom } \beta \cap \text{span } \eta \neq \emptyset$ , then there is  $t \in \{0, \dots, u+k-1\}$  such that  $\beta | \text{span } \eta = \alpha^t | \text{span } \eta$ .*

*Proof.* Let  $\lambda$  be the cell in  $\alpha$  that has  $\varrho$  as the circuit. By Theorem 1 and Lemma 5,  $i_1 \in \text{dom } \beta$  and one of the following four cases holds.

*Case 1.*  $i_1\beta = i_p$  for some  $p \in \{1, \dots, u\}$ .

Then  $\beta | \text{span } \eta = \alpha^{p-1} | \text{span } \eta$  by Theorem 1.



*Case 2.*  $i_1\beta = x_p$  for some  $p \in \{0, \dots, k-1\}$ .

Then  $\beta | \text{span } \eta = \alpha^{u+p} | \text{span } \eta$  by Theorem 1.

*Case 3.* There is a cilium  $\xi = (j_1 \dots j_v x_s)$  in  $\lambda$  such that for some  $p \in \{1, \dots, v\}$ ,  $i_1\beta = j_p$  and  $j_p \notin \{i_1, \dots, i_u\}$ .

We consider two cases.

*Case 3.1.*  $s \neq 0$ , i.e.,  $\eta$  and  $\xi$  meet  $\varrho$  at different points.

We will construct  $\gamma \in C(\alpha)$  such that  $i_1\gamma = i_1$  and  $j_p\gamma \neq j_p$ . Set  $\text{dom } \gamma = \text{dom } \lambda$ . Let  $\mu = (m_1 \dots m_d x_s)$  be any cilium in  $\lambda$  attached to  $\varrho$  at  $x_s$  and let  $x_h \in \text{dom } \varrho$  be such that  $x_h \alpha^d = x_s$ . Define  $\gamma$  so that it maps the points  $m_1, m_2, \dots, m_d, x_s$  of  $\text{span } \mu$  to  $x_h, x_h \alpha, \dots, x_h \alpha^{d-1}, x_h \alpha^d = x_s$ . If  $y \in \text{dom } \lambda$  is not in the span of any cilium attached to  $\varrho$  at  $x_s$ , define  $y\gamma = y$ . By Theorem 1 and the construction of  $\gamma$ , we have  $\gamma \in C(\alpha)$ ,  $i_1\gamma = i_1$  (since  $\eta$  is not attached to  $\varrho$  at  $x_s$ ), and  $j_p\gamma \neq j_p$  (since  $\xi$  is attached to  $\varrho$  at  $x_s$  and so  $j_p\gamma \in \text{dom } \varrho$ ). Since  $i_1(\beta \circ \gamma) = j_p\gamma \neq j_p$  and  $i_1(\gamma \circ \beta) = i_1\beta = j_p$ ,  $\gamma \notin C(\beta)$ , which is a contradiction.

*Case 3.2.*  $s = 0$ , i.e.,  $\eta$  and  $\xi$  are attached to  $\varrho$  at the same point.

Since both  $\eta$  and  $\xi$  meet  $\varrho$  at  $x_0$  and  $j_p \notin \{1, \dots, u\}$ , we have  $\eta = (i_1 \dots i_q z \dots)$ ,  $\xi = (j_1 \dots j_p \dots j_r z \dots)$  ( $q \geq 1, r \geq p$ ), and  $\{i_1, \dots, i_q\} \cap \{j_1, \dots, j_r\} = \emptyset$ . (Note that  $z$  may be equal to  $x_0$ .) Let  $\tau = (k_1 \dots k_b j_r z \dots)$  ( $b \geq 0$ ) be a longest cilium in  $\lambda$  whose span contains  $j_r$ . If  $j_p \in \text{span } \tau$ , we may assume that  $\xi = \tau$ . We consider three cases.

*Case 3.2.1.*  $\tau \neq \xi$  (which implies  $j_p \notin \text{span } \tau$ ).

We will construct  $\gamma \in C(\alpha)$  such that  $i_1\gamma = i_1$  and  $j_p\gamma \neq j_p$ . Set  $\text{dom } \gamma = \text{dom } \lambda$ . Let  $\mu = (m_1 \dots m_d x_0)$  be any cilium in  $\lambda$  whose span contains  $j_r$ . Since  $\ell(\tau) \geq \ell(\mu)$ , we can define  $\gamma$  so that it maps  $\mu$  onto a terminal segment of  $\tau$ . If  $y \in \text{dom } \lambda$  is not in the span of any cilium in  $\lambda$  whose span contains  $j_r$ , define  $y\gamma = y$ . By Theorem 1 and the construction of  $\gamma$ , we have  $\gamma \in C(\alpha)$ ,  $i_1\gamma = i_1$  (since  $j_r \notin \text{span } \eta$ ), and  $j_p\gamma \neq j_p$  (since  $j_p\gamma \in \text{span } \tau$  and  $j_p \notin \text{span } \tau$ ), which leads to a contradiction as in Case 3.1.

*Case 3.2.2.*  $\tau = \xi$  and  $\ell(\eta) \geq \ell(\xi)$ .

Again, we will construct  $\gamma \in C(\alpha)$  such that  $i_1\gamma = i_1$  and  $j_p\gamma \neq j_p$ . Let  $\mu = (m_1 \dots m_d x_0)$  be any cilium in  $\lambda$  whose span contains  $j_r$ . Since  $\ell(\eta) \geq \ell(\xi) \geq \ell(\mu)$ , we can define  $\gamma$  so that it maps  $\mu$  onto a terminal segment of  $\eta$ . If  $y \in \text{dom } \lambda$  is not in the span of any cilium in  $\lambda$  whose span contains  $j_r$ , define  $y\gamma = y$ . By Theorem 1 and the construction of  $\gamma$ , we have  $\gamma \in C(\alpha)$ ,  $i_1\gamma = i_1$  (since  $j_r \notin \text{span } \eta$ ), and  $j_p\gamma \neq j_p$  (since  $j_p\gamma \in \text{span } \eta$  and  $j_p \notin \text{span } \eta$ ), which leads to a contradiction as in Case 3.1.

*Case 3.2.3.*  $\tau = \xi$  and  $\ell(\eta) < \ell(\xi)$ .

We will construct  $\gamma \in C(\alpha)$  such that  $j_1\gamma = i_1$  and  $j_p \notin \text{ran } \gamma$ . Set  $\text{dom } \gamma = \text{dom } \lambda$ . Let  $a \in \{0, \dots, k-1\}$  be such that  $a \equiv v - u \pmod{k}$ . Let  $\mu = (m_1 \dots m_d x_0)$  be any cilium in  $\lambda$  whose span contains  $j_r$  and let  $c = v - d + 1$ . Since  $\ell(\xi) \geq \ell(\mu)$ ,  $c \geq 1$ . If  $c \leq u$ , define  $\gamma$  so that it maps the points  $m_1, m_2, \dots, m_d, x_0$  of  $\text{dom } \mu$  to

$i_c, i_c\alpha, \dots, i_c\alpha^{d-1}, i_c\alpha^d$ . Note that  $i_c\alpha^d = x_a$ . If  $c > u$ , select  $x_h \in \text{dom } \varrho$  so that  $x_h\alpha^d = x_a$  and define  $\gamma$  so that it maps the points  $m_1, m_2, \dots, m_d, x_0$  of  $\text{dom } \mu$  to  $x_h, x_h\alpha, \dots, x_h\alpha^{d-1}, x_h\alpha^d = x_a$ . Note that if  $\mu = \xi$ , then  $c = v-d+1 = v-v+1 = 1$  and  $j_1\gamma = m_1\gamma = i_c = i_1$ . If  $y \in \text{dom } \lambda$  is not in the span of any cilium whose span contains  $j_r$ , we define  $y\gamma = y\alpha^a$ .

By Theorem 1 and the construction of  $\gamma$ , we have  $\gamma \in C(\alpha)$ ,  $j_1\gamma = i_1$ , and  $j_p \notin \text{ran } \gamma$ . (Indeed, let  $y \in \text{dom } \lambda$ . If  $y$  is in the span of a cilium whose span contains  $j_r$ , then  $y\gamma$  is in the set  $\{i_1, \dots, i_u\} \cup \text{dom } \varrho$ . Thus  $y\gamma \neq j_p$  since  $j_p$  is not in that set. If  $y$  is not in the span of any such cilium, then  $y\gamma = y\alpha^a \neq j_p$  since otherwise we would have  $y\alpha^{a+r-p} = j_r$ , which cannot happen if  $y$  is not in the span of a cilium whose span contains  $j_r$ . Hence  $j_p \notin \text{ran } \gamma$ .) Since  $\text{ran}(\beta \circ \gamma) \subseteq \text{ran } \gamma$ ,  $j_p \notin \text{ran}(\beta \circ \gamma)$ . Since  $j_1(\gamma \circ \beta) = i_1\beta = j_p$ ,  $j_p \in \text{ran}(\gamma \circ \beta)$ . It follows that  $\gamma \notin C(\beta)$ , which is a contradiction.

*Case 4.* There is a maximal chain  $\xi = (j_1 \dots j_v]$  in  $\alpha$  such that  $i_1\beta = j_p$  for some  $p \in \{1, \dots, v\}$ .

Define  $\gamma \in PT_n$  by:  $\text{dom } \gamma$  is the union of spans of all maximal chains in  $\alpha$ , and  $y\gamma = y$  for all  $y \in \text{dom } \gamma$ . By Theorem 1 and the construction of  $\gamma$ , we have  $\gamma \in C(\alpha)$ ,  $j_p \in \text{dom } \gamma$ , and  $i_1 \notin \text{dom } \gamma$ . Since  $i_1(\beta \circ \gamma) = j_p\gamma$  is defined and  $i_1(\gamma \circ \beta) = (i_1\gamma)\beta$  is undefined,  $\gamma \notin C(\beta)$ , which is a contradiction.  $\square$

**Lemma 7.** Let  $\alpha, \beta \in PT_n$  be such that  $\beta \in C^2(\alpha)$ , and let  $\eta = (i_1 \dots i_u x_0)$  and  $\xi = (j_1 \dots j_v x_s)$  be cilia in  $\alpha$  attached to a circuit  $\varrho = (x_0 \dots x_{k-1})$  such that  $\beta| \text{dom } \varrho = \alpha^e| \text{dom } \varrho$  for some  $e \in \{0, \dots, k-1\}$ . Suppose that  $t \in \{0, \dots, u+k-1\}$  and  $w \in \{0, \dots, v+k-1\}$  are integers such that  $\beta| \text{span } \eta = \alpha^t| \text{span } \eta$  and  $\beta| \text{span } \xi = \alpha^w| \text{span } \xi$ . If  $t > w$ , then  $\beta| \text{span } \xi = \alpha^t| \text{span } \xi$ .

*Proof.* Suppose  $t > w$  and let  $\lambda$  be the cell in  $\alpha$  that has  $\varrho$  as the circuit. Proceeding by induction on  $\ell(\eta) + \ell(\xi)$ , we assume that the lemma is true for all cilia  $\eta'$  and  $\xi'$  in  $\lambda$  with  $\ell(\eta') + \ell(\xi') > \ell(\eta) + \ell(\xi)$ .

Since  $x_0\alpha^t = x_0\alpha^e$  and  $x_s\alpha^w = x_s\alpha^e$ , we have  $t \equiv e \pmod{k}$  and  $w \equiv e \pmod{k}$ . Thus  $t \equiv w \pmod{k}$  and so, since  $t > w$ ,  $t = w + lk$  for some  $l \geq 1$ . We consider three cases.

*Case 1.*  $w = 0$ .

Then  $i_u\beta = i_u\alpha^{lk} = x_{k-1}$  and  $j_1\beta = j_1\alpha^0 = j_1$ . We will construct  $\gamma \in C(\alpha)$  such that  $j_1\gamma = i_u$ . Set  $\text{dom } \gamma = \text{dom } \lambda$ . Select  $q \in \{0, \dots, k-1\}$  so that  $q \equiv v-1 \pmod{k}$  and define  $\gamma$  so that it maps the points  $j_1, j_2, \dots, j_v, x_s$  of  $\text{span } \xi$  to  $i_u, i_u\alpha = x_0, \dots, i_u\alpha^{v-1}, i_u\alpha^v = x_q$ . Let  $a \in \{0, \dots, k-1\}$  be such that  $a \equiv q-s \pmod{k}$ . For any cilium  $\mu = (m_1 \dots m_d x_c)$  in  $\lambda$  with  $\mu \neq \xi$ , select  $x_h \in \text{dom } \varrho$  so that  $x_h\alpha^d = x_{a+c}$  and define  $\gamma$  so that it maps the points  $m_1, m_2, \dots, m_d, x_c$  of  $\text{span } \mu$  to  $x_h, x_h\alpha, \dots, x_h\alpha^{d-1}, x_h\alpha^d = x_{a+c}$ . By Theorem 1 and the construction of  $\gamma$ , we

have  $\gamma \in C(\alpha)$  and  $j_1\gamma = i_u$ . Since  $j_1(\beta \circ \gamma) = j_1\gamma = i_u$  and  $j_1(\gamma \circ \beta) = i_u\beta = x_{k-1}$ ,  $\gamma \notin C(\beta)$ , which is a contradiction.

*Case 2.*  $w \geq v$ .

Then for each  $p \in \{1, \dots, v\}$ ,  $j_p\beta = j_p\alpha^w = x_q$  for some  $q \in \{0, \dots, k-1\}$ . Thus  $j_p\alpha^t = j_p\alpha^{w+lk} = (j_p\alpha^w)\alpha^{lk} = x_q\alpha^{lk} = x_q = j_p\beta$ . Similarly,  $x_s\alpha^t = x_s\beta$  and so  $\beta|_{\text{span } \xi} = \alpha^t|_{\text{span } \xi}$ .

*Case 3.*  $1 \leq w < v$ .

Then  $j_1\beta = j_1\alpha^w = j_{w+1}$ . Let  $m = \min\{u, v\}$ . Since  $w < v$  and  $w = t - lk \leq u + k - 1 - lk = u - (l-1)k - 1 < u$ ,  $w + 1 \leq m$ . Let  $\tau = (k_1 \dots k_b j_m \dots j_v x_s)$  ( $b \geq 0$ ) be a longest cilium in  $\lambda$  whose span contains  $j_m$ . We consider two cases.

*Case 3.1.*  $b \leq u - 1$ .

Then we can construct  $\gamma \in C(\alpha)$  that maps the initial segment  $(j_1 \dots j_m)$  of  $\xi$  onto a terminal segment of  $(i_1 \dots i_u)$ . Set  $\text{dom } \gamma = \text{dom } \lambda$ . Select  $q \in \{0, \dots, k-1\}$  such that  $q \equiv v - m \pmod{k}$ . Let  $\mu = (m_1 \dots m_d j_m \dots j_v x_s)$  ( $d \geq 0$ ) be any cilium in  $\lambda$  whose span contains  $j_m$ . Since  $\ell(\tau) \geq \ell(\mu)$  and  $b \leq u - 1$ , we have  $u \geq b + 1 \geq d + 1$ . Thus we can define  $\gamma$  so that it maps the initial segment  $(m_1 \dots m_d j_m)$  of  $\mu$  onto a terminal segment of  $(i_1 \dots i_u)$  and the remaining points of  $\text{span } \mu$ ,  $j_{m+1}, j_{m+2}, \dots, j_v, x_s$ , to  $x_0, x_0\alpha, \dots, x_0\alpha^{v-m-1}, x_0\alpha^{v-m} = x_q$ . In particular,  $\gamma$  maps the initial segment  $(j_1 \dots j_m)$  of  $\xi$  onto a terminal segment of  $(i_1 \dots i_u)$ , say  $(i_r \dots i_u)$ . Let  $\mu = (m_1 \dots m_d x_c)$  be any cilium in  $\lambda$  whose span does not contain  $j_m$ . Let  $a \in \{0, \dots, k-1\}$  be such that  $a \equiv q - s \pmod{k}$ . Select  $x_h \in \text{dom } \rho$  so that  $x_h\alpha^d = x_{c+a}$  and define  $\gamma$  so that it maps the points  $m_1, m_2, \dots, m_d, x_c$  of  $\text{span } \mu$  to  $x_h, x_h\alpha, \dots, x_h\alpha^{d-1}, x_h\alpha^d = x_{c+a}$ .

By Theorem 1 and the construction of  $\gamma$ , we have  $\gamma \in C(\alpha)$ ,  $j_1\gamma = i_r$ , and  $j_{w+1}\gamma = i_{r+w}$ . Since  $j_1(\beta \circ \gamma) = j_{w+1}\gamma = i_{r+w}$  and  $j_1(\gamma \circ \beta) = i_r\beta = i_r\alpha^t \neq i_{r+w}$  (since  $t > w$ ),  $\gamma \notin C(\beta)$ , which is a contradiction.

*Case 3.2.*  $b \geq u$ .

Then  $\ell(\tau) = b + v - m + 2 \geq u + v - m + 2 \geq m + v - m + 2 = v + 2 > \ell(\xi)$  and  $\ell(\tau) = b + v - m + 2 \geq u + v - m + 2 \geq u + m - m + 2 = u + 2 > \ell(\eta)$ . By Lemma 6,  $\beta|_{\text{span } \tau} = \alpha^p|_{\text{span } \tau}$  for some  $p \in \{0, \dots, \ell(\tau) + k - 2\}$ . Suppose  $p > w$ . Then, by the inductive hypothesis applied to  $\tau$  and  $\xi$ ,  $\beta|_{\text{span } \xi} = \alpha^p|_{\text{span } \xi}$ . It follows that  $j_1\beta = j_1\alpha^p \neq j_{w+1} = j_1\beta$ , which is a contradiction. Thus  $p \leq w$ . Then  $t > p$  and so, by the inductive hypothesis applied to  $\eta$  and  $\tau$ ,  $\beta|_{\text{span } \tau} = \alpha^t|_{\text{span } \tau}$ . Note that, since  $\ell(\tau) > \ell(\eta)$  and  $t \in \{0, \dots, \ell(\eta) + k - 2\}$ , we also have  $t \in \{0, \dots, \ell(\tau) + k - 2\}$ . Now repeat the argument used in the case  $p > w$  above (with  $p$  replaced by  $t$ ) to obtain a contradiction. This concludes the proof.  $\square$

Lemmas 3–7 imply that if  $\alpha \in PT_n$  and  $\beta \in C^2(\alpha)$ , then  $\beta$  satisfies (1) and (2) of Theorem 2, that is,  $\beta|_N = \alpha^t|_N$  and  $\beta|_{\text{dom } \lambda_i} = \alpha^{t_i}|_{\text{dom } \lambda_i}$  for some  $t \in \mathbb{N}$  and

$t_i \in \mathbb{N} \cup \{-\infty\}$  ( $i = 1, \dots, m$ ). The next two lemmas show that the exponents  $t$  and  $t_i$  satisfy (3) of Theorem 2.

**Lemma 8.** *Let  $\alpha, \beta \in PT_n$  be such that  $\beta \in C^2(\alpha)$ , let  $\eta = (i_1 \dots i_u]$  be a maximal chain in  $\alpha$ , let  $\varrho = (x_0 \dots x_{k-1})$  be a circuit in  $\alpha$ , and let  $\xi = (j_1 \dots j_v x_0)$  be a longest cilium attached to  $\varrho$ . Suppose that  $t$  is a nonnegative integer such that  $\beta| \text{span } \eta = \alpha^t| \text{span } \eta$ . If  $t < \min\{u, v\}$ , then  $\text{dom } \beta \cap \text{span } \xi \neq \emptyset$ .*

*Proof.* Let  $t < \min\{u, v\}$ . Suppose, by way of contradiction, that  $\text{dom } \beta \cap \text{span } \xi = \emptyset$ . Let  $m = \min\{u, v\}$ . We will construct  $\gamma \in C(\alpha)$  that maps the initial segment  $(j_1 \dots j_m]$  of  $\xi$  onto a terminal segment of  $\eta$ . Let  $\lambda$  be the cell in  $\alpha$  that has  $\varrho$  as the circuit. Set  $\text{dom } \gamma = \{x \in \text{dom } \lambda: x\alpha^q = j_m \text{ for some } q \geq 0\}$ . Let  $\mu = (m_1 \dots m_d j_m \dots j_v x_0)$  ( $d \geq 0$ ) be a cilium in  $\lambda$  whose span contains  $j_m$ . Since  $\ell(\mu) \leq \ell(\xi)$ ,  $d+1 \leq m \leq u$ . Thus we can define  $\gamma$  so that it maps the initial segment  $(m_1 \dots m_d j_m]$  of  $\mu$  onto a terminal segment of  $\eta$ . In particular,  $\gamma$  maps the initial segment  $(j_1 \dots j_m]$  of  $\xi$  onto a terminal segment of  $\eta$ , say  $(i_r \dots i_u]$ .

By Theorem 1 and the construction of  $\gamma$ , we have  $\gamma \in C(\alpha)$  and  $j_1 \gamma = i_r$ . Since  $j_1(\beta \circ \gamma)$  is undefined (since  $j_1 \notin \text{dom } \beta$ ) and  $j_1(\gamma \circ \beta) = i_r \beta = i_r \alpha^t = i_{r+t}$  (since  $t < m$ ),  $\gamma \notin C(\beta)$ , which is a contradiction. Thus  $\text{dom } \beta \cap \text{span } \xi \neq \emptyset$ .  $\square$

**Lemma 9.** *Let  $\alpha, \beta \in PT_n$  be such that  $\beta \in C^2(\alpha)$ , let  $\eta = (i_1 \dots i_u]$  be a maximal chain in  $\alpha$ , let  $\varrho = (x_0 \dots x_{k-1})$  be a circuit in  $\alpha$ , and let  $\xi = (j_1 \dots j_v x_0)$  be a longest cilium attached to  $\varrho$ . Suppose that  $t$  and  $w$  are nonnegative integers such that  $\beta| \text{span } \eta = \alpha^t| \text{span } \eta$  and  $\beta| \text{span } \xi = \alpha^w| \text{span } \xi$ . If either  $t$  or  $w$  is less than  $\min\{u, v\}$ , then  $t = w$ .*

*Proof.* Let  $m = \min\{u, v\}$ . Let  $\lambda$  be the cell in  $\alpha$  that has  $\varrho$  as the circuit. As in the proof of Lemma 8, we can construct  $\gamma \in C(\alpha)$  such that  $\text{dom } \gamma = \{x \in \text{dom } \lambda: x\alpha^q = j_m \text{ for some } q \geq 0\}$  and  $\gamma$  maps the initial segment  $(j_1 \dots j_m]$  of  $\xi$  onto a terminal segment of  $\eta$ , say  $(i_r \dots i_u]$ . Then  $j_1(\beta \circ \gamma) = (j_1 \alpha^w) \gamma$  and  $j_1(\gamma \circ \beta) = i_r \beta = i_r \alpha^t$ . Since  $\gamma \in C(\beta)$ ,  $(j_1 \alpha^w) \gamma = i_r \alpha^t$ .

Suppose  $t < m$ . Then  $i_r \alpha^t$  is defined and  $i_r \alpha^t = i_{r+t}$ . Thus  $w$  must be less than  $m$  (otherwise  $j_1 \alpha^w$  would not be in  $\text{dom } \gamma$ ) and so  $(j_1 \alpha^w) \gamma = j_{w+1} \gamma = i_{r+w}$ . Hence  $i_{r+t} = i_{r+w}$  and so  $t = w$ .

Suppose  $w < m$ . Then  $(j_1 \alpha^w) \gamma$  is defined and  $(j_1 \alpha^w) \gamma = j_{w+1} \gamma = i_{r+w}$ . Thus  $t$  must be less than  $m$  (otherwise  $i_r \alpha^t$  would be undefined) and so  $i_r \alpha^t = i_{r+t}$ . Hence  $i_{r+t} = i_{r+w}$  and so  $t = w$ .  $\square$

We already proved (Lemmas 5–7) that if  $\alpha \in PT_n$  and  $\beta \in C^2(\alpha)$ , then  $\beta$  satisfies (2) of Theorem 2, that is,  $\beta| \text{dom } \lambda_i = \alpha^{t_i}| \text{dom } \lambda_i$  for some  $t_i \in \mathbb{N} \cup \{-\infty\}$  ( $i = 1, \dots, m$ ). The next three lemmas show that the exponents  $t_i$  satisfy (4) of Theorem 2.

**Lemma 10.** *Let  $\alpha, \beta \in PT_n$  be such that  $\beta \in C^2(\alpha)$ , and let  $\varrho = (x_0 \dots x_{k-1})$  and  $\delta = (y_0 \dots y_{m-1})$  be circuits in  $\alpha$  such that  $k$  divides  $m$ . Then  $\text{dom } \beta \cap \text{dom } \varrho = \emptyset$  if and only if  $\text{dom } \beta \cap \text{dom } \delta = \emptyset$ .*

**P r o o f.** We will construct  $\gamma \in C(\alpha)$  such that  $y_0\gamma = x_0$ . Let  $\lambda$  be the cell in  $\alpha$  that has  $\delta$  as the circuit. Set  $\text{dom } \gamma = \text{dom } \lambda$ . Define  $\gamma$  so that it maps the points  $y_0, y_1, \dots, y_{m-1}$  of  $\text{dom } \delta$  to  $x_0, x_0\alpha, \dots, x_0\alpha^{m-1}$ . Let  $\xi = (j_1 \dots j_v y_p)$  be any cilium in  $\alpha$  attached to  $\delta$ . Select  $x_h \in \text{dom } \varrho$  so that  $x_h\alpha^v = x_0\alpha^p$  and define  $\gamma$  so that it maps the points  $j_1, j_2, \dots, j_v, y_p$  of  $\text{span } \xi$  to  $x_h, x_h\alpha, \dots, x_h\alpha^{v-1}, x_h\alpha^v$ .

By Theorem 1 and the construction of  $\gamma$ , we have  $\gamma \in C(\alpha)$  and  $y_0\gamma = x_0$ . Suppose  $\text{dom } \beta \cap \text{dom } \varrho = \emptyset$  and  $\text{dom } \beta \cap \text{dom } \delta \neq \emptyset$ . Then  $y_0 \in \text{dom } \beta$  and  $y_0\beta \in \text{dom } \delta$  (by Lemma 5). Thus  $y_0(\beta \circ \gamma)$  is defined and  $y_0(\gamma \circ \beta) = x_0\beta$  is undefined. Suppose  $\text{dom } \beta \cap \text{dom } \delta = \emptyset$  and  $\text{dom } \beta \cap \text{dom } \varrho \neq \emptyset$ . Then  $x_0 \in \text{dom } \beta$  (by Theorem 1). Thus  $y_0(\beta \circ \gamma)$  is undefined and  $y_0(\gamma \circ \beta) = x_0\beta$  is defined. In either case,  $\gamma \notin C(\beta)$ , which is a contradiction. The result follows.  $\square$

**Lemma 11.** *Let  $\alpha, \beta \in PT_n$  be such that  $\beta \in C^2(\alpha)$ , and let  $\varrho = (x_0 \dots x_{k-1})$  and  $\delta = (y_0 \dots y_{m-1})$  be circuits in  $\alpha$  such that  $k$  divides  $m$ . Suppose that  $t$  and  $w$  are nonnegative integers such that  $\beta \mid \text{dom } \varrho = \alpha^t \mid \text{dom } \varrho$  and  $\beta \mid \text{dom } \delta = \alpha^w \mid \text{dom } \delta$ . Then  $w \equiv t \pmod{k}$ .*

**P r o o f.** Let  $t' \in \{0, 1, \dots, k-1\}$  and  $w' \in \{0, 1, \dots, m-1\}$  be such that  $t' \equiv t \pmod{k}$  and  $w' \equiv w \pmod{m}$ . Note that  $w' \equiv w \pmod{k}$  (since  $k$  divides  $m$ ),  $x_0\beta = x_{t'}$ , and  $y_0\beta = y_{w'}$ . As in the proof of Lemma 10, we can construct  $\gamma \in C(\alpha)$  that maps  $y_0, y_1, \dots, y_{m-1}$  to  $x_0, x_0\alpha, \dots, x_0\alpha^{m-1}$ . Note that  $y_{w'}\gamma = x_{w''}$ , where  $w'' \in \{0, 1, \dots, k-1\}$  and  $w'' \equiv w' \pmod{k}$ . On the other hand,  $y_{w'}\gamma = y_0(\beta \circ \gamma) = y_0(\gamma \circ \beta) = x_0\beta = x_{t'}$ . Hence  $t' = w''$  and so  $t \equiv t' = w'' \equiv w' \equiv w \pmod{k}$ .  $\square$

**Lemma 12.** *Let  $\alpha, \beta \in PT_n$  be such that  $\beta \in C^2(\alpha)$ , let  $\varrho = (x_0 \dots x_{k-1})$  and  $\delta = (y_0 \dots y_{l-1})$  be circuits in  $\alpha$  such that  $k$  divides  $l$ , let  $\eta = (i_1 \dots i_u x_0)$  be a cilium attached to  $\varrho$ , and let  $\xi = (j_1 \dots j_v y_0)$  be a longest cilium attached to  $\delta$ . Suppose that  $t$  and  $w$  are nonnegative integers such that  $\beta \mid \text{span } \eta = \alpha^t \mid \text{span } \eta$  and  $\beta \mid \text{span } \xi = \alpha^w \mid \text{span } \xi$ . If either  $t < \min\{u, v\}$  or  $w < \min\{u, v\}$ , then  $w = t$ .*

**P r o o f.** Let  $m = \min\{u, v\}$ . We will construct  $\gamma \in C(\alpha)$  that maps the initial segment  $(j_1 \dots j_m)$  of  $\xi$  onto a terminal segment of  $(i_1 \dots i_u)$ . Set  $\text{dom } \gamma = \text{dom } \lambda$ , where  $\lambda$  is the cell in  $\alpha$  that has  $\delta$  as the circuit. Let  $q \in \{0, \dots, k-1\}$  be such that  $q \equiv v - m \pmod{k}$ .

Let  $\mu = (m_1 \dots m_d j_m \dots j_v y_0)$  ( $d \geq 0$ ) be any cilium in  $\lambda$  whose span contains  $j_m$ . Since  $\ell(\xi) \geq \ell(\mu)$ ,  $m-1 \geq d$  and so  $u \geq m \geq d+1$ . Thus we can define  $\gamma$  so that it maps the initial segment  $(m_1 \dots m_d j_m)$  of  $\mu$  onto a terminal segment of  $(i_1 \dots i_u)$

and the remaining points of span  $\mu, j_{m+1}, j_{m+2}, \dots, j_v, y_0$ , to  $x_0, x_0\alpha, \dots, x_0\alpha^{v-m-1}, x_0\alpha^{v-m} = x_q$ . In particular,  $\gamma$  maps the initial segment  $(j_1 \dots j_m]$  of  $\xi$  onto a terminal segment of  $(i_1 \dots i_u]$ , say  $(i_r \dots i_u]$ . Let  $\mu = (m_1 \dots m_d y_s)$  be any cilium in  $\lambda$  whose span does not contain  $j_m$ . Select  $a \in \{0, \dots, k-1\}$  such that  $a \equiv q+s \pmod{k}$  and  $x_h \in \text{dom } \varrho$  such that  $x_h\alpha^d = x_a$ . Define  $\gamma$  so that it maps the points  $m_1, m_2, \dots, m_d, y_s$  of span  $\mu$  to the points  $x_h, x_h\alpha, \dots, x_h\alpha^{d-1}, x_h\alpha^d = x_a$ .

By Theorem 1 and the construction of  $\gamma, \gamma \in C(\alpha)$  and it maps  $(j_1 \dots j_m]$  onto  $(i_r \dots i_u]$ . (Note that this implies  $u-r = m-1$  and so  $r+m-1 = u$ .) Then  $j_1(\beta \circ \gamma) = (j_1\alpha^w)\gamma$  and  $j_1(\gamma \circ \beta) = i_r\beta = i_r\alpha^t$ . Since  $\gamma \in C(\beta), (j_1\alpha^w)\gamma = i_r\alpha^t$ .

Suppose  $t < m$ . Then  $r+t \leq r+m-1 = u$  and so  $i_r\alpha^t = i_{r+t}$ . Thus  $w$  must be less than  $m$  (otherwise, by the construction of  $\gamma, (j_1\alpha^w)\gamma$  would be in  $\text{dom } \varrho$  and so it could not be equal to  $i_{r+t}$ ) and so  $(j_1\alpha^w)\gamma = j_{w+1}\gamma = i_{r+w}$ . Hence  $i_{r+t} = i_{r+w}$  and so  $t = w$ .

Suppose  $w < m$ . Then  $(j_1\alpha^w)\gamma = j_{w+1}\gamma = i_{r+w}$ . Thus  $t$  must be less than  $m$  (otherwise  $i_r\alpha^t$  would be in  $\text{dom } \varrho$  and so it could not be equal to  $i_{r+w}$ ) and so  $i_r\alpha^t = i_{r+t}$ . Hence  $i_{r+t} = i_{r+w}$  and so  $t = w$ .  $\square$

Now we are in a position to prove Theorem 2.

**P r o o f** of Theorem 2. Suppose that  $\beta \in C^2(\alpha)$ . Suppose that  $k \geq 1$ , that is,  $\alpha$  has at least one maximal chain. Let  $i \in \{1, \dots, k\}$ . By Lemma 3,  $\beta| \text{span } \eta_i = \alpha^{w_i}| \text{span } \eta_i$  for some  $w_i \in \{0, \dots, \ell(\eta_i)\}$ . Let  $t = \max\{w_1, \dots, w_k\}$ . By Lemma 4,  $\beta| \text{span } \eta_i = \alpha^t| \text{span } \eta_i$  for each  $i \in \{1, \dots, k\}$ . Since  $N = \text{span } \eta_1 \cup \dots \cup \text{span } \eta_k, \beta|N = \alpha^t|N$ . If  $k = 0$ , that is,  $N = \emptyset$ , then  $\beta|N = \alpha^0|N$ . Thus, in any case, there is an integer  $t$  that satisfies condition (1).

Let  $i \in \{1, \dots, m\}$ . By Lemma 5,  $\beta| \text{dom } \varrho_i = \alpha^w| \text{dom } \varrho_i$  for some  $w \in \{0, \dots, \ell(\varrho_i) - 1\} \cup \{-\infty\}$ . If  $\lambda_i = \varrho_i$ , that is, if  $\varrho_i$  is an isolated circuit, take  $t_i = w$ . Suppose that  $\lambda_i \neq \varrho_i$ , that is,  $\lambda_i$  has at least one cilium. Let  $\eta_1, \dots, \eta_b$  be the cilia in  $\lambda_i$ . Suppose  $w = -\infty$ , that is,  $\text{dom } \beta \cap \text{dom } \varrho_i = \emptyset$ . Then  $\text{dom } \beta \cap \text{dom } \eta_p = \emptyset$  for each  $p \in \{1, \dots, b\}$  (by Lemma 6), and so  $\beta| \text{dom } \lambda_i = \alpha^w$ . Thus if  $w = -\infty$ , take  $t_i = w$ . Suppose  $w \neq -\infty$ . Then, by Lemma 6, for each  $p \in \{1, \dots, b\}$ , there is  $w_p \in \{0, \dots, \ell(\eta_p) + \ell(\varrho_i) - 2\}$  such that  $\beta| \text{span } \eta_p = \alpha^{w_p}| \text{span } \eta_p$ . Let  $t_i = \max\{w_1, \dots, w_b\}$ . By Lemma 7,  $\beta| \text{span } \eta_p = \alpha^{t_i}| \text{span } \eta_p$  for each  $p \in \{1, \dots, b\}$ . Let  $y$  be the point at which  $\eta_1$  meets  $\varrho_i$ . Then  $y\alpha^w = y\beta = y\alpha^{t_i}$ , which implies  $t_i \equiv w \pmod{\ell(\varrho_i)}$ . Let  $x \in \text{dom } \lambda_i$ . If  $x \in \text{span } \eta_p$  for some  $p \in \{1, \dots, b\}$ , then  $x\beta = x\alpha^{t_i}$ . If  $x \in \text{dom } \varrho_i$ , then  $x\beta = x\alpha^w = x\alpha^{t_i}$  (since  $t_i \equiv w \pmod{\ell(\varrho_i)}$ ). Since  $\text{dom } \lambda_i = \text{span } \eta_1 \cup \dots \cup \text{span } \eta_b \cup \text{dom } \varrho_i, \beta| \text{dom } \lambda_i = \alpha^{t_i}| \text{dom } \lambda_i$ . Thus for each  $i \in \{1, \dots, m\}$ , there is  $t_i \in \mathbb{N} \cup \{-\infty\}$  that satisfies condition (2). Moreover, it follows from Lemma 8 and Lemma 9 that for each  $i \in \{1, \dots, m\}, t$  and  $t_i$  satisfy condition (3), and it follows from Lemma 10, Lemma 11, and Lemma 12 that for all  $i, j \in \{1, \dots, m\}, t_i$  and  $t_j$  satisfy condition (4).

Conversely, suppose that there are  $t \in \mathbb{N}$  and  $t_1, \dots, t_m \in \mathbb{N} \cup \{-\infty\}$  such that conditions (1)–(4) are satisfied for all  $i, j \in \{1, \dots, m\}$ . Let  $\gamma \in C(\alpha)$ . We need to prove that  $\beta \circ \gamma = \gamma \circ \beta$ . Let  $x \in X$  and consider four cases.

*Case 1.*  $x \in N$  and  $x \in \text{dom}(\gamma \circ \beta)$ .

Then  $x \in \text{dom} \gamma$  and  $x\gamma \in \text{dom} \beta$ . Since  $x\gamma \in N$  (by Theorem 1),  $x\gamma \in \text{dom} \alpha^t$  (by (1)). Thus  $x \in \text{dom}(\gamma \circ \alpha^t)$  and so, since  $\gamma$  commutes with  $\alpha^t$ ,  $x \in \text{dom}(\alpha^t \circ \gamma)$ . Thus, since  $\beta|N = \alpha^t|N$ ,  $x \in \text{dom}(\beta \circ \gamma)$  and  $x(\beta \circ \gamma) = (x\beta)\gamma = (x\alpha^t)\gamma = x(\alpha^t \circ \gamma) = x(\gamma \circ \alpha^t) = (x\gamma)\alpha^t = (x\gamma)\beta = x(\gamma \circ \beta)$ .

*Case 2.*  $x \in N$  and  $x \in \text{dom}(\beta \circ \gamma)$ .

By an argument similar to that used in Case 1,  $x \in \text{dom}(\gamma \circ \beta)$  and  $x(\beta \circ \gamma) = x(\gamma \circ \beta)$ .

*Case 3.*  $x \in \text{dom} \lambda_j$  for some  $j \in \{1, \dots, m\}$  and  $x \in \text{dom}(\gamma \circ \beta)$ .

Then  $x \in \text{dom} \gamma$  and  $x\gamma \in \text{dom} \beta$ . By Theorem 1, one of the following two cases holds.

*Case 3.1.*  $x\gamma \in \text{dom} \lambda_i$  for some  $i \in \{1, \dots, m\}$ .

Then, by Theorem 1,  $\ell(\varrho_i)$  divides  $\ell(\varrho_j)$  and  $\text{dom} \lambda_j \subseteq \text{dom} \gamma$ . Since  $\beta| \text{dom} \lambda_i = \alpha^{t_i}| \text{dom} \lambda_i$  (by (2)) and  $x\gamma \in \text{dom} \beta$ ,  $t_i$  cannot be  $-\infty$ . Thus  $t_j \neq -\infty$  by (4b). It follows that  $x \in \text{dom}(\alpha^{t_j} \circ \gamma)$  and so, since  $\beta| \text{dom} \lambda_j = \alpha^{t_j}| \text{dom} \lambda_j$ ,  $x \in \text{dom}(\beta \circ \gamma)$ .

Since  $x\gamma \in \text{dom} \lambda_i$ ,  $x(\gamma \circ \beta) = (x\gamma)\beta = (x\gamma)\alpha^{t_i} \in \text{dom} \lambda_i$ . Since  $x \in \text{dom} \lambda_j$ ,  $x(\beta \circ \gamma) = (x\beta)\gamma = (x\alpha^{t_j})\gamma \in \text{dom} \lambda_i$ . Thus  $x\gamma$ ,  $x(\gamma \circ \beta)$ , and  $x(\beta \circ \gamma)$  are all in  $\text{dom} \lambda_i$ . Let  $\varrho_i = (x_0 \dots x_{a-1})$ , let  $\varrho_j = (y_0 \dots y_{b-1})$ , and consider two cases.

*Case 3.1.1.*  $x(\gamma \circ \beta) \in \text{dom} \varrho_i$ .

We claim that  $x(\beta \circ \gamma)$  is also in  $\text{dom} \varrho_i$ . Suppose, by way of contradiction, that  $x(\beta \circ \gamma) \notin \text{dom} \varrho_i$ . Then  $x \notin \text{dom} \varrho_j$  since otherwise  $x\beta = x\alpha^{t_j}$  would be in  $\text{dom} \varrho_j$  and so  $x(\beta \circ \gamma) = (x\beta)\gamma$  would be in  $\text{dom} \varrho_i$  (by Theorem 1). Thus there is a cilium  $\xi = (m_1 \dots m_v y_r)$  in  $\lambda_j$  such that  $x = m_p$  for some  $p \in \{1, \dots, v\}$ . We observed in the foregoing argument that  $m_p\beta = x\beta$  cannot be in  $\text{dom} \varrho_j$ . It follows that  $p + t_j \leq v$  and  $m_p\beta = m_p\alpha^{t_j} = m_{p+t_j}$ . Since  $p + t_j \leq v$ ,  $t_j \leq v - p < v$ . Since  $m_{p+t_j}\gamma = (m_p\beta)\gamma = m_p(\beta \circ \gamma) = x(\beta \circ \gamma) \notin \text{dom} \varrho_i$ , it follows by Theorem 1 that there is a cilium  $\eta = (k_1 \dots k_u x_s)$  in  $\lambda_i$  such that for some  $q \in \{1, \dots, u\}$ ,  $m_p\gamma = k_q$ ,  $q + t_j \leq u$ , and  $m_{p+t_j}\gamma = k_{q+t_j}$ . Since  $q + t_j \leq u$ ,  $t_j \leq u - q < u$ . Hence  $t_j < \min\{u, v\} \leq \min\{r(\lambda_i), r(\lambda_j)\}$  and so  $t_i = t_j$  by (4b). But then  $x(\gamma \circ \beta) = (m_p\gamma)\beta = k_q\beta = k_q\alpha^{t_i} = k_q\alpha^{t_j} = k_{q+t_j} \notin \text{dom} \varrho_i$ , which is a contradiction.

Thus both  $x(\gamma \circ \beta)$  and  $x(\beta \circ \gamma)$  are in  $\text{dom} \varrho_i$  and so  $x(\gamma \circ \beta) = x_p$  and  $x(\beta \circ \gamma) = x_q$  for some  $p, q \in \{0, \dots, a-1\}$ . By (4a),  $t_i \equiv t_j \pmod{a}$  and so there is an integer  $l \geq 0$  such that either  $t_i = t_j + lk$  or  $t_j = t_i + lk$ . In the former case, we have:

$$\begin{aligned} x_p &= x(\gamma \circ \beta) = (x\gamma)\beta = (x\gamma)\alpha^{t_i} = x(\gamma \circ \alpha^{t_j} \circ \alpha^{lk}) = x(\alpha^{t_j} \circ \gamma \circ \alpha^{lk}) \\ &= (x\alpha^{t_j})(\gamma \circ \alpha^{lk}) = (x\beta)(\gamma \circ \alpha^{lk}) = (x(\beta \circ \gamma))\alpha^{lk} = x_q\alpha^{lk} = x_q. \end{aligned}$$

And in the latter case, we have:

$$\begin{aligned} x_q &= x(\beta \circ \gamma) = (x\beta)\gamma = (x\alpha^{t_j})\gamma = x(\alpha^{t_j} \circ \gamma) = x(\gamma \circ \alpha^{t_j}) = x(\gamma \circ \alpha^{t_i} \circ \alpha^{lk}) \\ &= ((x\gamma)\alpha^{t_i})\alpha^{lk} = ((x\gamma)\beta)\alpha^{lk} = (x(\gamma \circ \beta))\alpha^{lk} = x_p\alpha^{lk} = x_p. \end{aligned}$$

Thus  $x(\gamma \circ \beta) = x_p = x_q = x(\beta \circ \gamma)$ .

*Case 3.1.2.*  $x(\gamma \circ \beta) \notin \text{dom } \varrho_i$ .

Then  $x \notin \text{dom } \varrho_j$  since otherwise  $x\gamma$  would be in  $\text{dom } \varrho_i$  (by Theorem 1) and so  $x(\gamma \circ \beta) = (x\gamma)\beta = (x\gamma)\alpha^{t_i}$  would also be in  $\text{dom } \varrho_i$ . Thus there is a cilium  $\xi = (m_1 \dots m_v y_r)$  in  $\lambda_j$  such that  $x = m_p$  for some  $p \in \{1, \dots, v\}$ . We observed in the foregoing argument that  $x\gamma$  cannot be in  $\text{dom } \varrho_i$ . It follows that there is a cilium  $\eta = (k_1 \dots k_u x_s)$  in  $\lambda_i$  such that  $x\gamma = m_p\gamma = k_q$  for some  $q \in \{1, \dots, u\}$ . Since  $k_q\alpha^{t_i} = k_q\beta = (m_p\gamma)\beta = m_p(\gamma \circ \beta) \notin \text{dom } \varrho_i$ , we must have  $q + t_i \leq u$  and  $m_p(\gamma \circ \beta) = k_q\alpha^{t_i} = k_{q+t_i}$ . Since  $q + t_i \leq u$ ,  $t_i \leq u - q < u$ . Since (by Theorem 1) either  $m_v\gamma = k_u$  or  $m_v\gamma \in \text{dom } \varrho_i$ , the fact that  $m_p\gamma = k_q$  coupled with Theorem 1 implies that  $u - q \leq v - p$ . Thus  $t_i \leq u - q \leq v - p < v$ . Hence  $t_i < \min\{u, v\}$  and so  $t_i = t_j$  by (4b). Thus

$$\begin{aligned} x(\beta \circ \gamma) &= (x\beta)\gamma = (x\alpha^{t_j})\gamma = x(\alpha^{t_j} \circ \gamma) = x(\gamma \circ \alpha^{t_j}) = (x\gamma)\alpha^{t_j} = (x\gamma)\alpha^{t_i} \\ &= (x\gamma)\beta = x(\gamma \circ \beta). \end{aligned}$$

*Case 3.2.*  $x\gamma \in N$ .

Then, by Theorem 1, there is a cilium  $\xi = (m_1 \dots m_v y_r)$  in  $\lambda_j$  and a maximal chain  $\eta_i = (k_1 \dots k_u)$  in  $\alpha$  such that for some  $p \in \{1, \dots, v\}$ ,  $x = m_p$  and  $\gamma$  maps an initial segment  $(m_1 \dots m_p \dots)$  of  $(m_1 \dots m_v)$  onto a terminal segment of  $\eta_i$ . Let  $m_p\gamma = k_q$  ( $q \in \{1, \dots, u\}$ ). Since  $k_q = m_p\gamma = x\gamma \in \text{dom } \beta$  and  $\beta|N = \alpha^t|N$ ,  $k_q \in \text{dom } \alpha^t$ , which implies  $q + t \leq u$  and  $k_q\beta = k_q\alpha^t = k_{q+t}$ . Since  $\gamma$  maps an initial segment of  $(m_1 \dots m_v)$  onto a terminal segment of  $(k_1 \dots k_u)$ ,  $m_p\gamma = k_q$  and  $q + t \leq u$  imply that  $p + t \leq v$  and  $m_{p+t} \in \text{dom } \gamma$ . Thus  $t < \min\{u, v\} \leq \min\{d(N), r(\lambda_j)\}$  and so  $t = t_j$  (by (3)). Hence  $x \in \text{dom } \beta$  (since  $t_j = t \geq 0$  and  $\beta| \text{dom } \lambda_j = \alpha^{t_j}| \text{dom } \lambda_j$ ) and  $x\beta = m_p\beta = m_p\alpha^{t_j} = m_p\alpha^t = m_{p+t} \in \text{dom } \gamma$ . Thus  $x \in \text{dom}(\beta \circ \gamma)$  and, since  $\gamma$  commutes with  $\alpha^t$ ,  $x(\beta \circ \gamma) = (x\beta)\gamma = (x\alpha^{t_j})\gamma = (x\alpha^t)\gamma = x(\alpha^t \circ \gamma) = x(\gamma \circ \alpha^t) = (x\gamma)\alpha^t = (x\gamma)\beta = x(\gamma \circ \beta)$ .

*Case 4.*  $x \in \text{dom } \lambda_j$  for some  $j \in \{1, \dots, m\}$  and  $x \in \text{dom}(\beta \circ \gamma)$ .

Then  $x \in \text{dom } \beta$  and  $y = x\beta \in \text{dom } \gamma$ . Since, by (2),  $\beta| \text{dom } \lambda_j = \alpha^{t_j}| \text{dom } \lambda_j$ ,  $t_j \geq 0$  and  $y \in \text{dom } \lambda_j$ . By Theorem 1, one of the following two cases holds.

*Case 4.1.*  $y\gamma \in \text{dom } \lambda_i$  for some  $i \in \{1, \dots, m\}$ .

Then, by Theorem 1,  $\ell(\varrho_i)$  divides  $\ell(\varrho_j)$ ,  $\text{dom } \lambda_j \subseteq \text{dom } \gamma$ , and  $x\gamma \in \text{dom } \lambda_i$ . Since  $t_j \neq -\infty$ ,  $t_i \neq -\infty$  by (4b). Thus, since  $\beta| \text{dom } \lambda_i = \alpha^{t_i}| \text{dom } \lambda_i$ ,  $\text{dom } \lambda_i \subseteq \text{dom } \beta$ .



Hence  $x \in \text{dom } \lambda_j \subseteq \text{dom } \gamma$  and  $x\gamma \in \text{dom } \lambda_i \subseteq \text{dom } \beta$ , which implies  $x \in \text{dom}(\gamma \circ \beta)$ . It follows by Case 3 that  $x(\beta \circ \gamma) = x(\gamma \circ \beta)$ .

*Case 4.2.*  $y\gamma \in N$ .

Then, by Theorem 1,  $y \notin \text{dom } \varrho_j$ . Thus, since  $y = x\beta = x\alpha^{t_j}$ , there is a cilium  $\xi = (m_1 \dots m_v y_r)$  in  $\lambda_j$  such that for some  $p \in \{1, \dots, v\}$ ,  $y = m_p$ ,  $p - t_j \geq 1$ , and  $x = m_{p-t_j}$ . Since  $y\gamma \in N$ , it follows by Theorem 1 that there is a maximal chain  $\eta_i = (k_1 \dots k_u)$  in  $\alpha$  such that  $\gamma$  maps an initial segment  $(m_1 \dots m_{p-t_j} \dots m_p \dots)$  of  $(m_1 \dots m_v)$  onto a terminal segment of  $\eta_i$ . Let  $m_p\gamma = k_q$  ( $q \in \{1, \dots, u\}$ ). Then, since  $\gamma$  maps an initial segment of  $(m_1 \dots m_v)$  onto a terminal segment of  $\eta_i$ ,  $q - t_j \geq 1$  and  $m_{p-t_j}\gamma = k_{q-t_j}$ . Since  $q - t_j \geq 1$  and  $p - t_j \geq 1$ ,  $t_j < \min\{q, p\} \leq \min\{u, v\} \leq \min\{d(N), r(\lambda_j)\}$ . Thus, by (3),  $t_j = t$  and so  $k_{q-t_j} = k_{q-t} \in \text{dom } \beta$  (since  $k_{q-t}\alpha^t = k_q$  and  $\text{dom } \beta|N = \text{dom } \alpha^t|N$ ). Hence  $x = m_{p-t_j} \in \text{dom } \gamma$  and  $x\gamma = m_{p-t_j}\gamma = k_{q-t_j} \in \text{dom } \beta$ , which implies  $x \in \text{dom}(\gamma \circ \beta)$ . It follows by Case 3 that  $x(\beta \circ \gamma) = x(\gamma \circ \beta)$ .

This concludes the proof. □

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