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CONNECTED DOMATIC NUMBER IN PLANAR GRAPHS

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Abstract. A dominating set in a graph G is a *connected dominating set* of G if it induces a connected subgraph of G. The *connected domatic number* of G is the maximum number of pairwise disjoint, connected dominating sets in $V(G)$. We establish a sharp lower bound on the number of edges in a connected graph with a given order and given connected domatic number. We also show that a planar graph has connected domatic number at most 4 and give a characterization of planar graphs having connected domatic number 3.

Keywords: connected dominating set, connected domatic number, planar

MSC 2000: 05C70, 05C99

1. Introduction

A set of vertices D in a graph $G = (V, E)$ is a dominating set if every vertex in $V - D$ has at least one neighbour in D. Such a dominating set D is called a *connected dominating set* if the subgraph induced by D, $\langle D\rangle$, is a connected subgraph of G. In this paper we assume that all graphs are connected since we are interested in connected dominating sets. The minimum number of vertices in a connected dominating set of G is called the *connected domination number* of G, and is denoted by $\gamma_c(G)$. A *connected domatic partition* of G is a partition of the vertex set, V, into connected dominating sets. The maximum number of subsets in such a partition is called the *connected domatic number* of G and is denoted by $d_c(G)$. Equivalently, $d_c(G)$ is the maximum number of pairwise disjoint, connected dominating sets which can be found in $V(G)$. The concept of a connected dominating set was defined in [5] and the connected domatic number was introduced by S. T. Hedetniemi and R. Laskar in [4].

A graph $G = (V, E)$ is *planar* if it is possible to establish a one-to-one correspondence $(v_i \leftrightarrow p_i)$ between V and a set $\{p_1, p_2, \ldots, p_n\}$ of points in the plane in such a way that if $v_r v_s \in E$ then a curve can be drawn joining p_r and p_s such that the interiors of distinct curves do not intersect. Such an embedding of a planar graph is called a *plane* graph. G is called *outerplanar* if the embedding can be chosen so that the boundary of one of the planar regions contains every vertex of G. A graph G_2 is called an *elementary contraction* of G_1 if there is an edge uv of G_1 such that $V(G_2) = (V(G_1) - \{u, v\}) \cup \{x\}$ (where x does not belong to $V(G_1)$, and the edge set of G_2 consists of all edges of G_1 which are not incident with either of u or v together with all edges of the form xy where at least one of uy or vy is an edge of G_1. If the graph H is isomorphic to G or is obtainable from G by a finite sequence of elementary contractions, then we say that H is a *contraction* of G. Perhaps a more intuitive way to think of a contraction H of a graph G is to consider a partition of $V(G)$ into subsets each of which induces a connected subgraph of G. Each member of the partition corresponds to a vertex of H, and two vertices of H are adjacent if the union of the corresponding subsets of G induces a connected subgraph of G. In effect each member of the partition has been shrunk to a single vertex and multiple edges have then been removed. It is clear that the property of being planar is preserved under contractions.

For our purposes here we need the following result concerning planar and outerplanar graphs. See chapter 4 of [1].

Theorem 1.1. *A graph G is planar (outerplanar) if and only if neither K_5 nor $K_{3,3}$ (K_4 nor $K_{2,3}$) is a contraction of a subgraph of G.*

Hedetniemi and Laskar observed that $d_c(G) \leqslant \delta(G)$ unless G is a complete graph. If G is not complete, there is a proper subset A of $V(G)$ such that the subgraph $G - A$ is disconnected. Every connected dominating set D of G must intersect A nontrivally since D induces a connected subgraph and dominates the vertices in each component of $G - A$. B. Zelinka used this idea to prove the following result.

Theorem 1.2. ([6]) *If G is a connected graph which is not complete, then the connected domatic number is no larger than $\kappa(G)$, the vertex connectivity of G.*

For general graphs the upper bound of $\kappa(G)$ for $d_c(G)$ is the best known. In what follows we will prove a better bound for planar graphs.

2. Size and Connected Domatic Number

We begin by establishing a sharp lower bound on the number of edges in a connected graph G having a given order and given connected domatic number. Unless otherwise noted we will use n to denote the order of a graph G. If A and B are disjoint subsets of $V(G)$, then we use $E(A, B)$ to denote the set of all edges of G which join a vertex in A and a vertex in B.

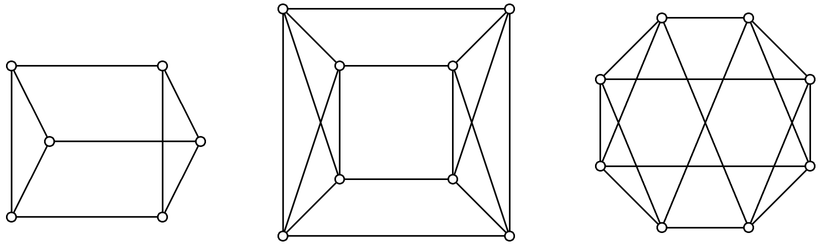
Lemma 2.1. *Let G be a connected graph of order n with connected domatic number $k \geqslant 1$. Then G must have at least $\frac{k+1}{2}n - k$ edges.*

P r o o f. Since G is connected of order n it must have at least $n - 1 = \frac{1+1}{2}n - 1$ edges, so the result is true for $k = 1$. Thus we assume $k \geqslant 2$. Let $D_1, D_2, \ldots, D_k$ be a connected domatic partition of G, and let $G_i = \langle D_i \rangle$ have order n_i, for each i. Then since D_i is a dominating set for G it follows that $|E(D_i, V(G) - D_i)| \geqslant n - n_i$ and so

$$\begin{aligned}
|E(G)| &\geqslant \sum_{i=1}^{k} |E(G_i)| + \frac{1}{2} \sum_{i=1}^{k} |E(D_i, V(G) - D_i)| \\
&\geqslant \sum_{i=1}^{k} (n_i - 1) + \frac{1}{2} \sum_{i=1}^{k} (n - n_i) \\
&\geqslant (n - k) + \frac{1}{2} n(k - 1) \\
&= \frac{k+1}{2} n - k.
\end{aligned}$$

□

For each value of k the complete bipartite graph $K_{k,k}$ shows that the lower bound of Lemma 2.1 is sharp. Figure 1 contains three other graphs showing the sharpness of the bound.



$K_3 \square K_2$ G_1 G_2

Figure 1. Regular Graphs Achieving Bound of Lemma 2.1.

By Theorem 1.2, $d_c(G) \leqslant r$ if G is an r-regular graph, unless G is the complete graph K_{r+1}.

Corollary 2.2. *If G is a cubic graph and $d_c(G) = 3$, then G is one of $K_{3,3}$ or $K_3 \square K_2$. If G is 4-regular such that $d_c(G) = 4$, then G is one of $K_{4,4}$, G_1 or G_2 of Figure* 1.

P r o o f. Assume G is a cubic graph such that $d_c(G) = 3$. By Lemma 2.1 G must have at least $2n - 3$ edges. Thus $\frac{3}{2}n \geqslant 2n - 3$, and so G has order at most six. It is straightforward to check that $K_{3,3}$ and $K_3 \square K_2$ are the only such cubic graphs with connected domatic number three. Similar reasoning establishes the second statement of the corollary. $\square$

In general, if G is r-regular and $d_c(G) = r$, then by Lemma 2.1 $\frac{r}{2}n = |E(G)| \geqslant \frac{r+1}{2}n - r$, and so G has order at most $2r$. The following result then follows.

Corollary 2.3. *For each positive integer r there are a finite number of r-regular graphs with connected domatic number r. Each of these graphs must have order $2r$, and each of the connected dominating sets in the connected domatic partition induces a path of order* 2.

3. Planar Graphs

We first learned of the connected domatic number 'problem' from Steve Hedetniemi [2], who was considering various domination problems on chessboard graphs. A chessboard graph is a graph whos vertices are those of a square grid graph with edges corresponding to legal moves of a particular chess piece. Individuals working on these problems consider chessboard graphs corresponding to queens, kings, knights, rooks and bishops. In addition the usual grid graph, which is the Cartesian product of two paths of order n, is included in this class of graphs and is the chessboard graph of a 'chess piece', often called the cross, which can move one square horizontally or vertically. See [3] for a survey. We will concentrate here on planar graphs.

We consider first the more general $r \times s$ grid graph, $G_{r,s} = P_r \square P_s$, the Cartesian product of two paths. Specifically, let P_r be the path $v_1, v_2, \ldots, v_r$ and let P_s be the path $w_1, w_2, \ldots, w_s$. Then $V(G_{r,s}) = \{(v_i, w_j) | 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s\}$. Assume $2 \leqslant r \leqslant s$. Since $\kappa(G_{r,s}) = 2$ it follows from 1.2 that $d_c(G_{r,s}) \leqslant 2$. When $r = 2$ the subsets $D_1 = \{(v_1, w_i) | 1 \leqslant i \leqslant s\}$ and $D_2 = \{(v_2, w_i) | 1 \leqslant i \leqslant s\}$ partition the vertex set of the $r \times s$ grid into two connected dominating sets. These are the only grid graphs with connected domatic number larger than one.

Theorem 3.1. *Let $2 \leqslant r \leqslant s$. If $r = 2$, then $d_c(G_{r,s}) = 2$. For $r \geqslant 3$, $d_c(G_{r,s}) = 1$.*

Proof. The case $r = 2$ is covered above. Assume $r \geqslant 3$ and that $\{D_1, D_2\}$ is a partition of $V(G_{r,s})$ into two connected dominating sets. For ease of reference let a, b, c, d, e, f be the vertices (v_1, w_2), (v_2, w_1), (v_1, w_{s-1}), (v_2, w_s), (v_{r-1}, w_s), (v_r, w_{s-1}), respectively. Note that if $r = 3$, then $d = e$. (If, in addition, $s = 3$, then $a = c$ as well.) We may assume without loss of generality that $a \in D_1$ and $b \in D_2$. Since D_1 and D_2 are connected dominating sets of $G_{r,s}$, some neighbour of (v_1, w_s) must belong to D_2, and some neighbour of (v_r, w_s) must belong to D_1. Let $\{x\} = D_2 \cap \{c, d\}$ and let $\{y\} = D_1 \cap \{e, f\}$.

If $r = 3 = s$, then $f \in D_1$ and $d \in D_2$. But then $(v_2, w_2) \in D_1$ and so $\langle D_2 \rangle$ is not connected, a contradiction. Therefore, assume $s \geqslant 4$. There are three cases to consider. If $r = 3$ (so that $d = e$) and $y = e$, then it follows that $x = c$ and $f \in D_2$. Thus the vertices of any $a - y$ path form a cutset which separates vertices b and c. If $r = 3$ and $y = f$, then the vertex set of any $a - y$ path separates vertices b and d. If $r \geqslant 4$, then vertices b and x are separated by the vertex set of any $a - y$ path. In each case there is a contradiction to the assumption that $\langle D_2 \rangle$ is connected. □

Any tree T has exactly one minimum connected dominating set, namely the set consisting of all vertices of T having degree larger than one. Since every planar graph has a vertex of degree at most five, it follows from Theorem 1.2 that if G is planar then $1 \leqslant d_c(G) \leqslant 5$. We conclude by showing that $d_c(G) \leqslant 4$ and by specifying the structure of the subgraphs induced by the subsets in a connected domatic partition when $d_c(G)$ is three or four.

Theorem 3.2. *Let G be a planar graph. The connected domatic number of G is at most 4, and K_4 is the only planar graph achieving this bound.*

Proof. Assume G is a planar graph of order n such that $d_c(G) \geqslant 5$. By Lemma 2.1, G has at least $\frac{5+1}{2}n - 5$ edges. But this contradicts the well known upper bound of $3n - 6$ for a planar graph of order n. Thus, $d_c(G) \leqslant 4$.

Now assume that $d_c(G) = 4$ but that G is not isomorphic to K_4. Let $D_1 \cup D_2 \cup D_3 \cup D_4$ be a connected domatic partition of G. We may assume that $|D_1| \geqslant 2$. Let a and b be adjacent vertices in D_1 and let $u_i \in D_i$ for $i = 2, 3, 4$. Let H be the contraction of G formed by identifying the vertices in each of D_2, D_3 and D_4 to a single vertex x_2, x_3 and x_4, respectively. H is planar since it is a contraction of a planar graph. But since D_2 is a dominating set in G it follows that x_2 is adjacent to each of a, b, x_3, x_4 in H. Similar statements hold for x_3 and x_4. But then H contains the subgraph $\langle\{a, b, x_2, x_3, x_4\}\rangle$ which has been shown to be isomorphic to K_5. This contradiction shows that G must be isomorphic to K_4. □

The planar graph G in Figure 2 is an example of a graph with connected domatic number three. The three "vertical" paths are connected dominating sets. In this particular example the three connected dominating sets in the partition have orders four, five and seven.

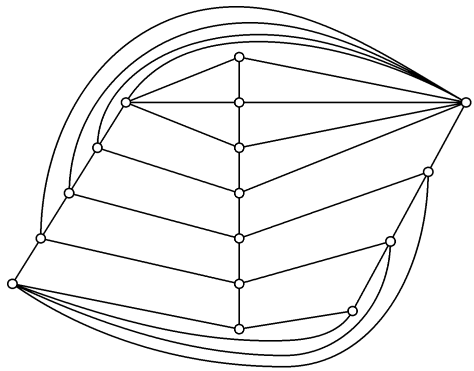


Figure 2. Planar Graph G such that $d_c(G) = 3$.

The following result shows that the structure present in the connected dominating sets of any planar graph having connected domatic number three must be similar to that of the graph in Figure 2.

Theorem 3.3. *Let G be a planar graph such that $d_c(G) = 3$ and let $D_1 \cup D_2 \cup D_3$ be any connected domatic partition of G. Each of the induced subgraphs $\langle D_1 \rangle$, $\langle D_2 \rangle$ and $\langle D_3 \rangle$ is a path.*

P r o o f. Assume first that G is a planar graph and $D_1 \cup D_2 \cup D_3$ is a connected domatic partition of G such that $\langle D_1 \rangle$ has a vertex a of degree at least three. Let b, c, d be three of its neighbours in D_1 and let $u_2 \in D_2$ and $u_3 \in D_3$. Let H be the planar graph obtained from G by contracting D_i onto u_i for $i = 2, 3$, and by removing any of the edges bc, bd, cd which is present in G. Since D_2 dominates D_1, each of a, b, c and d is adjacent to u_2 in H. Consider any planar embedding of H. By Euler's formula, the subgraph $K = \langle \{a, b, c, d, u_2\} \rangle$ has four regions, and u_2 belongs to the boundary of each of these regions. Since K contains a subgraph isomorphic to $K_{2,3}$, it is not outerplanar, and so none of the four regions has a boundary which contains all of a, b, c and d. But u_3 lies in one of these regions and can be adjacent only to vertices on the boundary of this region. But then u_3 does not dominate D_1, contradicting our assumption above. Therefore, the maximum degree of any of the induced subgraphs $\langle D_1 \rangle$, $\langle D_2 \rangle$ and $\langle D_3 \rangle$ is no more than two.

Now assume that one of these subgraphs, say $\langle D_1 \rangle$, is a cycle of length $k \geqslant 3$. By contracting this cycle to K_3 and the other two subgraphs $\langle D_2 \rangle$ and $\langle D_3 \rangle$ to single vertices it follows that G has a subgraph which can be contracted to K_5. This contradiction establishes the theorem. $\square$

For any three given positive integers r, s and t it is straightforward to see how to modify the construction in Figure 2 to embed a planar graph with a connected domatic partition whose dominating sets induce the paths P_r, P_s and P_t. Similarly, for any two given outerplanar graphs H_1 and H_2, Figure 3 illustrates how to construct a plane embedding of a graph with connected domatic number 2 such that the two connected dominating sets in the partition induce subgraphs isomorphic to H_1 and H_2.

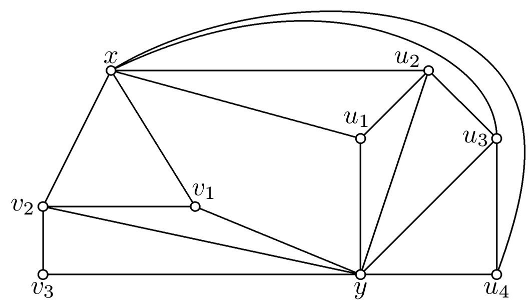


Figure 3. Planar Graph G such that $d_c(G) = 2$.

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