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DEGREE-CONTINUOUS GRAPHS

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Abstract. A graph G is degree-continuous if the degrees of every two adjacent vertices of G differ by at most 1. A finite nonempty set S of integers is convex if $k \in S$ for every integer k with $\min(S) \leqslant k \leqslant \max(S)$. It is shown that for all integers $r > 0$ and $s \geqslant 0$ and a convex set S with $\min(S) = r$ and $\max(S) = r + s$, there exists a connected degree-continuous graph G with the degree set S and diameter $2s + 2$. The minimum order of a degree-continuous graph with a prescribed degree set is studied. Furthermore, it is shown that for every graph G and convex set S of positive integers containing the integer 2, there exists a connected degree-continuous graph H with the degree set S and containing G as an induced subgraph if and only if $\max(S) \geqslant \Delta(G)$ and G contains no $r-$regular component where $r = \max(S)$.

Keywords: distance, degree-continuous

MSC 2000: 05C12

1. Introduction

In the newly introduced area of analytic graph theory (see [1]), an integer-valued function f defined on a metric space $\mathcal{M}$ associated with a graph G, where there is a symmetric adjacency relation defined on $\mathcal{M}$, is said to be *continuous* on $\mathcal{M}$ if $|f(x) - f(y)| \leqslant 1$ for every two adjacent elements x and y of $\mathcal{M}$. Certainly, one of the best known and most studied integer-valued parameters associated with a graph is the degree of a vertex. Inspired by the terminology just described, we begin a study of this topic. Consequently, we define a graph G to be *degree-continuous* if the degrees of every two adjacent vertices of G differ by at most 1. Observe that a graph G is degree-continuous if and only if every component of G is degree-continuous.

A finite nonempty set S of integers is *convex* if $k \in S$ for every integer k with $\min(S) \leqslant k \leqslant \max(S)$. The *degree set* $\mathcal{D}(G)$ of a graph G is the set of the degrees of the vertices of G. Necessarily, the degree set of every connected degree-continuous graph is convex. In this paper, we assume that a convex set S is of the form

$$S = \{r, r+1, r+2, \ldots, r+s\} \tag{1}$$

where r and s are nonnegative integers with $s \geqslant 0$.

In fact, every convex set S as given by (1) is the degree set of some connected degree-continuous graph. For example, let $T_{r,s}$ be a rooted tree with root v such that if the distance $d(u,v)$ between u and v in $T_{r,s}$ is i, then $\deg_{T_{r,s}} u = r+i$, where $0 \leqslant i \leqslant s-1$. In particular, $\deg_{T_{r,s}} v = r$. Furthermore, the distance between v and every end-vertex of $T_{r,s}$ is s. It follows that

$$\mathcal{D}(T_{r,s}) = \{r, r+1, r+2, \ldots, r+s-1\} \cup \{1\}.$$

If $r+s \leqslant 3$, then $T_{r,s}$ is degree-continuous since the degree of each of its vertices is 1 or 2. Suppose that $r+s > 3$. Since every vertex u that is adjacent to an end-vertex w has degree $r+s-1$, it follows that $\deg_{T_{r,s}} u - \deg_{T_{r,s}} w = r+s-2 > 1$ and so $T_{r,s}$ is not degree-continuous. However, $|\deg_{T_{r,s}} u_1 - \deg_{T_{r,s}} u_2| = 1$ for every two adjacent vertices u_1 and u_2 that are not end-vertices. For each end-vertex of $T_{r,s}$ we construct a complete graph K_{r+s-1} of order $r+s-1$ and add an edge between this end-vertex and each vertex in the corresponding K_{r+s-1}. Denote the resulting graph by $G_{r,s}$. It is not difficult to see that:

$$\begin{array}{ll} \deg_{T_{r,s}} u = \deg_{G_{r,s}} u & \text{for every vertex } u \text{ of } T_{r,s} \text{ that is not an end-vertex,} \\ \deg_{G_{r,s}} w = r+s & \text{for every end-vertex } w \text{ of } T_{r,s}\text{, and} \\ \deg_{G_{r,s}} u = r+s-1 & \text{for every vertex } u \text{ of } G_{r,s} \text{ that does not belong to } T_{r,s}. \end{array}$$

It follows that

$$\mathcal{D}(G_{r,s}) = (\mathcal{D}(T_{r,s}) - \{1\}) \cup \{r+s\} = S.$$

Note that if $d(u,v)$ is $r+s-1$ or $r+s+1$, then $\deg_{G_{r,s}} u = r+s-1$. Moreover, $|\deg_{G_{r,s}} u_1 - \deg_{G_{r,s}} u_2| = 1$ for every two adjacent vertices u_1 and u_2 of $G_{r,s}$. Hence the graph $G_{r,s}$ is degree-continuous. From the structure of $G_{r,s}$ it follows that $\operatorname{diam} G_{r,s} = 2s+2$ and $G_{r,s}$ contains a path P of length s such that the degrees of its vertices attain each element of the set S exactly once. In fact, every $v-w$ path, where w is an end-vertex of $T_{r,s}$, has this property. Figure 1 illustrates the structure of $G_{2,4}$ and so $S = \{2,3,4,5,6\}$. Clearly, $\operatorname{diam} G_{2,4} = 10$ and the path $P\colon\ v_2, v_3 \ldots, v_6$ of G has the property that $\deg v_i = i$ for $2 \leqslant i \leqslant 6$.

The discussion above gives the following theorem.

Theorem 1.1. *For all integers $r \geqslant 1$ and $s \geqslant 0$ and a convex set S of integers with $\min(S) = r$ and $\max(S) = r+s$, there exists a connected degree-continuous*

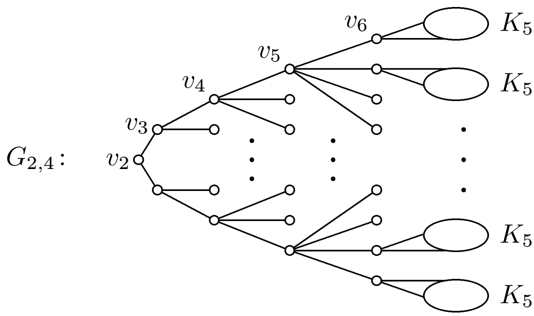

Figure 1. The degree-continuous graph $G_{2,4}$

graph G with the degree set S and $\operatorname{diam}(G) = 2s + 2$. *Moreover, G contains a path*

$$P\colon\ v_r, v_{r+1}, \ldots, v_{r+s}$$

of length s with $\deg v_i = i$ *for* $r \leqslant i \leqslant r + s$.

2. The minimum order of a degree-continuous graph

In this section we investigate the minimum order of a degree-continuous graph with some prescribed properties. First, we study the minimum order of a degree-continuous tree with a given maximum degree.

A fact from number theory will be useful to us here. Let a_n denote the number of nonempty words that can be formed from n given characters, where no character is repeated in the word. The number a_n (see [4], p. M3503) is given by

$$a_n = n + n(n-1) + n(n-1)(n-2) + \ldots + n!.$$

Alternatively, the sequence $\{a_n\}$ can be defined recursively by the initial value

$$a_1 = 1$$

and the recursive relation

$$a_n = n(1 + a_{n-1}) \quad \text{for } n \geqslant 2.$$

We are now prepared to present a result on the order of a degree-continuous tree that is not a path.

Theorem 2.1. *The order of every degree-continuous tree with maximum degree* $\Delta \geqslant 3$ *is at least* $1 + \Delta + \Delta a_{\Delta-2}$.

P r o o f. Let v be a vertex of a degree-continuous tree T with $\deg v = \Delta$. We partition $V(T)$ as $\{V_1, V_2, \ldots, V_\Delta\}$, where V_k consists of the vertices of degree k in T. We now root T at v and orient each edge of T away from v. We note that each vertex in V_k has at least $k-1$ descendants in V_{k-1}. Hence,

$$|V_{k-1}| \geqslant (k-1)|V_k|.$$

Furthermore, $V_{\Delta-1}$ has cardinality at least Δ. Thus,

$$\begin{aligned}|V_\Delta| + |V_{\Delta-1}| + \ldots + |V_2| + |V_1| &\geqslant 1 + \Delta + \Delta(\Delta-2)\\ &\qquad + \Delta(\Delta-2)(\Delta-3) + \ldots + \Delta[(\Delta-2)!]\\ &= 1 + \Delta + \Delta a_{\Delta-2}\end{aligned}$$

and the proof is complete. □

We see that this result is sharp by considering a tree that is rooted at a vertex of degree Δ and in which each vertex of degree $k \geqslant 2$ has $k-1$ children of degree $k-1$.

Now we consider the minimum orders of more general degree-continuous graphs. We begin by stating some additional definitions. Let $G_1, G_2, \ldots, G_k$ be k graphs with disjoint vertex sets. Then the k-path composition $G = P_k[G_1, G_2, \ldots, G_k]$ has

$$V(G) = \bigcup_{i=1}^{k} V(G_i)$$

and

$$E(G) = \left(\bigcup_{i=1}^{k} E(G_i)\right) \cup \left(\bigcup_{i=1}^{k-1} \{v_i v_{i+1} \colon\ v_i \in V(G_i) \text{ and } v_{i+1} \in V(G_{i+1})\}\right).$$

Observe that $P_k[K_1, K_1, \ldots, K_1]$ is simply a path of order k. Moreover, for $v_1 \in V(G_1)$,

$$\deg_G v_1 = \deg_{G_1} v_1 + |V(G_2)|. \tag{2}$$

For $v_i \in V(G_i)$ $(2 \leqslant i \leqslant k-1)$,

$$\deg_G v_i = |V(G_{i-1})| + \deg_{G_i} v_i + |V(G_{i+1})|. \tag{3}$$

For $v_k \in V(G_k)$,

$$\deg_G v_k = |V(G_{k-1})| + \deg_{G_k} v_k. \tag{4}$$

For a set S as given by (1), define $m(S)$ to be the *minimum order* of a degree-continuous graph G having S as its degree set. The following theorems (see [2], pp. 227, 272) will be useful.

Theorem A. *For integers r and n with $0 \leqslant r < n$, there exists an r-regular graph of order n if and only if r and n are not both odd.*

Theorem B. *Every r-regular bipartite graph with $r \geqslant 1$ is 1-factorable.*

The following theorem gives the minimum orders for degree-continuous graphs with given degree sets.

Theorem 2.2. *Let S be a convex set as described in* (1) *and let $m = \lceil \frac{s}{3} \rceil$.*

1. *If $s \equiv 0 \pmod 3$, then*

$$(m+1)\left(r+1+\frac{3m}{2}\right) \leqslant m(S) \leqslant 1+(m+1)\left(r+1+\frac{3m}{2}\right).$$

Moreover, if r and m are not both even, then

$$m(S) = (m+1)\left(r+1+\frac{3m}{2}\right).$$

2. *If $s \equiv 1 \pmod 3$, then*

$$m(S) = (m+1)\left(r+2+\frac{3m}{2}\right).$$

3. *If $s \equiv 2 \pmod 3$, then*

$$1+(m+1)\left(r+\frac{3(m+2)}{2}\right) \leqslant m(S) \leqslant 2+(m+1)\left(r+\frac{3(m+2)}{2}\right).$$

Moreover, if r is even or m is odd, then

$$m(S) = 1+(m+1)\left(r+\frac{3(m+2)}{2}\right).$$

Proof. We will only prove the theorem in the case when $s \equiv 0 \pmod 3$ since the proofs of the remaining cases are similar. In this case,

$$S = \{r, r+1, r+2, \ldots, r+3m\}$$

where $m \geqslant 0$. Let G be a connected degree-continuous graph with the degree set S. First, we show that

$$|V(G)| \geqslant (m+1)\left(r+1+\frac{3m}{2}\right).$$

For each j $(r \leqslant j \leqslant r+3m)$, define

$$V_j = \{v \in V(G)\colon\ \deg v = j\}. \tag{5}$$

Then $V(G) = \bigcup\limits_{j=r}^{r+s} V_j$. Let $N(v_j)$ denote the neighborhood of a vertex $v_j \in V_j$. Then $|N(v_j)| = \deg v_j = j$, where $r \leqslant j \leqslant r+3m$. Assume first that $m = 0$. Since every graph with the degree set $\{r\}$ has at least $r+1$ vertices and the complete graph K_{r+1} is a degree-continuous graph with the degree set $\{r\}$, it follows that $m(S) = r+1$. Next assume that $m > 0$. Let $v_r \in V_r$. Since G is degree-continuous, it follows that

$$\{v_r\} \cup N(v_r) \subseteq V_r \cup V_{r+1}$$

and so

$$|V_r \cup V_{r+1}| \geqslant |N(v_r)| + 1 = r+1. \tag{6}$$

For each i $(1 \leqslant i \leqslant m-1)$, since

$$\{v_{r+3i}\} \cup N(v_{r+3i}) \subseteq V_{r+3i-1} \cup V_{r+3i} \cup V_{r+3i+1},$$

it follows that

$$|V_{r+3i-1} \cup V_{r+3i} \cup V_{r+3i+1}| \geqslant |N(v_{r+3i})| + 1 = r+3i+1. \tag{7}$$

Similarly, since $\{v_{r+3m}\} \cup N(v_{r+3m}) \subseteq V_{r+3m-1} \cup V_{r+3m}$, we have that

$$|V_{r+3m-1} \cup V_{r+3m}| \geqslant |N(v_{r+3m})| + 1 = r+3m+1. \tag{8}$$

Combining (6), (8), and (7), we obtain

$$|V(G)| \geqslant \sum_{i=0}^{m}(r+3i+1) = (m+1)\left(r+1+\frac{3m}{2}\right).$$

In order to construct the desired degree-continuous graph, we consider two cases.

Case 1. At least one of r and m is odd. In this case, we can construct a degree-continuous graph G of order $(m+1)\left(r+1+\frac{3m}{2}\right)$ with the degree set S. Let H_1 be the $(3m-1)$-path composition

$$H_1 = P_{3m-1}\left[K_1, K_r, K_1, K_2, K_{r+1}, K_2, K_3, K_{r+2}, \dots, K_{m-1}, K_m, K_{r+m-1}\right].$$

By Theorem A, there exists an $(r-3)$-regular graph F of order $r+m-1$. Let

$$H_2 = P_2[P_2[F, K_{2m+1}], K_1].$$

By Theorem B, we can construct a graph G from H_1 and H_2 by connecting the graph K_{r+m-1} in H_1 with the graph F in H_2 using an m-regular bipartite graph B such that the partite sets of B are $V_1 = V(K_{r+m-1})$ and $V_2 = V(F)$, where $|V_1| = |V_2| = r+m-1$. The graph G is shown in Figure 2, where then $|V(G)| = (m+1)\left(r+1+\frac{3m}{2}\right)$ and $\mathcal{D}(G) = S$.

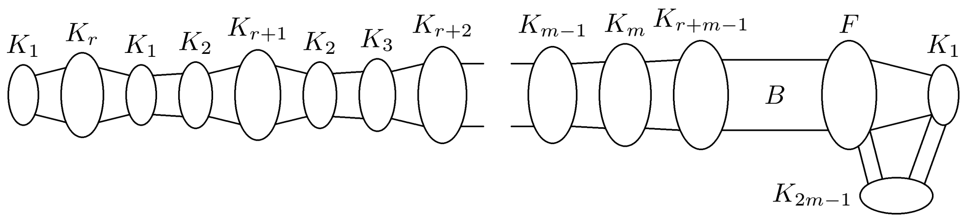

Figure 2. A degree-continuous graph G of order $(m+1)\left(r+1+\frac{3m}{2}\right)$

Case 2. r and m are both even. In this case, we construct a degree-continuous graph G' of order $1+(m+1)\left(r+1+\frac{3m}{2}\right)$ and with the degree set S. Let $H_1' = H_1$ be as described in Case 1, F' an $(r-2)$-regular graph of order $r+m-1$, and $L = K_{2m+1} - \{e_1, e_2, \dots, e_m\}$, where the edges $e_i = v_iu_i$ $(1 \leqslant i \leqslant m)$ are independent in K_{2m+1}. Now let

$$H_2' = P_2[P_2[F', L], K_1].$$

The desired graph G' is obtained from H_1' and H_2' by first connecting the graph K_{r+m-1} in H_1' with the graph F' in H_2' using an $(m-1)$-regular bipartite graph B' with its partite sets $V_1 = V(K_{r+m-1})$ and $V_2 = V(F')$ and then adding a new vertex x and edges $\{xv\colon v \in V(K_{r+m-1}) \cup \{v_1, u_1, v_2, u_2, \dots, v_m, u_m\}\}$ where K_{r+m-1} is the subgraph of H_1'. $\square$

The following corollary is a direct result of the manner in which degree-continuous graphs were constructed in Theorem 2.2.

Corollary 2.3. *For each set $S = \{r, r+1, r+2, \dots, r+s\}$ of positive integers with $s \geqslant 0$, there exists a degree-continuous graph G of minimum order such that $\operatorname{diam} G = 2s+2$.*

3. Degree-continuous graphs with prescribed induced subgraphs

It is a well known result of König [3] that every graph G can be embedded as an induced subgraph in an r-regular graph H for every integer $r \geqslant \Delta(G)$. Of course, H is a degree-continuous graph with the degree set $\mathcal{D}(H) = \{r\}$. We now provide an extension of König's theorem, the proof of which is similar to König's.

Lemma 3.1. *If G is a connected, non-regular graph and S is a convex set of positive integers where $\max\{2, \Delta(G)\} \leqslant \max(S)$, then there exists a connected degree-continuous graph H with $\mathcal{D}(H) = S$ such that H contains G as an induced subgraph.*

Proof. Let $M = \max(S)$ and $m = \min(S)$. Let G' be a second copy of the graph G. If a vertex v has degree less than M, then join v in G to the vertex corresponding to v in G'. This procedure is repeated until an M-regular graph H_M is produced. The graph H_M contains at least two disjoint copies of G as induced subgraphs. If $S = \{M\}$, then $H = H_M$. Suppose then that $S \neq \{M\}$. For $m \leqslant r < M$, let H_r denote an r-regular graph. For each r with $m \leqslant r \leqslant M-1$, delete an edge $u_r v_r$ from H_r. Furthermore, for each r with $m+1 \leqslant r \leqslant M$, delete an edge $w_r x_r$ (distinct from $u_r v_r$) from H_r. For $k = m, m+1, \ldots, M-1$, the edges $u_k w_{k+1}$ and $v_k x_{k+1}$ are added, denoting the resulting graph by H. The graph H has the desired properties. $\square$

We now present the following result.

Theorem 3.2. *For a non-regular graph G and a convex set S of positive integers containing the number 2, there exists a connected degree-continuous graph H with $\mathcal{D}(H) = S$ which contains G as an induced subgraph if and only if*
(a) *G contains no r-regular component where $r = \max(S)$, and*
(b) $\max(S) \geqslant \Delta(G)$.

Proof. First, we establish the necessity of condition (a). Suppose that G is a non-regular subgraph of a connected degree-continuous graph H. Clearly, if G is connected, then G contains no regular component. Suppose that G is disconnected and contains an r-regular component G_1 with $r = \max(S)$. Since H is connected, there exists a vertex of H not in G_1 that is adjacent to a vertex in G_1, implying that $\max(S) > r$, a contradiction. Since $\Delta(H) \geqslant \Delta(G)$, it follows that (b) is necessary as well.

For the converse, if G is connected, then the result follows from Lemma 3.1. Therefore, we assume that G is disconnected. Let $G_1, G_2, \ldots, G_k$ be components of G. We consider two cases.

Case 1: $\max(S) = 2$. Then every component of G is a path. So suppose that G_i is a path with end-vertices u_i and v_i, where $1 \leqslant i \leqslant k$. If $S = \{1, 2\}$, then we add $k-1$ new vertices w_i $(1 \leqslant i \leqslant k-1)$ and the new edges $v_i w_i$ and $w_i u_{i+1}$ to G. The resulting graph H is a path. If $S = \{2\}$, we also add a vertex w_k and two edges $u_1 w_k$ and $w_k v_k$ to G. The resulting graph H is a cycle. So H is degree-continuous with $\mathcal{D}(H) = S$. Moreover, H contains G as an induced subgraph.

Case 2: $\max(S) \geqslant 3$. By (a), every component of G contains at least one vertex of degree less than $\max(S)$. Let $u_i \in V(G_i)$ with $\deg_G u_i < \max(S)$. We add a new vertex w_{k-1} and new edges $u_{k-1} w_{k-1}$ and $w_{k-1} u_k$ to G to produce a graph G^*. Then G^* contains $k-1$ components, namely $G_1, G_2, \ldots, G_{k-2}, G'_{k-1}$, where G'_{k-1} consists of G_{k-1}, G_k, and the edges $u_{k-1} w_{k-1}$, $w_{k-1} u_k$. Since $w_{k-1} \in V(G'_{k-1})$ and $\deg_{G^*} w_{k-1} = 2 < \max(S)$, we can repeat this procedure with G^*, producing a graph with $k-2$ components. In fact, if we repeat this procedure $k-1$ times, we obtain a connected graph G' containing G as an induced subgraph. Since G' satisfies the conditions described in Lemma 3.1 we can apply Lemma 3.1 to G' and produce a graph H with the desired properties. $\square$

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