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ON THE GENERALIZED DRAZIN INVERSE AND GENERALIZED RESOLVENT

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Abstract. We investigate the generalized Drazin inverse and the generalized resolvent in Banach algebras. The Laurent expansion of the generalized resolvent in Banach algebras is introduced. The Drazin index of a Banach algebra element is characterized in terms of the existence of a particularly chosen limit process. As an application, the computing of the Moore-Penrose inverse in C^*-algebras is considered. We investigate the generalized Drazin inverse as an outer inverse with prescribed range and kernel. Also, 2×2 operator matrices are considered. As corollaries, we get some well-known results.

Keywords: Drazin inverse, generalized resolvent, limit processes, outer inverses, operator matrices

MSC 2000: 47A05, 47A10, 46L05

0. Introduction

Let $\mathscr{A}$ be a Banach algebra with the unit 1. Recall that an element $b \in \mathscr{A}$ is the Drazin inverse of $a \in \mathscr{A}$ provided that

$$a^{k+1}b = a^k, \quad bab = b, \quad ab = ba$$

holds for some nonnegative integer k. The least k in the previous definition is called the Drazin index of a, and will be denoted by $\operatorname{ind}(a)$. If a has the Drazin inverse, then the Drazin inverse of a is unique and is denoted by a^D. It is well-known that $a \in \mathscr{A}$ has the Drazin inverse if and only if the point $\lambda = 0$ is a pole of the resolvent $\lambda \mapsto (\lambda - a)^{-1}$. The order of this pole is equal to $\operatorname{ind}(a)$. Particularly, it follows that 0 is not the point of accumulation of the spectrum $\sigma(a)$.

The Drazin inverse is investigated in the matrix theory [2, 3, 17, 26, 27], in the ring theory [10, 11, 12]. In [5, 20] the Drazin inverse for bounded linear operators on complex Banach spaces is investigated.

Recently, Koliha introduced the concept of a generalized Drazin inverse [15]. The generalized Drazin inverse of an element $a \in \mathscr{A}$ exists if and only if $0 \notin \operatorname{acc}\sigma(a)$ and is described as follows. If $0 \notin \operatorname{acc}\sigma(a)$, then there exist open subsets U and V of $\mathbb{C}$, such that $0 \in U$, $\sigma(a) \setminus \{0\} \subset V$ and $U \cap V = \emptyset$. Define a function f in the following way:

$$f(\lambda) = \begin{cases} 0, & \lambda \in U, \\ 1/\lambda, & \lambda \in V. \end{cases}$$

The function f is regular in a neighbourhood of $\sigma(a)$. The generalized Drazin inverse of a is defined as $a^d = f(a)$. Notice that a^d is a double commutant of a. The generalized Drazin inverse retains some nontrivial nice properties of the ordinary Drazin inverse. For example, the continuity properties of the generalized and ordinary Drazin inverses are similar (see [16, 20]).

We mention that Harte also gave an alternative definition of a generalized Drazin inverse in a ring [10, 11, 12]. These two concepts are equivalent in the case when the ring is actually a Banach algebra.

On the other hand, in [21] Rose considered the Laurent expansion of the generalized resolvent $\lambda \mapsto (A + \lambda B)^{-1}$ for square matrices A and B, and found several useful applications.

The purpose of this paper is to introduce several results which connect the generalized Drazin inverse and the generalized resolvent of an element of a Banach algebra.

In Section 1 we introduce the Laurent expansion of the generalized resolvent using the generalized Drazin inverse. In Section 2 we characterize the Drazin index in terms of the existence of a particularly chosen limit process. As corollaries we get some well-known results of Koliha [15], Rose [21], Meyer [18], Rothblum [22], Ji [14]. In Section 3 we use the resolvent expansion to compute the Moore-Penrose inverse in C^*-algebras. In Section 4 we investigate outer inverses with prescribed range and kernel. In particular, we prove that the generalized Drazin inverse has similar properties. Finally, in Section 5 we give a brief generalization of the well-known result of Meyer and Rose [19] concerning the Drazin inverse of a block 2×2 upper triangular operator matrix. Also, we consider one special case with non-zero entries of a 2×2 operator matrix.

Let $\mathscr{A}$ be a Banach algebra and $a \in \mathscr{A}$. We use $\sigma(a)$ to denote the spectrum of a. If K is a compact subset of $\mathbb{C}$, then $\operatorname{acc} K$ and $\operatorname{iso} K$, respectively, denote the set of all accumulation points and isolated points of K. Also, $\mathscr{H}(K)$ denotes the set of all complex functions which are defined and regular in a neighbourhood of K.

Let $0 \notin \operatorname{acc}\sigma(a)$. If $p = p(a,0)$ is the spectral idempotent of a corresponding to 0, then ap is quasinilpotent, and $a(1-p)$ is invertible in the Banach algebra $(1-p)\mathscr{A}(1-p)$. Using the well-known properties of the functional calculus, it can be easily seen that a^d is equal to the ordinary inverse of $a(1-p)$ in $(1-p)\mathscr{A}(1-p)$, i.e. $a^d = [a(1-p)]^{-1}_{(1-p)\mathscr{A}(1-p)}$. We can write

$$a = ap + a(1-p) \tag{1}$$

and (1) is called the core-quasinilpotent decomposition of a. Also, $p = 1 - aa^d$. If a has the ordinary Drazin inverse, then $a^d = a^D$. In this case the core-quasinilpotent decomposition reduces to the well-known core-nilpotent decomposition.

Recall that $a \in \mathscr{A}$ is g-invertible provided there exists $b \in \mathscr{A}$, such that $aba = a$. In this case b is a g-inverse, or an inner inverse of a. On the other hand, b is an outer inverse of a, if $bab = b$. Notice that a^d (if it exists) is an outer inverse of a.

Various expressions and applications of the resolvent $\lambda \mapsto (\lambda - a)^{-1}$ are known in the literature (for example, see [22]). We shall generalize Rose's [21] and Koliha's [15] results.

Let $b \in \mathscr{A}$ be such that $ab = ba$ and $0 \notin \operatorname{acc}\sigma(a) \cup \operatorname{acc}\sigma(b)$. Then there exist generalized Drazin inverses of a and b, denoted by a^d and b^d, respectively.

We shall prove the following result, originally proved in [4] for complex matrices. Notice that the proof in [4] is essentially based on the fact $\operatorname{ind}(a) < \infty$.

Theorem 1.1. *Let $a, b \in \mathscr{A}$, $ab = ba$, $0 \notin \operatorname{acc}\sigma(a) \cup \operatorname{acc}\sigma(b)$ and let there exist $(a+\lambda b)^{-1}$ for some $\lambda \in \mathbb{C}$. Then $(1-aa^d)bb^d = (1-aa^d)$.*

Proof. Let $p_a = p(a,0)$ and $p_b = p(b,0)$ denote, respectively, the spectral idempotents of a and b corresponding to the point $z = 0$. Then $1 - aa^d = p_a$ and $bb^d = 1 - p_b$. We have to prove $p_a(1-p_b) = p_a$, or, equivalently, $p_a p_b = 0$.

If a is invertible, then $p_a = 0$, so the statement of Theorem 1.1 holds.

Suppose that a is not invertible. Then $(\lambda b + a)$ is invertible for some $\lambda \in \mathbb{C} \setminus \{0\}$. Since $(\lambda b)^d = \lambda^{-1} b^d$, we may assume that $\lambda = -1$, so let $(a-b)$ be invertible. We have to prove

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma (z-a)^{-1}\,\mathrm{d}z \; \frac{1}{2\pi i} \int_\gamma (u-b)^{-1}\,\mathrm{d}u = 0$$

for a suitably chosen contour γ around the point $z = 0$. Notice that

$$(z-a)^{-1} - (u-b)^{-1} = [(u-z)+(a-b)][(z-a)^{-1}(u-b)^{-1}].$$

Since $a-b$ is invertible, it follows that $(u-z)+(a-b)$ is invertible for small values of u and z. We may take γ such that $(u-z)+(a-b)$ is invertible for all $z, u \in \gamma$. Consider the function

$$F(z,u) = [(u-z)+(a-b)]^{-1}[(z-a)^{-1} - (u-b)^{-1}] = (z-a)^{-1}(u-b)^{-1},$$

which is continuous on the set $\gamma \times \gamma$, so the order of integration may be reversed if necessary. Obviously,

$$\left(\frac{1}{2\pi \mathrm{i}}\right)^2 \iint_{\gamma\times\gamma} F(z,u)\,\mathrm{d}z\,\mathrm{d}u = p_a p_b.$$

On the other hand, the functions $z \mapsto [(u-z)+(a-b)]^{-1}(u-b)^{-1}$ and $u \mapsto [(u-z)+(a-b)]^{-1}(z-a)^{-1}$ are regular in a neighbourhood of 0, so

$$\begin{aligned}\left(\frac{1}{2\pi \mathrm{i}}\right)^2 &\iint_{\gamma\times\gamma} F(z,u)\,\mathrm{d}z\,\mathrm{d}u = \\ &= \frac{1}{2\pi i}\int_\gamma \left[\frac{1}{2\pi \mathrm{i}}\int_\gamma [(u-z)+(a-b)]^{-1}(z-a)^{-1}\,\mathrm{d}u\right]\mathrm{d}z \\ &- \frac{1}{2\pi \mathrm{i}}\int_\gamma \left[\frac{1}{2\pi \mathrm{i}}\int_\gamma [(u-z)+(a-b)]^{-1}(u-b)^{-1}\,\mathrm{d}z\right]\mathrm{d}u = 0.\end{aligned}$$

It follows that $p_a p_b = 0$. $\square$

The analogous statement in [4] is formulated using the condition $\mathscr{N}(A) \cap \mathscr{N}(B) = \{0\}$, where $\mathscr{N}(A)$ denotes the kernel of a matrix A. However, if A and B are square matrices, then the following holds: $\mathscr{N}(A) \cap \mathscr{N}(B) = \{0\}$ if and only if there exists some $\lambda \in \mathbb{C}$ such that $A + \lambda B$ is invertible.

Using Theorem 1.1 we prove a generalization of the well-known results of Rose [21] and Koliha [15].

Theorem 1.2. *Suppose the conditions from Theorem* 1.1 *are satisfied. Then in a punctured neighbourhood of* $\lambda = 0$ *the following holds:*

$$(\lambda b - a)^{-1} = b^d(1 - aa^d)\sum_{n=1}^{\infty}(ab^d)^{n-1}\lambda^{-n} - a^d\sum_{n=0}^{\infty}(a^d b)^n\lambda^n.$$

P r o o f. The expansion $\sum\limits_{n=0}^{\infty}(a^d b)^n\lambda^n$ exists for small λ, since it represents the function $\lambda\mapsto(1-\lambda a^d b)^{-1}$. From $1-aa^d=p_a$ and $ab=ba$ we know that $ab^d p_a$ is quasinilpotent. The expansion $\sum\limits_{n=1}^{\infty}(ab^d)^{n-1}\lambda^{-n}(1-aa^d)$ exists for $|\lambda|>0$, since it represents the function $\lambda\mapsto(\lambda-ab^d p_a)^{-1}$.

Using Theorem 1.1 we obtain

$$\begin{aligned}
(\lambda b-a)&\Big[b^d(1-aa^d)\sum_{n=1}^{\infty}(ab^d)^{n-1}\lambda^{-n}-a^d\sum_{n=0}^{\infty}(a^d b)^n\lambda^n\Big]\\
&=bb^d(1-aa^d)\sum_{n=1}^{\infty}(ab^d)^{n-1}\lambda^{-n+1}-\sum_{n=0}^{\infty}(a^d b)^{n+1}\lambda^{n+1}\\
&\quad-(1-aa^d)\sum_{n=1}^{\infty}(ab^d)^n\lambda^{-n}+aa^d\sum_{n=0}^{\infty}(a^d b)^n\lambda^n\\
&=(1-aa^d)\Big[\sum_{n=0}^{\infty}(ab^d)^n\lambda^{-n}-\sum_{n=1}^{\infty}(ab^d)^n\lambda^{-n}\Big]\\
&\quad+aa^d+\sum_{n=1}^{\infty}a(a^d)^{n+1}b^n\lambda^n-\sum_{n=1}^{\infty}(a^d b)^n\lambda^n=1.
\end{aligned}$$

This completes the proof. □

If $\operatorname{ind}(a)<\infty$, we get Rose's expansion, established in [21] for complex square matrices. If $b=1$, we get Koliha's expansion in Banach algebras [15].

2. Limits and characterizations of the index

In this section we will consider various limit processes which are related with the generalized and ordinary Drazin inverses and the index of a Banach algebra element.

We state several algebraic results.

Lemma 2.1. *If $a,b,p\in\mathscr{A}$ are mutually commuting elements such that $p^2=p$, and a,b are invertible, then*

$$[ap+b(1-p)]^{-1}=a^{-1}p+b^{-1}(1-p)\quad\text{and}\quad[ap]^{-1}_{p\mathscr{A}p}=a^{-1}p.$$

Lemma 2.2. (a) *The number of elements of the set*

$$\big\{(i_1,\dots,i_l)\colon\ i_1,\dots,i_l\in\{1,\dots,n\},i_1+\dots+i_l=n\big\}$$

is equal to $\binom{n-1}{l-1}$, where $n\geqslant1$ and $1\leqslant l\leqslant n$ are arbitrary integers.

(b) *Let* $v(n,l)$ *denote the number of elements of the set*

$$\{(i_1,\dots,i_l)\colon\ i_1+\dots+i_l=n,\ i_1,\dots,i_l\in\{0,1,\dots,n\}\},$$

where $n\geqslant 0$ *and* $n+1\geqslant l\geqslant 1$. *Then*

$$v(n,l)=\begin{cases}1, & n=0,\\ \displaystyle\sum_{i=1}^{l}\binom{l}{i}\binom{n-1}{i-1}, & n>0.\end{cases}$$

The following result is a generalization of a large class of known results for matrices and very special elements of Banach algebras (see [17]). We frequently use $(a^s)^d=(a^d)^s$ for an arbitrary integer $s\geqslant 0$, which follows from the definition of the generalized Drazin inverse and from the well-known properties of the functional calculus.

Theorem 2.3. *Let* $a\in\mathscr{A}$, $0\notin\operatorname{acc}\sigma(a)$, *and let* s,l,t *be positive integers. Then*

$$\lim_{\lambda\to 0}(\lambda+a^s)^{-l}(a^d)^t=(a^d)^{sl+t}.$$

Proof. Let $p=p(a,0)$ be the spectral idempotent of a, corresponding to 0. Since $a^d=a^d(1-p)=[a(1-p)]^{-1}_{(1-p)\mathscr{A}(1-p)}$, using Lemma 2.1 we obtain

$$\begin{aligned}(\lambda+a^s)^{-l}(a^d)^t&=[(\lambda+a^s)^{-l}p+(\lambda+a^s)^{-l}(1-p)](a^d)^t(1-p)\\&=[(\lambda+a^s)(1-p)]^{-l}_{(1-p)\mathscr{A}(1-p)}(a^d)^t(1-p).\end{aligned}$$

Since the limit $\lim\limits_{\lambda\to 0}[(\lambda+a^s)(1-p)]^{-l}_{(1-p)\mathscr{A}(1-p)}$ exists and is equal to $((a^s)^d)^l=(a^d)^{sl}$, it follows that

$$\lim_{\lambda\to 0}(\lambda+a^s)^{-l}(a^d)^t=(a^d)^{sl+t}.$$

□

It is also possible to consider the limit of the type $\lim\limits_{\lambda\to 0}(\lambda+a^s)^{-l}(a^d)^t a^r$, $r\geqslant 0$, in Theorem 2.3, and repeat $a^daa^d=a^d$ several times, to get the known results for matrices [17].

We arrive at the main result of this section. The next theorem contains all known results for the limit expressions characterizing the Drazin index of an arbitrary square matrix. We will also use the expansion of the generalized resolvent $\lambda\mapsto(\lambda b-a)^{-1}$ from Theorems 1.2 and 1.1.

Theorem 2.4. *Let $a, b \in \mathscr{A}$ satisfy the conditions from Theorem* 1.1 *and let us consider the limit*

$$w = \lim_{\lambda \to 0} w(\lambda), \quad w(\lambda) = \lambda^m (\lambda b - a^s)^{-l} a^k, \quad m, k \geqslant 0,\ s, l > 0.$$

If $m < l$, then the limit $\lim_{\lambda \to 0} w(\lambda)$ exists if and only if $\operatorname{ind}(a) \leqslant k$. If $m \geqslant l$, then the limit $\lim_{\lambda \to 0} w(\lambda)$ exists if and only if $\operatorname{ind}(a) \leqslant s(m-l)+k+s$. In the case when $w = \lim_{\lambda \to 0} w(\lambda)$ exists, it is given in the following way:

$$w = \begin{cases} 0, & 0 < m < l \text{ and } \operatorname{ind}(a) \leqslant k; \\ 0, & m \geqslant l \text{ and } s(m-l)+k \geqslant \operatorname{ind}(a); \\ (-1)^l (a^D)^{sl} a^k, & m = 0 \text{ and } \operatorname{ind}(a) \leqslant k; \\ \binom{m-1}{l-1} a^{s(m-l)+k}(1-aa^D)(b^d)^m, & m \geqslant l \text{ and} \\ & \operatorname{ind}(a) - s \leqslant s(m-l)+k < \operatorname{ind}(a). \end{cases}$$

P r o o f. Obviously, $0 \notin \operatorname{acc} \sigma(a^s)$ for any integer $s \geqslant 1$. Theorem 1.2 yields that the following holds in a punctured neighbourhood of 0:

$$(\lambda b - a^s)^{-1} = \sum_{n=1}^{\infty} \lambda^{-n} (a^s b^d)^{n-1} b^d (1 - aa^d) - \sum_{n=0}^{\infty} \lambda^n [(a^s)^d]^{n+1} b^n.$$

We know that

$$(a^s)^{n-1}(1-aa^d)[(a^s)^d]^{j+1} = 0, \quad n \geqslant 1,\ j \geqslant 0. \tag{2}$$

Recall $v(n,l)$ from Lemma 2.2. For any integer $l \geqslant 1$, using Lemma 2.2 and (2), we conclude that

$$\begin{aligned}(\lambda b &- a^s)^{-l} \\ &= \sum_{n=l}^{\infty} \lambda^{-n} \binom{n-1}{l-1} (a^s b^d)^{n-l} (b^d)^l (1-aa^d) + (-1)^l \sum_{n=0}^{\infty} \lambda^n v(n,l) [(a^s)^d]^{n+l} b^n \\ &= \sum_{n=l}^{\infty} \lambda^{-n} \binom{n-1}{l-1} (a^s)^{n-l} (b^d)^n (1-aa^d) + (-1)^l \sum_{n=0}^{\infty} \lambda^n v(n,l) [(a^s)^d]^{n+l} b^n.\end{aligned}$$

Now, for arbitrary integers $m \geqslant 0$ and $k \geqslant 0$ we get

$$\begin{aligned} w(\lambda) = \sum_{n=l}^{\infty} \lambda^{m-n} \binom{n-1}{l-1} (a^s)^{n-l} a^k (b^d)^n (1-aa^d) \\ + (-1)^l \sum_{n=0}^{\infty} \lambda^{m+n} v(n,l) [(a^s)^d]^{n+l} a^k b^n. \end{aligned} \tag{3}$$

We consider several cases.

Case I. Let $m = 0$. Since $v(0,l) = 1$, (3) becomes

$$w(\lambda) = \sum_{n=l}^{\infty} \lambda^{-n} \binom{n-1}{l-1} (a^s)^{n-l} a^k (b^d)^n (1 - aa^d) + (-1)^l [(a^s)^d]^l a^k \tag{4}$$
$$+ (-1)^l \lambda \sum_{n=1}^{\infty} \lambda^{n-1} v(n,l) [(a^s)^d]^{n+l} a^k b^n.$$

Obviously, the limit $\lim_{\lambda \to 0} w(\lambda)$ exists if and only if the principal part of the Laurent series (4) vanishes. It is enough to assume that the first coefficient of the principal part of (4) is equal to 0, i.e. $(b^d)^l a^k (1 - aa^d) = 0$. If $\operatorname{ind}(a) \leqslant k$, then $a^k(1 - aa^d) = a^k(1 - aa^D) = 0$ and the limit $\lim_{\lambda \to 0} w(\lambda)$ exists. On the other hand, if $(b^d)^l a^k (1 - aa^d) = 0$, using Theorem 1.1 we conclude

$$0 = b^l (b^d)^l a^k (1 - aa^d) = a^k (1 - aa^d),$$

so $\operatorname{ind}(a) \leqslant k$.

It is easy to see that $\operatorname{ind}(a) \leqslant k$ implies

$$w = (-1)^l [(a^s)^D]^l a^k = (-1)^l (a^D)^{sl} a^k.$$

Case II. Let $0 < m < l$. Then it is obvious that $m - n < 0$ for all $n \geqslant l$. It follows that $\lim_{\lambda \to 0} w(\lambda)$ exists if and only if the principal part of the Laurent series (3) vanishes. As in Case I, using Theorem 1.1 we can prove that the principal part of (3) vanishes if and only if $(1 - aa^d) a^k = 0$, i.e. $\operatorname{ind}(a) \leqslant k$. Since the regular part of (3) has the form $\lambda B(\lambda)$, where $\lambda \mapsto B(\lambda)$ is a regular function in a neighbourhood of 0, it is easy to conclude $w = \lim_{\lambda \to 0} w(\lambda) = 0$.

Case III. Let $m \geqslant l$. It follows that (3) has the form

$$w(\lambda) = \sum_{n=m+1}^{\infty} \lambda^{m-n} \binom{n-1}{l-1} a^{s(n-l)+k} (b^d)^n (1 - aa^d) \tag{5}$$
$$+ \binom{m-1}{l-1} a^{s(m-l)+k} (1 - aa^d)(b^d)^m + \lambda C(\lambda),$$

where $\lambda \mapsto C(\lambda)$ is a regular function in a neighborhood of 0. The limit $\lim_{\lambda \to 0} w(\lambda)$ exists if and only if the principal part of (5) vanishes, i.e.

$$a^{s(n-l)+k} (1 - aa^d)(b^d)^n = 0 \quad \text{for all } n \geqslant m+1.$$

If $\operatorname{ind}(a) \leqslant s(m-l)+k+s$, then $a^{s(n-l)+k}(1-aa^d)(b^d)^n = 0$ holds for all $n \geqslant m+1$ and the limit $\lim\limits_{\lambda\to 0} w(\lambda)$ exists.

Now, suppose that $a^{s(n-l)+k}(1-aa^d)(b^d)^n = 0$ holds for $n \geqslant m+1$. Multiplying this equality by b^n and applying Theorem 1.1, we conclude that $a^{s(n-l)+k}(1-aa^d) = 0$ for all $n \geqslant m+1$, so $\operatorname{ind}(a) \leqslant s(m-l)+k+s$. Now it is easy to verify that $w = 0$ if $s(m-l)+k \geqslant \operatorname{ind}(a)$, and $w = \binom{m-1}{l-1} a^{s(m-l)+k}(1-aa^d)(b^d)^m$ if $\operatorname{ind}(a) - s \leqslant s(m-l)+k < \operatorname{ind}(a)$. $\square$

In [21] Rose proved the existence of the limit

$$\lim_{\lambda\to 0} \lambda^m A^k (A+\lambda I)^{-l},$$

where A is a square matrix, $l \geqslant 1$ and $m+k \geqslant \operatorname{ind}(A)$. In the case when a belongs to a Banach algebra $\mathscr{A}$, the main part of our Theorem 2.2 is that the existence of the limit

$$\lim_{\lambda\to 0} \lambda^m (\lambda b - a^s)^{-l} a^k$$

implies $\operatorname{ind}(a) < \infty$.

As corollaries, we mention the most important results, which are well-known for matrices and for a tiny class of bounded operators on an arbitrary Banach space. We point out the well-known results from the papers [1, 14, 17, 18, 21, 22]. Notice that all of these corollaries are proved for complex square matrices in the original papers.

Corollary 2.5. (Ji [14]) *If $a \in \mathscr{A}$, then $\operatorname{ind}(a) \leqslant k$ if and only if the limit*

$$w = \lim_{\lambda\to 0} (\lambda + a)^{-(k+1)} a^k$$

exists. In this case $w = a^D$.

Corollary 2.6. (Meyer [18], Rothblum [22]) *Let $a \in \mathscr{A}$ and let m, k be non-negative integers. Then $\operatorname{ind}(a) \leqslant m+k$ if and only if the limit*

$$w = \lim_{\lambda\to 0} \lambda^m (\lambda + a)^{-1} a^k$$

exists. In this case

$$w = \begin{cases} 0, & m > 0,\ m+k > \operatorname{ind}(a), \\ (-1)^{m-1}(1-aa^d)a^{\operatorname{ind}(a)-1}, & m > 0,\ m+k = \operatorname{ind}(a), \\ a^k a^D, & m = 0,\ k \geqslant \operatorname{ind}(a). \end{cases}$$

Analogous results for matrices of index zero or one are proved by Ben-Israel in [1].

Corollary 2.7. (Meyer [18]) *Let $a \in \mathscr{A}$. Then* $\operatorname{ind}(a) \leqslant k < \infty$ *if and only if the limit*

$$w = \lim_{\lambda \to 0} (\lambda + a^{k+1})^{-1} a^k$$

exists. In this case $w = a^D$.

Some corollaries are also mentioned in [21].

3. Computing the Moore-Penrose inverse in C^*-algebras

In this section we introduce further applications of the generalized Drazin inverse, such as the computation of the Moore-Penrose generalized inverse in C^*-algebras.

If $\mathscr{A}$ is a C^*-algebra, then the Moore-Penrose inverse of a, denoted by $a^\dagger$, exists if and only if a is g-invertible, as is shown in a paper of Harte and Mbekhta [13].

We need the following result from the functional calculus in C^*-algebras.

Theorem 3.1. *Let $a \in \mathscr{A}$ and $f \in \mathscr{H}(\sigma(a) \cup \sigma(a^*))$. If $D(f)$ is the domain of definition of f and $f(\overline{z}) = \overline{f(z)}$ holds for all $z \in D(f)$, then $f(a^*) = f(a)^*$.*

Proof. Notice that we can take $D(f)$ symmetrically with respect to the real axis. Let γ^+ denote the finite union of disjoint contours around $\sigma(a^*)$, positively oriented with respect to $\sigma(a^*)$. Then $\gamma^{*-} = \{\overline{z}\colon\ z \in \gamma^+\}$ is negatively oriented with respect to $\sigma(a)$. We may assume that γ^+ and γ^{*-} are contained in $D(f)$. Using $\overline{f(z)} = f(\overline{z})$ we compute

$$\begin{aligned} f(a^*) &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma^+} f(z)(z - a^*)^{-1} \, \mathrm{d}z = \left[-\frac{1}{2\pi i} \int_{\gamma^{*-}} f(\overline{z})(\overline{z} - a)^{-1} \, \mathrm{d}\overline{z} \right]^* \\ &= \left[\frac{1}{2\pi \mathrm{i}} \int_{\gamma^{*+}} f(z)(z - a)^{-1} \, \mathrm{d}z \right]^* = f(a)^*. \end{aligned}$$

□

As a corollary, we get some useful properties concerning the Drazin inverse of selfadjoint and positive elements in C^*-algebras.

Theorem 3.2. *If $a \in \mathscr{A}$ is selfadjoint and $0 \notin \operatorname{acc} \sigma(a)$, then a^d is self-adjoint. Moreover, if $a \geqslant 0$, then $a^d \geqslant 0$.*

If a is selfadjoint and Δ is an arbitrary spectral subset of $\sigma(a)$, then the spectral idempotent of a corresponding to Δ is positive, i.e. $p(a, \Delta) \geqslant 0$.

P r o o f. Since $a = a^*$, the function f from the definition of the generalized Drazin inverse of a satisfies the conditions from Theorem 3.1, so $a^d = f(a)$ is selfadjoint.

Moreover, if $a \geqslant 0$, then

$$\sigma(a^d) \setminus \{0\} = \{1/z \colon z \in \sigma(a) \setminus \{0\}\} \subset (0, +\infty), \quad [15]$$

so a^d is a spectral inverse of a (for the definition of spectral inverses see [2, 5]). We conclude that $a^d \geqslant 0$.

The rest of the theorem follows from the definition of the spectral idempotent and Theorem 3.1. $\square$

We will use Theorem 1.2 and Theorem 3.2 to get some limit results concerning the Moore-Penrose inverse in C^*-algebras.

Theorem 3.3. (a) *Suppose* $a \in \mathscr{A}$, *where* $\mathscr{A}$ *is a* C^**-algebra and* $0 \notin \operatorname{acc} \sigma(r^* s)$, *where* $r, s, t \in \mathscr{A}$ *are arbitrary. Then the limit*

$$\lim_{\lambda \to 0} (\lambda + r^* s)^{-1} t = h \tag{6}$$

exists if and only if

$$(1 - r^* s (r^* s)^d) t = 0.$$

In this case $h = (r^* s)^d t$.

(b) *If* $0 \notin \operatorname{acc} \sigma(r^* s)$, *then*

$$\lim_{\lambda \to 0} r^* (\lambda + s r^*)^{-1} = r^* (s r^*)^d = (r^* s)^d r^* = \lim_{\lambda \to 0} (\lambda + r^* s)^{-1} r^*$$

if and only if

$$0 = r^* (1 - s r^* (s r^*)^d) = (1 - r^* s (r^* s)^d) r^*.$$

(c) *If* $0 \notin \operatorname{acc} \sigma(a^* a)$, *then*

$$\lim_{\lambda \to 0} (\lambda + a^* a)^{-1} a^* = \lim_{\lambda \to 0} a^* (\lambda + a a^*)^{-1} = a^{\dagger}$$

if and only if $a^* = a^* a (a^* a)^d a^* = a^* a a^* (a a^*)^d$.

P r o o f. (a) Consider the expansion from Theorem 1.2 (for $b = 1$):

$$\begin{aligned} (\lambda + r^* s)^{-1} t = \Bigg[&(r^* s)^d \sum_{n=0}^{\infty} (-1)^n [(r^* s)^d]^n \lambda^n \\ &+ (1 - r^* s (r^* s)^d) \sum_{n=0}^{\infty} (-1)^n (r^* s)^n \lambda^{-n-1} \Bigg] t. \end{aligned} \tag{7}$$

Obviously, the limit (6) exists if and only if the principal part of the Laurent series (7) vanishes, i.e. $(1 - r^*s(r^*s)^d)t = 0$. In this case $h = (r^*s)^d t$.

(b) In this case we have $t = r^*$ and (a) holds. Since $\sigma(r^*s) \cup \{0\} = \sigma(sr^*) \cup \{0\}$, we get $0 \notin \operatorname{acc} \sigma(sr^*)$. Consider the following expression in a punctured neighbourhood of $\lambda = 0$:

$$r^*(\lambda + sr^*)^{-1} = \frac{r^*}{\lambda}\Big(1 + \frac{1}{\lambda}sr^*\Big)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}(-1)^n \lambda^{-n} r^*(sr^*)^n$$
$$= \frac{1}{\lambda}\sum_{n=0}^{\infty}(-1)^n (r^*s)^n r^* = (\lambda + r^*s)^{-1} r^*.$$

The rest of the proof follows in the same way as in (a).

(c) Now $r = s = a = t^*$ and (a) and (b) hold. Then the limit

$$\lim_{\lambda \to 0} (\lambda + a^*a)^{-1} a^* = h$$

exists if and only if $a^* = a^*a(a^*a)^d a^*$ and in this case $h = (a^*a)^d a^*$. From Theorem 3.2 it follows that $(a^*a)^d$ is selfadjoint in $\mathscr{A}$. Now we verify $h = a^\dagger$. Obviously, $hah = h$ and $ah = a(a^*a)^d a^*$ is selfadjoint in $\mathscr{A}$. The equality $a^* = a^*a(a^*a)^d a^*$ implies $a = a(a^*a)^d a^*a = aha$. Finally, $ha = 1 - p_{a^*a}$, where p_{a^*a} is the spectral idempotent of a^*a corresponding to the point $\lambda = 0$. From Theorem 3.2 it follows that p_{a^*a} is a selfadjoint idempotent in $\mathscr{A}$, so we get $0 \leqslant p_{a^*a} \leqslant 1$. It follows that $ha \geqslant 0$ is a selfadjoint idempotent in $\mathscr{A}$. The rest of the proof follows from (b). $\square$

Notice that the limit in Theorem 3.3 (c) is well-known for square matrices (see for example [21, 25] and references cited there).

We mention a few computational methods related to the limit representations of generalized inverses of matrices. An imbedding method and a finite algorithm for computation of the Drazin inverse, based on the limit representation of the Drazin inverse given in Corollary 2.5, is introduced in [8]. Ji obtained our Corollary 2.5 in [14]. He used this result to develop an iterative method for computing the Drazin inverse of a given matrix. In [24] a more general method for computing the limit expression of the form

$$\lim_{\alpha \to 0} (\alpha I + R^*S)^{-l} R^*$$

is developed, where R and S are arbitrary complex matrices. Partially, the method from [24] can be applied for computing generalized inverses contained in the limit expression from our Theorem 3.3.

4. Outer inverses with prescribed range and kernel

In this section we will consider the generalized Drazin inverse as an outer inverse with prescribed range and kernel. We restate some facts about outer inverses. Let X and Y be Banach spaces and let $\mathscr{L}(X,Y)$ be the set of all bounded operators from X into Y. For $A \in \mathscr{L}(X,Y)$, we use $\mathscr{N}(A)$ to denote the kernel, and $\mathscr{R}(A)$ to denote the range of A. An operator $B \in \mathscr{L}(Y,X)$ is an outer inverse of A, if $BAB = B$.

Consider the following problem. Let closed subspaces be given: T is a subspace of X, and S is a subspace of Y. Can we choose $B \in \mathscr{L}(Y,X)$, such that $BAB = B$, $\mathscr{R}(B) = T$ and $\mathscr{N}(B) = S$? If such B exists, then B is denoted as $A^{(2)}_{T,S}$. It is well-known that for a given operator $A \in \mathscr{L}(X,Y)$ and closed subspaces T of X and S of Y, there exists an $A^{(2)}_{T,S}$ inverse of A if and only if T is a complemented subspace of X, the restriction $A|_T\colon T \to A(T)$ is invertible and $A(T) \oplus S = Y$. In this case the $A^{(2)}_{T,S}$ inverse is unique.

For example, if $\operatorname{ind}(A) = k$, we can take $T = \mathscr{R}(A^k)$ and $S = \mathscr{N}(A^k)$, to get $A^{(2)}_{T,S} = A^D$. In the case when X and Y are Hilbert spaces and A^* is the adjoint of A, we can take $T = \mathscr{R}(A^*)$ and $S = \mathscr{N}(A^*)$ to get $A^{(2)}_{T,S} = A^\dagger$, the Moore-Penrose inverse of A. We shall show that A^d has some similar properties.

The first result we state for a Banach algebra setting. We prove that the generalized Drazin inverse can be computed as an outer inverse with prescribed range and kernel, in form similar to [9, 26, 27].

Theorem 4.1. *If $a \in \mathscr{A}$, $0 \notin \operatorname{acc}\sigma(a)$ and $p = p(a,0)$, then*

$$a^d = \lim_{\lambda\to 0}(ga-\lambda)^{-1}g = \lim_{\lambda\to 0} g(ag-\lambda)^{-1},$$

where we take $g = 1-p$.

Proof. For an arbitrary $\lambda \in \mathbb{C}$ notice that

$$(ga-\lambda) = (a(1-p)-\lambda)p + (a(1-p)-\lambda)(1-p) = -\lambda p + (a-\lambda)(1-p).$$

There exists an $\varepsilon > 0$ such that for all $\lambda \in \mathbb{C}$, if $0 < |\lambda| < \varepsilon$ then $a-\lambda$ is invertible. Using Lemma 2.1 we conclude

$$\begin{aligned}(ga-\lambda)^{-1}g &= \left[-\frac{1}{\lambda}p + (a-\lambda)^{-1}(1-p)\right](1-p)\\ &= (a-\lambda)^{-1}(1-p) = [(a-\lambda)(1-p)]^{-1}_{(1-p)\mathscr{A}(1-p)}.\end{aligned}$$

Since $a(1-p)$ is invertible in $(1-p)\mathscr{A}(1-p)$, it follows that

$$\lim_{\lambda\to 0}(ga-\lambda)g = [a(1-p)]^{-1}_{(1-p)\mathscr{A}(1-p)} = a^d.$$

The second equality can be proved in a similar way. $\square$

In the case when $\operatorname{ind}(a)=1$, the Drazin inverse of a is known as the group inverse, denoted by $a^{\#}$. In the next result we shall show how to use the group inverse of an operator to get the $A^{(2)}_{T,S}$ inverse. This statement represents a generalization of the result from [26], stated for complex matrices.

Theorem 4.2. *Let T and S be closed subspaces of X and Y, respectively, such that for an operator $A \in \mathscr{L}(X,Y)$ the $A^{(2)}_{T,S}$ inverse exists. Let $G \in \mathscr{L}(Y,X)$ be an arbitrary operator which satisfies $\mathscr{N}(G)=S$ and $\mathscr{R}(G)=T$. Then*

$$\operatorname{ind}(AG)=\operatorname{ind}(GA)=1$$

and

$$A^{(2)}_{T,S}=G(AG)^{\#}=(GA)^{\#}G=J\left(GA|_T\right)^{-1}G.$$

Here $J\colon T\to X$ denotes the natural inclusion.

Proof. Since the $A^{(2)}_{T,S}$ inverse of A exists, we conclude $X=T\oplus T_1$ for some closed subspace T_1 of X. Also, the restriction $A|_T\colon T\to A(T)$ is invertible and $A(T)\oplus S=Y$. We can write A in the matrix form

$$A=\begin{bmatrix} A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix}\colon \begin{bmatrix} T\\ T_1\end{bmatrix}\to\begin{bmatrix} A(T)\\ S\end{bmatrix},$$

where $A_{11}\colon T\to A(T)$ is invertible. Also, G has the matrix form

$$G=\begin{bmatrix} G_1 & 0\\ 0 & 0\end{bmatrix}\colon \begin{bmatrix} A(T)\\ S\end{bmatrix}\to\begin{bmatrix} T\\ T_1\end{bmatrix},$$

where $G_1\colon A(T)\to T$ is invertible. We get

$$AG=\begin{bmatrix} A_{11}G_1 & 0\\ 0 & 0\end{bmatrix},\quad \operatorname{ind}(AG)=1\quad\text{and}\quad (AG)^{\#}=\begin{bmatrix} G_1^{-1}A_{11}^{-1} & 0\\ 0 & 0\end{bmatrix}.$$

It is easy to verify that

$$G(AG)^{\#}=\begin{bmatrix} A_{11}^{-1} & 0\\ 0 & 0\end{bmatrix}=A^{(2)}_{T,S}.$$

Notice that

$$GA=\begin{bmatrix} G_1A_{11} & G_1A_{12}\\ 0 & 0\end{bmatrix}.$$

It can be easily seen that

$$(GA)^{\#}=\begin{bmatrix} A_{11}^{-1}G_1^{-1} & A_{11}^{-1}G_1^{-1}A_{11}^{-1}A_{12}\\ 0 & 0\end{bmatrix}.$$

(Compare the last statement with our Corollary 5.2 in Section 5). It follows that $(GA)^{\#}G = A^{(2)}_{T,S}$.

To prove that $J(GA|_T)^{-1}G$ is also equal to $A^{(2)}_{T,S}$, we use slightly different matrix decompositions. Let

$$A|_T = \begin{bmatrix} A_{11} \\ 0 \end{bmatrix} : T \to \begin{bmatrix} A(T) \\ S \end{bmatrix}, \quad G = [\, G_1 \quad 0 \,] : \begin{bmatrix} A(T) \\ S \end{bmatrix} \to T.$$

Obviously,

$$J = \begin{bmatrix} I \\ 0 \end{bmatrix} : T \to \begin{bmatrix} T \\ T_1 \end{bmatrix},$$

where I denotes the identity operator on T. Since $GA|_T = G_1 A_{11} \colon T \to T$ is invertible, we compute

$$J(GA|_T)^{-1}G = \begin{bmatrix} A_{11}^{-1} G_1^{-1} \\ 0 \end{bmatrix} [\, G_1 \quad 0 \,] = A^{(2)}_{T,S}.$$

$\square$

Let $X = Y$, $A \in \mathscr{L}(X)$ and $0 \notin \operatorname{acc} \sigma(A)$. If P is the spectral idempotent of A corresponding to 0, then we can take $G = I - P$ in Theorem 4.2 (also $T = \mathscr{R}(G)$ and $S = \mathscr{N}(G)$), to get $A^{(2)}_{T,S} = A^d$.

Corollary 4.3. (i) *If X, Y are Hilbert spaces and $A \in \mathscr{L}(X, Y)$ has a closed range, then*

$$A^{\dagger} = [A^* A|_{\mathscr{R}(A^*)}]^{-1} A^* \quad [7],$$

(ii) $A^D = [A^{k+1}|_{\mathscr{R}(A^k)}]^{-1} A^k ([26]) \ = [A|_{\mathscr{R}(A^k)}]^{-(k+1)} A^k$,

(iii) $A^d = [(I-P)A|_{\mathscr{R}(I-P)}]^{-1}(I-P)$, *where P is the spectral idempotent of A corresponding to* 0.

5. Operator matrices

In this section we give a brief generalization of the well-known Meyer-Rose result [19] concerning the Drazin inverse of block 2×2 upper triangular matrices. A general problem with non-zero entries is also considered.

Let X, Y, Z be Banach spaces and $Z = X \oplus Y$. For $A \in \mathscr{L}(X)$, $B \in \mathscr{L}(Y)$ and $C \in \mathscr{L}(Y, X)$ consider the operator

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \mathscr{L}(Z).$$

Theorem 5.1. *If A and B have generalized Drazin inverses, then M has the generalized Drazin inverse and*

$$M^d = \begin{bmatrix} A^d & S \\ 0 & B^d \end{bmatrix}$$

where

$$S = (A^d)^2 \left[\sum_{n=0}^{\infty} (A^d)^n C B^n\right](I - BB^d) + (I - AA^d)\left[\sum_{n=0}^{\infty} A^n C (B^d)^n\right](B^d)^2 - A^d C B^d.$$

Proof. Since A^d and B^d exist, it follows that $0 \notin \operatorname{acc}\sigma(A) \cup \operatorname{acc}\sigma(B)$. Since $\sigma(M) \subset \sigma(A) \cup \sigma(B)$, we conclude $0 \notin \operatorname{acc}\sigma(M)$, so M^d exists.

Consider the Laurent expansion

$$(\lambda - M)^{-1} = \sum_{n=1}^{\infty} \lambda^{-n} M^{n-1}(I - MM^d) - \sum_{n=0}^{\infty} \lambda^n (M^d)^{n+1}$$

and similar expansions for $(\lambda - A)^{-1}$ and $(\lambda - B)^{-1}$ in a punctured neighbourhood of 0. Notice that

$$(\lambda - M)^{-1} = \begin{bmatrix} (\lambda - A)^{-1} & (\lambda - A)^{-1} C (\lambda - B)^{-1} \\ 0 & (\lambda - B)^{-1} \end{bmatrix}.$$

Comparing the coefficients at $\lambda^0 = 1$, we get the statement of the theorem. □

As a corollary we get the following well-known result [19].

Corollary 5.2. (Meyer and Rose [19]) *If $\operatorname{ind}(A) = k$ and $\operatorname{ind}(B) = l$, then the Drazin inverse of M exists and has the form*

$$M^D = \begin{bmatrix} A^D & S \\ 0 & B^D \end{bmatrix},$$

where

$$S = (A^D)^2 \left[\sum_{n=0}^{l-1} (A^D)^n C B^n\right](I - BB^D) + (I - AA^D)\left[\sum_{n=0}^{k-1} A^n C (B^D)^n\right](B^D)^2 - A^D C B^D.$$

If M has the Drazin inverse and $0 \notin \operatorname{acc}\sigma(A) \cup \operatorname{acc}\sigma(B)$, then A and B also have the Drazin inverses.

Finally, we use a recent result of Förster and Nagy [6], to prove another representation of the generalized Drazin inverse of operator matrices.

Let $V = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathscr{L}(Z)$ be a bounded operator on $Z = X \oplus Y$. The following result holds.

Theorem 5.3. *If $BC = BD = DC = 0$ and $0 \notin \operatorname{acc}\sigma(A) \cup \operatorname{acc}\sigma(D)$, then $0 \notin \operatorname{acc}\sigma(V)$ and the generalized Drazin inverse of V has the form*

$$V^d = \begin{bmatrix} A^d & (A^d)^2 B \\ C(A^d)^2 & D^d + C(A^d)^3 B \end{bmatrix}.$$

Proof. In [6] it is proved that $\sigma(A) \cup \sigma(V) = \sigma(A) \cup \sigma(D)$, so we conclude $0 \notin \operatorname{acc}\sigma(V)$. We use $\varrho(A) = \mathbb{C} \setminus \sigma(A)$ to denote the resolvent set of A. Using [6, Lemma and Theorem] we know that for $\lambda \in (\varrho(A) \cap \varrho(V)) \setminus \{0\}$ the resolvent operator of V has the form

$$(\lambda - V)^{-1} = \begin{bmatrix} (\lambda - A)^{-1} & \lambda^{-1}(\lambda - A)^{-1}B \\ \lambda^{-1}C(\lambda - A)^{-1} & (\lambda - D)^{-1} + \lambda^{-2}C(\lambda - A)^{-1}B \end{bmatrix}. \tag{8}$$

There exist open sets $U, W \subset \mathbb{C}$, such that $0 \in U$ and $[\sigma(A) \cup \sigma(D)] \setminus \{0\} \subset W$. Define a function $f(\lambda)$ in the following way:

$$f(\lambda) = \begin{cases} 0, & \lambda \in U, \\ \frac{1}{\lambda}, & \lambda \in W. \end{cases}$$

Taking a suitable contour γ around V, we obtain

$$V^d = f(V) = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(\lambda)(\lambda - V)^{-1}\,\mathrm{d}\lambda.$$

Using the well-known properties of the functional calculus and (8), we conclude that the statement of the theorem holds. $\square$

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