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Ivan Chajda; Jiří Rachůnek

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ANNIHILATORS IN NORMAL AUTOMETRIZED ALGEBRAS

Ivan Chajda and Jiří Rachůnek, Olomouc

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Abstract. The concepts of an annihilator and a relative annihilator in an autometrized l-algebra are introduced. It is shown that every relative annihilator in a normal autometrized l-algebra $\mathcal{A}$ is an ideal of $\mathcal{A}$ and every principal ideal of $\mathcal{A}$ is an annihilator of $\mathcal{A}$. The set of all annihilators of $\mathcal{A}$ forms a complete lattice. The concept of an I-polar is introduced for every ideal I of $\mathcal{A}$. The set of all I-polars is a complete lattice which becomes a two-element chain provided I is prime. The I-polars are characterized as pseudocomplements in the lattice of all ideals of $\mathcal{A}$ containing I.

Keywords: autometrized algebra, annihilator, relative annihilator, ideal, polar

MSC 2000: 06F05

1. Autometrized l-algebras, basic concepts

The concept of an annihilator was introduced for lattices by M. Mandelker [5] as a generalization of the concept of a pseudocomplement. Since the set of all annihilators of a lattice $\mathcal{L}$ need not form a lattice with respect to inclusion, the first author introduced in [2] the concept of the so called indexed annihilator; the set of indexed annihilators in $\mathcal{L}$ does form a lattice. Both the annihilators and the indexed annihilators characterize distributive and modular lattices. Recall the that for a lattice $\mathcal{L} = (\mathcal{L}; \vee, \wedge)$ and elements $a, b \in L$ the annihilator $\langle a, b\rangle$ is the set $\langle a, b\rangle = \{x \in L;\ a \wedge x \leqslant b\}$; an indexed annihilator in $\mathcal{L}$ is every subset of L which is the intersection of a system of annihilators of $\mathcal{L}$.

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Autometrized algebras were introduced by K. L. N. Swamy [8] as a common generalization of Brouwerian algebras and commutative lattice ordered groups (l-groups, for short). Let us recall this basic concept:

Definition. An algebraic system $\mathcal{A} = (\mathcal{A}; +, \prime, \leqslant, *)$ is called an *autometrized algebra* if

(1) $(A; +, 0)$ is a commutative monoid;
(2) $(A; +, \leqslant)$ is an ordered semigroup, i.e. $\leqslant$ is an order on A and $a \leqslant b \Longrightarrow a + c \leqslant b + c$ for all $a, b, c \in A$;
(3) $*$ is a binary operation on A satisfying

$$\begin{aligned} &a * b \geqslant 0, \\ &a * b = 0 \quad \text{if and only if} \quad a = b, \\ &a * b = b * a, \\ &a * c \leqslant (a * b) + (b * c) \end{aligned}$$

for all a, b, c in A; $*$ is called an *autometric* on A.

If, moreover, $(A, \leqslant)$ is a lattice whose operations are denoted by $\vee$ and $\wedge$ and

$$\begin{aligned} a + (b \vee c) &= (a + b) \vee (a + c), \\ a + (b \wedge c) &= (a + b) \wedge (a + c) \end{aligned}$$

for every $a, b, c \in A$, then $\mathcal{A}$ is called an *autometrized lattice algebra*, briefly an *Al-algebra*.

In this case $\mathcal{A}$ is considered to be also equipped by the lattice operations and this fact is expressed by the notation $\mathcal{A} = (\mathcal{A}; +, 0, \vee, \wedge, *)$.

However, the concept of an Al-algebra can be too general for our purpose, so we use the following specification (which was introduced by Swamy [8]):

Definition. An Al-algebra $\mathcal{A} = (\mathcal{A}; +, 0, \vee, \wedge, *)$ is called *normal* (briefly an *NAl-algebra* if

$$\begin{gathered} a \leqslant a * 0, \\ (a + c) * (b + d) \leqslant (a * b) + (c * d), \\ (a * c) * (b * d) \leqslant (a * b) + (c * d), \\ a \leqslant b \Longrightarrow \exists x \geqslant 0 \ \text{such that} \ a + x = b \end{gathered}$$

for all $a, b, c, d \in A$.

Remark.

(a) Having an abelian l-group $\mathcal{G} = (\mathcal{G}; +, 0, -, \vee, \wedge)$ we can set

$$a * b = |a - b| = (a - b) \vee (b - a)$$

for $a, b \in G$. Then $(G; +, 0, \vee, \wedge, *)$ is an *NAl*-algebra.

(b) Having a Brouwerian algebra $\mathcal{B} = (B; \vee, \wedge)$, i.e. a dually relative pseudocomplemented lattice with the greatest element (it means that for each $a, b \in B$ there is a least $x \in B$ with $b \vee x \geqslant a$), denote by $a - b$ this relative pseudocomplement x of b with respect to a and set $a * b = (a - b) \vee (b - a)$. Thus also $(B; +, 0, \vee, \wedge, *)$ is an *NAl*-algebra where $+$ denotes the lattice join $\vee$.

The concept of an ideal of an *NAl*-algebra was introduced in [9]:

Definition. Let $\mathcal{A} = (\mathcal{A}; +, 0, \vee, \wedge, *)$ be an *NAl*-algebra and $\emptyset \neq I \subseteq A$. The set I is called an *ideal* of $\mathcal{A}$ if it satisfies

$$a, b \in I \Longrightarrow a + b \in I,$$
$$a \in I, \;\; x \in A, \;\; x * 0 \leqslant a * 0 \Longrightarrow x \in I$$

for all $a, b, x \in A$.

Denote by $\mathcal{I}(\mathcal{A})$ the set of all ideals of an *NAl*-algebra $\mathcal{A}$. Following Theorem 1 in [9], $\mathcal{I}(\mathcal{A})$ is an algebraic lattice with respect to set inclusion where $\inf M = \bigcap M$ for every subset $M \subseteq \mathcal{I}(\mathcal{A})$. If $B \subseteq A$, denote by $I(B)$ the ideal of A generated by B, i.e. the least ideal of A containing B; if B is a singleton, say $\{b\}$, we will write briefly $I(b)$. Then $I(b)$ is called a *principal ideal* of $\mathcal{A}$ generated by b.

It is easy to verify that

$$I(B) = \{x \in A; \;\; x * 0 \leqslant (b_1 * 0) + \ldots + (b_n * 0); \;\; b_1, \ldots, b_n \in B\},$$
$$I(b) = \{x \in A; \;\; x * 0 \leqslant m(b * 0), \;\; \text{for} \;\; m \in \mathbb{N}\}.$$

Two elements a, b in any *NAl*-algebra $\mathcal{A}$ are said to be *orthogonal* (denoted by $a \perp b$) if

$$(a * 0) \wedge (b * 0) = 0.$$

For a subset B of A we denote by $B^{\perp}$ the set of all elements of A which are orthogonal to every element of B, i.e.

$$B^{\perp} = \{x \in A; \;\; x \perp b \;\; \text{for each} \;\; b \in B\}.$$

The set $B^{\perp}$ is called the *polar of* B. For $B = \{b\}$ we will write briefly $b^{\perp}$ instead of $\{b\}^{\perp}$. A subset C of A is called a *polar in* $\mathcal{A}$ if $C = B^{\perp}$ for some subset B of A.

Now, we specify some kinds of NAl-algebras: An NAl-algebra $\mathcal{A}$ is called

(a) *semiregular* if for every $a \in A$

$$a \geqslant 0 \Longrightarrow a * 0 = a;$$

(b) *interpolation* if for all $a, b, c \in A$, $0 \leqslant a, b, c$ and $a \leqslant b + c$ imply the existence of $b_1, c_1 \in A$ such that $0 \leqslant b_1 \leqslant b$, $0 \leqslant c_1 \leqslant c$ and $a = b_1 + c_1$.

Denote by $\mathcal{P}(\mathcal{A})$ the set of all polars of an NAl-algebra $\mathcal{A}$. It was proved in [9], Theorem 7, that for a semiregular $\mathcal{A}$ the set $\mathcal{P}(\mathcal{A})$ ordered by inclusion is a complete Boolean algebra. The properties of $\mathcal{P}(\mathcal{A})$ for an interpolation semiregular NAl-algebra $\mathcal{A}$ were investigated in [7].

On the other hand, the assumption "to be interpolation" can be omitted by virtue of Lemma 1.2 in [3]. Further, Lemma 5 in [9] enables us to omit the assumption of semiregularity in the most cases as it was done in [4], where some results on lattices $\mathcal{I}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$ are generalized to arbitrary NAl-algebras. This way will be used also here for an investigation of the above introduced concepts in a general setting.

2. Annihilators and relative annihilators

Definition. Let a, b be elements in an NAl-algebra $\mathcal{A}$. A subset

$$\langle a, b\rangle = \{x \in A;\ \ (a * 0) \wedge (x * 0) \leqslant n(b * 0)\ \text{ for some }\ n \in \mathbb{N}\}$$

will be called the *relative annihilator of a with respect to b.*

A subset B of $\mathcal{A}$ is a *relative annihilator in* $\mathcal{A}$ if $B = \langle a, b\rangle$ for some elements $a, b \in A$.

Theorem 1. *Every relative annihilator of an NAl-algebra $\mathcal{A}$ is an ideal of $\mathcal{A}$.*

P r o o f. Let $a, b, x, y \in A$ and suppose $x, y \in \langle a, b\rangle$. Then there are $n_1, n_2 \in \mathbb{N}$ such that

$$(a * 0) \wedge (x * 0) \leqslant n_1(b * 0),$$
$$(a * 0) \wedge (y * 0) \leqslant n_2(b * 0).$$

On account of normality of $\mathcal{A}$ we have

$$(a * 0) \wedge \big((x + y) * 0\big) \leqslant (a * 0) \wedge \big((x * 0) + (y * 0)\big).$$

By Lemma 1.2 in [3], this yields

$$\begin{aligned}(a*0)\wedge\big((x*0)+(y*0)\big) &\leqslant \big((a*0)\wedge(x*0)\big)+\big((a*0)\wedge(y*0)\big)\\ &\leqslant n_1(b*0)+n_2(b*0)\\ &=(n_1+n_2)(b*0),\end{aligned}$$

whence $x+y\in\langle a,b\rangle$.

It is obvious that for $z\in A$ we have $z*0\leqslant x*0\Longrightarrow z\in\langle a,b\rangle$. □

Remark.

(a) Of course, $\langle a,a\rangle=A$ for each $a\in A$, thus A is a relative annihilator of $\mathcal{A}$ for each NAl-algebra $\mathcal{A}$.

(b) If $a\in A$ then $\langle a,0\rangle=a^{\perp}$, the polar of a.

(c) The set of all relative annihilators of $\mathcal{A}$ need not be a complete lattice with respect to set inclusion. We can illuminate this fact by the following example:

Let G be an abelian l-group. For $a\in G$ we denote $|a|=a\vee -a$. Then $a*b=|a-b|$ is an autometric on G with $a*0=|a|$, thus

$$\langle a,b\rangle=\{x\in G;\ \ |a|\wedge|x|\leqslant n|b|,\ \ n\in\mathbb{N}\},$$

and hence $a^{\perp}=\{x\in G;\ |x|\wedge|a|=0\}$. Therefore polars in the autometrized algebra $\mathcal{G}$ coincide with polars in the l-group G. Recall that an element b in an l-group G is a weak unit of G if $b^{\perp}=\{0\}$.

Suppose now that the l-group G contains no weak units and let $a,b\in G$ be elements with $\langle a,b\rangle=\{0\}$. Since $|a|\wedge|b|\leqslant n|b|$ for each $n\in\mathbb{N}$, we have $n|b|=0$. Since G is torsion free, this yields $b=0$. Then $\langle a,b\rangle=\langle a,0\rangle=a^{\perp}$, i.e. $a^{\perp}=\{0\}$, a contradiction. Hence there are no elements $a,b\in G$ with $\langle a,b\rangle=\{0\}$, i.e. $\{0\}$ is not a relative annihilator of G.

On the other hand, $\{0\}=I(0)$, and, as will be shown in Theorem 4 later, every ideal generated by a singleton is the intersection of a set of relative annihilators. Altogether, $\{0\}$ is the intersection of all relative annihilators of G but it is not a relative annihilator of G.

The foregoing Remark (c) motivates us to introduce the following concept:

Definition. A subset B of an NAl-algebra $\mathcal{A}$ is called an *annihilator of* $\mathcal{A}$ if $B=\bigcap\{B_\gamma;\ \gamma\in\Gamma\}$ for a system of relative annihilators in $\mathcal{A}$.

Let us note that for lattices a different terminology was used, see [2] and [5], namely, relative annihilators in our sense are annihilators in [5] and annihilators in our sense are called indexed annihilators in [2].

Corollary 2. *Every annihilator of an NAl-algebra $\mathcal{A}$ is an ideal of $\mathcal{A}$.*

Proof. It follows from Theorem 1 and the fact that $\mathcal{I}(\mathcal{A})$ forms a lattice where meets are intersections. □

Corollary 3. *The set $Ann(\mathcal{A})$ of all annihilators of an NAl-algebra $\mathcal{A}$ forms a complete lattice with respect to set inclusion. For $B_\gamma \in Ann(\mathcal{A})$, $\gamma \in \Gamma$, we have*

$$\inf\{B_\gamma;\ \gamma \in \Gamma\} = \bigcap\{B_\gamma;\ \gamma \in \Gamma\}.$$

Applying Corollary 3, we conclude that for every NAl-algebra $\mathcal{A}$ and each subset M of A there exists the least annihilator of $\mathcal{A}$ containing M. We denote it by $A(M)$ and call it the *annihilator generated by* M.

For principal ideals of $\mathcal{A}$, we can prove

Theorem 4. *Every principal ideal of an NAl-algebra $\mathcal{A}$ is an annihilator of $\mathcal{A}$.*

Proof. Let $c \in A$ and $A(c) = A(\{c\})$. For the principal ideal $I(c)$ we clearly have $I(c) \subseteq A(c)$. Let us prove the converse inclusion. Let $z \in A(c)$. Then for every $a, b \in A$ we obviously have $c \in \langle a, b\rangle \Rightarrow z \in \langle a, b\rangle$. Since $(z * 0) \wedge (c * 0) \leqslant c * 0$, there must exist $s \in \mathbb{N}$ with $(z * 0) \wedge (z * 0) \leqslant s(c * 0)$, i.e. $z * 0 \leqslant s(c * 0)$. Then, of course, $z \in I(c)$. □

Remark.

(a) By Theorem 4, $I(0) = \{0\}$ is the least element of the lattice $Ann(\mathcal{A})$; of course, A is the greatest element of $Ann(\mathcal{A})$ by Remark after Theorem 1.

(b) By the proof of Theorem 4, $I(c) = A(c) = A(I(c))$ for each element $c \in A$.

The concept of a relative annihilator can be also generalized to subsets:

Definition. Let B, C be non-void subsets of an NAl-algebra $\mathcal{A}$. The set $\langle B, C\rangle = \bigcap\{\langle b, c\rangle;\ b \in B,\ c \in C\}$ is called the *generalized relative annihilator of B with respect to C*. A subset D of A is a *generalized relative annihilator of* $\mathcal{A}$ if $D = \langle B, C\rangle$ for some non-void subsets B, C of A.

Remark.

(a) Every relative annihilator of $\mathcal{A}$ is a generalized annihilator since $\langle a, b\rangle = \langle \{a\}, \{b\}\rangle$.

(b) Every generalized annihilator is an annihilator of $\mathcal{A}$.

(c) For every subset B of A we have $B^{\perp} = \langle B, \{0\}\rangle$, thus each polar of $\mathcal{A}$ is a generalized relative annihilator of $\mathcal{A}$.

It can be of some interest to study the set of generalized relative annihilators with a fixed second component:

Theorem 5. *Let B be a non-void subset of an NAl-algebra $\mathcal{A}$. The set of all generalized relative annihilators $\langle X, B\rangle$ where X runs over all non-void subsets of A forms a complete lattice with respect to set inclusion where infima coincide with intersections and A is the greatest element.*

Proof. Of course, $\langle\{0\}, B\rangle = A$, thus A is the greatest generalized annihilator of $\mathcal{A}$. It is an easy computation that for any non-void subsets C_γ of A we have $\bigcap\{\langle C_\gamma, B\rangle;\ \gamma \in \Gamma\} = \bigcap\{\langle c, b\rangle;\ c \in C_\gamma,\ b \in B;\ \gamma \in \Gamma\} = \bigcap\{\langle c, b\rangle;\ b \in B,\ c \in \bigcup\{C_\gamma;\ \gamma \in \Gamma\}\} = \langle\bigcup\{C_\gamma;\ \gamma \in \Gamma\}, B\rangle$. $\square$

3. I-polars

Let $\mathcal{A}$ be an NAl-algebra and a, b elements of $\mathcal{A}$. Using the concept of a principal ideal, we have

$$\langle a, b\rangle = \{x \in A;\ (a * 0) \wedge (x * 0) \in I(b)\}.$$

Since $I(0) = \{0\}$, the polar of a can be expressed by

$$a^\perp = \{x \in A;\ (a * 0) \wedge (x * 0) \in I(0)\}.$$

From this point of view, it is natural to substitute $I(0)$ by an arbitrary ideal I of $\mathcal{A}$ to obtain the following concept:

Definition. Let I be an ideal of an NAl-algebra $\mathcal{A}$ and let $a \in A$. By the *I-polar of a* we mean the set

$$a(I)^\perp = \{x \in A;\ (a * 0) \wedge (x * 0) \in I\}.$$

By the *I-polar of a non-void subset B* of $\mathcal{A}$ we mean the set

$$B(I)^\perp = \bigcap\{a(I)^\perp;\ a \in B\}.$$

A subset C is called an *I-polar of $\mathcal{A}$* if $C = B(I)^\perp$ for some non-void subset B of A.

Remark.

(a) Of course, if $I = I(0)$ then $a(I(0))^\perp = a^\perp$ and $B(I(0))^\perp = B^\perp$ for each $a \in A$ and every $\emptyset \neq B \subseteq \dot{A}$. Moreover, a subset C of A is an $I(0)$-polar of $\mathcal{A}$ if and only if C is a polar of $\mathcal{A}$.

(b) For every two elements $a, b \in A$ we have $a(I(b))^\perp = \langle a, b\rangle$ and for each subset $\emptyset \neq C \subseteq A$ we have $C(I(b))^\perp = \langle C, \{b\}\rangle$.

We are able to prove the following theorem.

Theorem 6. *Let I be an ideal of an NAl-algebra $\mathcal{A}$. The set $\mathcal{P}(I)$ of all I-polars of $\mathcal{A}$ forms a complete lattice with respect to set inclusion where infima coincide with intersections, the least element is I and the greatest one is A. Moreover, every I-polar of $\mathcal{A}$ is an ideal of $\mathcal{A}$ and for each non-void subset B of A we have $B(I)^{\perp} = \{x \in A;\ I(x) \cap I(B) \subseteq I\}$.*

Proof. Let I be an ideal of $\mathcal{A}$ and $B \subseteq A$. Denote

$$C = \{x \in A;\ I(x) \cap I(B) \subseteq I\}.$$

(a) Suppose $x \in B(I)^{\perp}$ and $z \in I(x) \cap I(B)$. Then there exist $m \in \mathbb{N}$ and elements $b_1, \ldots, b_n \in B$ such that

$$z * 0 \leqslant m(x * 0),$$
$$z * 0 \leqslant (b_1 * 0) + \ldots + (b_n * 0).$$

Hence

$$0 \leqslant z * 0 \leqslant m(x * 0) \wedge \big((b_1 * 0) + \ldots + (b_n * 0)\big)$$
$$\leqslant m\big((x * 0) \wedge (b_1 * 0)\big) + \ldots + m\big((x * 0) \wedge (b_n * 0)\big) \in I,$$

thus $z * 0 \in I$ and also $z \in I$. We have $I(x) \cap I(B) \subseteq I$, i.e. $B(I)^{\perp} \subseteq C$.

(b) Let $x \in A$ be an element satisfying $I(x) \cap I(B) \subseteq I$, let $b \in B$ and put $c = (x * 0) \wedge (b * 0)$. Then $0 \leqslant c \leqslant x * 0$ and, by Lemma 2 and Theorem 5 in [4], $I(x) = I(x * 0)$ and every ideal of $\mathcal{A}$ is a convex subset of A, i.e. $c \in I(x)$.

Analogously, $c \in I(b)$, which implies $c \in I$. Thus $x \in B(I)^{\perp}$ proving $C \subseteq B(I)^{\perp}$.

We conclude $B(I)^{\perp} = \{x \in A;\ I(x) \cap I(B) \subseteq I\}$. Suppose now $x \notin I$. Then $(x * 0) \wedge (x * 0) \notin I$ whence $x \notin A(I)^{\perp}$. Conversely, if $x \notin A(I)^{\perp}$ then there exists $a \in A$ with $(a * 0) \wedge (x * 0) \notin I$. Suppose $x \in I$. Then $x * 0 \in I$ and, on account of convexity of I, also $(a * 0) \wedge (x * 0) \in I$, a contradiction. Hence $x \notin I$. We have shown $A(I)^{\perp} = I$, i.e. $I \in \mathcal{P}(I)$. Since $B \subseteq C \subseteq A$ implies $C(I)^{\perp} \subseteq B(I)^{\perp}$, I is clearly the least element of $\mathcal{P}(I)$. Of course, A is the greatest element of $\mathcal{P}(I)$ because $\{0\}(I)^{\perp} = A$.

Let us prove that every I-polar is an ideal of $\mathcal{A}$. To this end, let $a \in A$ and $x, y \in a(I)^{\perp}$. Then $(a * 0) \wedge (x * 0) \in I$ and $(a * 0) \wedge (y * 0) \in I$. Applying the normality of $\mathcal{A}$ we have

$$0 \leqslant \big((x + y) * 0\big) \wedge (a * 0)$$
$$\leqslant \big((x * 0) + (y * 0)\big) \wedge (a * 0)$$
$$\leqslant \big((x * 0) \wedge (a * 0)\big) + \big((y * 0) \wedge (a * 0)\big) \in I.$$

Since I is convex, we obtain $\big((x+y)*0\big)\wedge(a*0)\in I$, whence $x+y\in a(I)^{\perp}$.

Suppose now $x\in a(I)^{\perp}$, $z\in A$, $z*0\leqslant x*0$. Then $0\leqslant(a*0)\wedge(z*0)\leqslant(a*0)\wedge(x*0)\in I$, i.e. also $(a*0)\wedge(z*0)\in I$, which implies $z\in a(I)^{\perp}$.

Hence $a(I)^{\perp}$ is an ideal of $\mathcal{A}$ and, moreover, for any non-void subset B of A we have $B(I)^{\perp}=\bigcap\{a(I)^{\perp};\ a\in B\}$, thus also $B(I)^{\perp}$ is an ideal of $\mathcal{A}$. This yields the fact that infima in $\mathcal{P}(I)$ coincide with intersections. □

Corollary 7. *Let I be an ideal of an NAl-algebra $\mathcal{A}$ and let C$\in\mathcal{P}(I)$. Then there exists an ideal J of $\mathcal{A}$ with $C=J(I)^{\perp}$.*

Proof. Of course, if $C=B(I)^{\perp}$ then $C=J(I)^{\perp}$ for $J=I(B)$. □

An ideal I of an *NAl*-algebra $\mathcal{A}$ is called a *prime ideal* if for each ideals J and K of $\mathcal{A}$ the implication $J\cap K=I\Longrightarrow J=I$ or $K=I$ holds. This concept was introduced by the second author in [6] where it was also shown that for $\mathcal{A}$ semiregular, I is a prime ideal of $\mathcal{A}$ if and only if $0\leqslant a\wedge b\in I\Longrightarrow a\in I$ or $b\in I$ for every a,b in A. On account of Theorem 9 in [4], this equivalent condition holds in every *NAl*-algebra. Hence we have

Corollary 8. *If I is a prime ideal of an NAl-algebra $\mathcal{A}$ then $\mathcal{P}(I)$ is the two-element chain $\{I,A\}$.*

Proof. Let I be a prime ideal of $\mathcal{A}$ and let $a\notin I$, $x\in A$. If $(a*0)\wedge(x*0)\in I$ then $x*0\in I$ and also $x\in I$. Hence $a(I)^{\perp}=I$. If $a\in I$ then $a(I)^{\perp}=A$. This yields that for $\emptyset\neq B\subseteq A$ we have only two possibilities:

$$B\nsubseteq I\Longrightarrow B(I)^{\perp}=I\quad\text{and}$$
$$B\subseteq I\Longrightarrow B(I)^{\perp}=A.$$

□

Remark. Applying Corollary 7, we can restrict ourselves to I-polars of ideals when investigating properties of arbitrary I-polars.

Let $\mathcal{A}$ be an *NAl*-algebra and I an ideal of $\mathcal{A}$. Denote

$$\mathcal{I}(\mathcal{A})_I=\{J\in\mathcal{I}(\mathcal{A});\ \ I\subseteq J\},$$

i.e. $\mathcal{I}(\mathcal{A})_I$ is the principal filter of the lattice $\mathcal{I}(\mathcal{A})$ generated by I. This fact together with Theorem 6 in [9] (stating that $\mathcal{I}(\mathcal{A})$ is a complete and Brouwerian lattice, i.e. $K\cap\bigvee_{\gamma\in\Gamma}J_\gamma=\bigvee_{\gamma\in\Gamma}(K\cap J_\gamma)$ for every $K,J_\gamma\in\mathcal{I}(\mathcal{A})$, $\gamma\in\Gamma$) immediately imply

Corollary 9. *For every ideal I of an NAl-algebra $\mathcal{A}$, $\mathcal{I}(\mathcal{A})_I$ is a complete Brouwerian lattice.*

Hence, we can ask about pseudocomplements in the lattice $\mathcal{I}(\mathcal{A})_I$.

Theorem 10. *Let I be an ideal of an NAl-algebra $\mathcal{A}$ and $J \in \mathcal{I}(\mathcal{A})_I$. Then the pseudocomplement of J in the lattice $\mathcal{I}(\mathcal{A})_I$ is $J(I)^{\perp}$.*

Proof. Since $J(I)^{\perp} \in \mathcal{P}(\mathcal{A})$, we have $I \subseteq J(I)^{\perp}$, i.e. $J(I)^{\perp} \in \mathcal{I}(\mathcal{A})_I$. Suppose $x \in J \cap J(I)^{\perp}$. Then $x \in J$ and $x \in J(I)^{\perp}$, thus $x * 0 = (x * 0) \wedge (x * 0) \in I$ whence $x \in I$. We have $J \cap J(I)^{\perp} = I$.

Let $K \in \mathcal{I}(\mathcal{A})_I$ with $J \cap K = I$. Let $x \in K$ and $a \in J$. Then $0 \leqslant (x*0) \wedge (a*0) \leqslant x * 0$. Since K is convex, this yields $(x * 0) \wedge (a * 0) \in K$. Analogously we obtain $(x * 0) \wedge (a * 0) \in J$, thus also $(x * 0) \wedge (a * 0) \in K \cap J = I$. However, this means $x \in J(I)^{\perp}$, i.e. $K \subseteq J(I)^{\perp}$. We have shown that $J(I)^{\perp}$ is the pseudocomplement of J in $\mathcal{I}(\mathcal{A})_I$. □

Applying Theorem 10 together with Glivenko's Theorem (see e.g. Theorem VIII. 4.3 in [1]), we immediately conclude

Corollary 11. *For every NAl-algebra $\mathcal{A}$ and $I \in \mathcal{I}(\mathcal{A})$, the mapping $J \longmapsto J(I)^{\perp\perp}$ is a closure operator on $\mathcal{I}(\mathcal{A})_I$. The closed subsets are just all I-polars of $\mathcal{A}$. The set $\mathcal{P}(\mathcal{A})_I$ of all I-polars of $\mathcal{A}$ is a complete Boolean algebra with respect to set inclusion.*

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Authors' address: Palacký University, Dept. of Algebra and Geometry, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mails: `chajda@risc.upol.cz`, `rachunek@risc.upol.cz`.