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ON DENSE SUBSPACES SATISFYING STRONGER SEPARATION AXIOMS

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Abstract. We prove that it is independent of ZFC whether every Hausdorff countable space of weight less than $\mathfrak{c}$ has a dense regular subspace. Examples are given of countable Hausdorff spaces of weight $\mathfrak{c}$ which do not have dense Urysohn subspaces. We also construct an example of a countable Urysohn space, which has no dense completely Hausdorff subspace.

On the other hand, we establish that every Hausdorff space of π-weight less than $\mathfrak{p}$ has a dense completely Hausdorff (and hence Urysohn) subspace. We show that there exists a Tychonoff space without dense normal subspaces and give other examples of spaces without "good" dense subsets.

Keywords: Hausdorff space, Urysohn space, completely Hausdorff space, filter of dense sets

MSC 2000: primary 54H11, 54C10, 22A05, 54D06, secondary 54D25, 54C25

0. Introduction

It is important to know whether or not a topological space has a dense subspace with "nice" properties. There are many papers devoted to prove the existence (or nonexistence) of dense subspaces which are in some way "better" than the original space. Namioka has shown [Na] that any Eberlein compact space has a dense metrizable subspace; it is independent of ZFC whether any Fréchet compact space has a dense first countable subspace [A], [M1]; Malyhin proved in [M2] that in some models

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of ZFC there are Tychonoff spaces without dense zero-dimensional subspaces, while it is not known in ZFC if any compact space has a dense zero-dimensional subspace.

In this paper we are interested in dense subspaces with stronger separation axioms (than those of the original space, of course). The first result in this direction was obtained by 80 % of the authors in [ATTW] where it was established that under Booth's lemma every Hausdorff countable space of weight less than $\mathfrak{c}$ has a dense regular (and hence Tychonoff and zero-dimensional) subspace. But the objective of the paper [ATTW] was different, so this result appears there only incidentally. In this article we give this matter a more systematic treatment considering spaces with practically all imaginable separation axioms and looking for their "nice" dense subspaces. We prove in particular that

1) any Hausdorff countable space of countable weight (respectively π-weight) has a dense regular (respectively, completely Hausdorff) subspace in ZFC;

2) there are (in ZFC) Hausdorff countable spaces without dense Urysohn (and hence without dense regular) subspaces;

3) there are countable Urysohn spaces without dense completely Hausdorff subspaces;

4) it is independent of ZFC whether or not a Hausdorff countable space of weight less than $\mathfrak{c}$ has a dense regular subspace;

5) there exists a Tychonoff space with no dense normal subspace.

1. Notation and terminology

If f and g are mapping from ω to ω (which we will denote by $f, g \in {}^\omega\omega$), then $f \leqslant^* g$ says that there exists an $N \in \omega$ such that $f(n) \leqslant g(n)$ for all $n \geqslant N$. A subset $\mathscr{F}$ of ${}^\omega\omega$ is called *unbounded* if it is not bounded with respect to the order $\leqslant^*$. The cardinal $\mathfrak{b}$ is the minimal power of an unbounded subset of ${}^\omega\omega$. A family $\mathscr{A}$ of subsets of ω is said to have *sfip* ($\equiv$strong finite intersection property) if for any finite $\mathscr{B} \subset \mathscr{A}$ the set $\cap\mathscr{B}$ is infinite. If $\mathscr{A}$ is a family of subsets of ω, then a set $B \subset \omega$ is a *pseudointersection* of $\mathscr{A}$ if $B \setminus A$ is finite for all $A \in \mathscr{A}$. The cardinal $\mathfrak{p}$ is the minimal power of a subfamily of infinite subsets of ω with *sfip* which has no infinite pseudointersection. It is known [vD] that both cardinals $\mathfrak{b}$ and $\mathfrak{p}$ are uncountable. Booth's lemma (denoted by BL) is the statement $\mathfrak{p} = \mathfrak{c}$. A space X is called *Urysohn* if any two distinct points of X have neighbourhoods with disjoint closures; if X has a weaker Tychonoff topology, then it is said to be *completely Hausdorff*; if regular open subsets of X constitute a base of X then X is referred to as *semiregular*. The symbol $\mathfrak{c}$ stands for the cardinality continuum, and $\mathbf{Q}$ denotes rational numbers. If $\{X_\alpha\colon \alpha \in A\}$ is a family of topological spaces and $y_\alpha \in X_\alpha$ for all α, then the σ-product (respectively Σ-product) with the basepoint $(y_\alpha\colon \alpha \in A)$ is the set $\{x = (x_\alpha : \alpha \in A)\colon x_\alpha = y_\alpha$ for all but finitely (countably) many $\alpha\}$. If we do not mention explicitly what the basepoint is, then it could be taken arbitrarily. If τ is a topology of a space X then $\tau^* = \tau \setminus \{\emptyset\}$.

All other notions are standard and can be found in [E].

2. The results

The following theorem has practically the same proof as Lemma 2.13 of [ATTW] but gives a result in ZFC.

2.1. Theorem. *Every countable Hausdorff space of weight less than* $\mathfrak{p}$ *has a dense regular subspace.*

Proof. Let X be a countable Hausdorff space with $w(X) < \mathfrak{p}$. Without loss of generality we may assume that X is dense in itself. It is known [vD] that $\mathfrak{p} = min\{|\mathscr{A}| + |\mathscr{B}|\colon \mathscr{A}, \mathscr{B} \subset 2^\omega$ and for any $B \in \mathscr{B}$ the family $\mathscr{A}$ restricted to B has the strong finite intersection property, while $\mathscr{A}$ has no pseudointersection A, with $A \cap B$ infinite for all $B \in \mathscr{B}\}$.

Now let $\mathscr{B}$ be a base of X of power less than $\mathfrak{p}$. Take as the family $\mathscr{A}$ the complements to the boundaries of the elements of $\mathscr{B}$. Therefore $\mathscr{A}$ has a pseudointersection A with $A \cap B$ infinite for any $B \in \mathscr{B}$. This means A is dense in X. Being the pseudointersection of $\mathscr{A}$ the set A meets the boundary of any element of $\mathscr{B}$ in a finite set. Hence A is a dense subspace of X which has a base with finite boundaries. It was proved in Lemma 2.13 of [ATTW] that this implies regularity of A. □

2.2. Corollary. *Every second countable Hausdorff space* X *contains a dense regular subspace.*

Proof. Indeed, such a space has a dense countable subspace Y. Now $w(Y) \leqslant \omega < \mathfrak{p}$ so Y and hence X has a dense regular subspace by Theorem 2.1. □

2.3. Theorem. *There exists a countable completely Hausdorff space of weight* $\mathfrak{b}$ *and countable* π*-weight with no dense regular subspace.*

Proof. By Lemma 8.4 of [vD], there is a family $\mathscr{F}$ of cardinality $\mathfrak{b}$ of compact subsets of $(\mathbf{Q}, t)$ (t is the usual topology on $\mathbf{Q}$) which has the following property:

$$(*) \qquad \begin{array}{l} \text{if } A \text{ is a non-trivial convergent sequence in } \mathbf{Q}, \\ \text{then } F \cap A \text{ is infinite for some } F \in \mathscr{F}. \end{array}$$

We define a subbase $\mathscr{B}$ for a new topology τ on $\mathbf{Q}$ as follows:

$$\mathscr{B} = t \cup \{(\mathbf{Q} \setminus \mathbf{F}) \cup \{\mathbf{x}\}\colon\ \mathbf{F} \in \mathscr{F} \ \textbf{and}\ \mathbf{x} \in \mathbf{F}\}.$$

Clearly $w(\mathbf{Q},\tau)=\mathfrak{b}$ and since $\mathbf{Q}\backslash F$ is t-open for each $F\in\mathscr{F}$, it follows that any base for t is a π-base for τ; hence $\pi w(\mathbf{Q},\tau)=\omega$. We will show that $(\mathbf{Q},\tau)$ has no regular dense subspace. To this end, suppose that D is a τ-dense subset of $\mathbf{Q}$. Fix $x\in D$; since D is dense in $(\mathbf{Q},t)$, we can find a non-trivial sequence $\{x_n\colon\ n\in\omega\}\subset D$ which is t-convergent to x. It follows from the definition of $\mathscr{F}$, that there is some $F\in\mathscr{F}$ such that $F\cap\{x_n\colon\ n\in\omega\}$ is infinite, so without loss of generality we may assume that $\{x\}\cup\{x_n\colon\ n\in\omega\}\subset F$, and hence $G=F\setminus\{x\}$ is τ-closed.

We claim that x and G have no disjoint τ-neighbourhoods. To prove this, it suffices to note that each compact subset of $(\mathbf{Q},t)$ is nowhere dense, and hence each τ-open set is τ-dense in some t-open set; thus any τ-open set containing x has τ-closure which meets G. Since $\tau\supset t$, it follows that $(\mathbf{Q},\tau)$ is completely Hausdorff and the result follows. □

2.4. Remark. In fact, even more is true. Since each τ-open set is τ-dense in some t-open set, it follows that t and τ have the same regular open sets. This is also true for any τ-dense subset D of $(\mathbf{Q},\tau)$. Since clearly $t|_D\neq\tau|_D$ and $(D,t|_D)$ is semi-regular, the space $(D,\tau|_D)$ is not semi-regular. Therefore $(\mathbf{Q},\tau)$ has no dense semi-regular subspace.

Since $\mathfrak{b}<\mathfrak{c}$ is consistent with ZFC, combining Theorem 2.3 with Lemma 2.13 of [ATTW], we obtain:

2.5. Corollary. *It is independent of ZFC whether or not every countable Hausdorff space of weight less than $\mathfrak{c}$ has a dense regular (or semi-regular) subspace.*

If a space is locally second countable, then it has a dense subspace which is the topological union of countable second countable spaces. Hence from Corollary 2.2 we obtain:

2.6. Corollary. *Every locally second countable Hausdorff space has a dense regular subspace.*

Similarly, if a space is locally separable, then it has a dense subspace which is a topological union of countable subspaces. Hence it follows immediately from Lemma 2.13 of [ATTW] that:

2.7. Theorem (BL). *Every locally separable Hausdorff space of local weight less than $\mathfrak{c}$ has a dense regular subspace.*

That these results cannot be significantly improved is clear from the following example constructed in *ZFC*.

2.8. Example. Let (X,μ) be a dense subspace of cardinality ω_1 of the Σ-product $\Sigma \subset \{0,1\}^{\omega_1}$ with base point $(0,\ldots)$ and where μ is the relative product topology. We define

$$\mathscr{F} = \{X \cap (\{0,1\}^{\alpha} \times \{0\}^{\omega_1 \setminus \alpha})\colon\ \alpha \in \omega_1\}.$$

We note that the elements of $\mathscr{F}$ are nowhere dense in X and that each separable subspace of X is contained in some element of $\mathscr{F}$. Furthermore, by a result of Noble [No], the space Σ, and hence X, is Fréchet-Urysohn.
Now define a new topology τ on X generated by the family

$$\mu \cup \{(X \setminus F) \cup \{x\}\colon\ x \in F \in \mathscr{F}\}.$$

Since (X,μ) has weight ω_1, it follows that $w(X,\tau) = \omega_1$. We claim that (X,τ) has no dense regular (nor even semiregular) subspace.

P r o o f. Suppose that D is a dense subspace of X and let $x \in D$. Since X is Fréchet-Urysohn there is some sequence $S = \{x_n\colon\ n \in \omega\}$ in D which converges to x and then there is some $F \in \mathscr{F}$ such that $S \cup \{x\} \subset F$. But then, $G = F \setminus \{x\}$ is τ-closed and the argument as in the final part of Theorem 2.3 shows that this set cannot be separated from x in (X,τ). □

2.9. Theorem. *Every countable Hausdorff space of π-weight less than $\mathfrak{p}$ has a dense completely Hausdorff (and hence Urysohn) subspace.*

P r o o f. Let A denote the set of isolated points of a countable Hausdorff space (Y,τ); then $X = Y \setminus \mathrm{cl}(A)$ is a countable dense-in-itself Hausdorff space and $\pi w(X) < \mathfrak{p}$ since X is open in Y. Let $\mathscr{P}$ be a π-base for X of cardinality less than $\mathfrak{p}$ and let $\mathscr{S}$ be a countable separating family of open sets for (X,τ), that is, $\mathscr{S} \subset \tau$ and for each pair of distinct points $x, y \in X$, there exist disjoint elements $S, T \in \mathscr{S}$ such that $x \in S$ and $y \in T$. Then $\mathscr{S} \cup \mathscr{P}$ is a subbase for a Hausdorff topology σ on X of weight less than $\mathfrak{p}$. Now by Theorem 2.1 the space (X,σ) has a dense regular subspace D which must meet each element of the π-base $\mathscr{P}$ of (X,τ); thus D is τ-dense in X. Furthermore, since $\tau \supset \sigma$ and σ is Tychonoff, it follows that $(D, \tau|_D)$ is completely Hausdorff (and hence Urysohn). Clearly $D \cup A$ is a τ-dense Urysohn subspace of Y. □

Note that if (X,τ) is a completely Hausdorff space, then X has a weaker Tychonoff topology and so if (X,τ) is countable, X has a weaker zero-dimensional topology, and hence (X,τ) is completely separated (that is, the quasicomponents of (X,τ) are singletons). The converse is clearly true as well.

2.10. Corollary. *Every Hausdorff space of countable π-weight has a dense completely Hausdorff (and hence Urysohn) subspace.*

Proof. Of course, such a space has a countable dense subspace of π-weight ω. Now use Theorem 2.9. $\square$

2.11. Corollary (BL). *Every countable Hausdorff space of π-weight less than $\mathfrak{c}$ has a dense completely Hausdorff subspace.*

We are going to prove that sufficiently large Hausdorff product spaces do have dense completely Hausdorff subspace.

2.12. Lemma. *Let κ be an infinite cardinal. Then there exists a family $\mathscr{A}$ on κ with the following properties:*
(1) $|\mathscr{A}| = 2^\kappa$;
(2) *if A and B are distinct elements of $\mathscr{A}$, then $|A \setminus B| = |B \setminus A| = \kappa$.*

Proof. It is easy to find a dense subspace T of 2^{2^κ} such that $|U \cap T| = \kappa$ for every open non-empty $U \subset 2^{2^\kappa}$. If p_α is the natural projection of 2^{2^κ} to its α-th coordinate, then let $H_\alpha = p^{-1}(0)$ for all $\alpha < 2^\kappa$. If j is a bijection between T and κ, then $\mathscr{A} = \{j(H_\alpha)\colon\ \alpha < 2^\kappa\}$ is the promised family. $\square$

2.13. Theorem. *Let (X, τ) be a Hausdorff space with $d(X) \leqslant 2^\kappa$. Then X^κ has a dense completely Hausdorff subspace.*

Proof. If X is discrete then there is nothing to prove. Fix a non-isolated point $r \in X$, distinct points $p, q \in X \setminus \{r\}$ and let U, V, W be disjoint open sets containing p, q and r respectively. Then $\mathrm{cl}(U) \cap \mathrm{cl}(V)$ is a nowhere dense closed subset of X and hence $Z = X \setminus (\mathrm{cl}(U) \cap \mathrm{cl}(V))$ is an open dense subset of X. Observe, that $r \in Z$. Let D be a dense subspace of $Z \setminus \{p, q, r\}$ of cardinality less than or equal to 2^κ.

Consider the set
$$\Sigma = \bigcup\{D^\alpha \times \{r\}^{\kappa\setminus\alpha}\colon\ \alpha \in \kappa\}.$$
Clearly Σ is dense in X^κ and has cardinality not greater than 2^κ.

The lemma 2.12 implies that there is an injective function $g\colon \Sigma \to \mathscr{A}$. Define a new subset $S \subset X^\kappa$ as follows:

For each $x \in \Sigma$, define $z_x = (z_x(\alpha)) \in X^\kappa$ by:
$z_x(\alpha) = x(\alpha)$ if $x(\alpha) \neq r$, or
$z_x(\alpha) = p$ if $x(\alpha) = r$ and $\alpha \in g(x)$, or
$z_x(\alpha) = q$ if $x_\alpha = r$ and $\alpha \notin g(x)$.
Now let $S = \{z_x\colon\ x \in \Sigma\}$.

Clearly S is a dense subspace of X and we further claim that S is completely Hausdorff. To show this, suppose that z_s and z_t are distinct points of S. Obviously $s \neq t$ and so $g(s) \neq g(t)$; thus there is some coordinate $\alpha \in \kappa$ such that $z_s(\alpha) = p$ and $z_t(\alpha) = q$. However, in this coordinate, the points p and q lie in different elements of a clopen partition of the space Z and $S \subset Z^\kappa$, so that there is a continuous real valued function on S which separates z_s and z_t. $\square$

Recall that a space is called κ-resolvable if has κ disjoint dense subspaces.

2.14. Theorem. *If X is a Hausdorff space, $d(X) = \kappa$ and Y is a κ-resolvable Urysohn space, then $X \times Y$ has a dense Urysohn subspace.*

P r o o f. Let $D = \{d_\alpha \colon \alpha \in \kappa\}$ be a dense subspace of X and let $\{D_\alpha \colon \alpha \in \kappa\}$ be a family of κ disjoint dense subspaces of Y. Define $\Delta \subset X \times Y$ by

$$\Delta = \bigcup \{\{d_\alpha\} \times D_\alpha \colon \alpha \in \kappa\}.$$

Clearly Δ is dense in $X \times Y$ and if $a, b \in \Delta$ are distinct, then their second coordinates a_2 and b_2 are distinct. Hence, since Y is Urysohn, there are disjoint closed neighbourhoods U and V of a_2 and b_2 in Y. Then $(X \times U) \cap \Delta$ and $(X \times V) \cap \Delta$ are the required closed disjoint neighbourhoods of a and b in Δ. $\square$

2.15. Corollary. *If X is a separable Hausdorff space, then $X \times \mathbf{Q}$ has a dense Urysohn (even completely separated) subspace.*

We now turn to the construction of Hausdorff spaces which have no dense Urysohn or completely Hausdorff subspaces. The following theorem shows us how such spaces can be constructed.

2.16. Theorem. *Suppose that (X, τ) is a Hausdorff (respectively, countable Hausdorff) space with the property that no non-empty open subset of X is a finite union of its Urysohn (respectively, completely Hausdorff) subspaces; then there exists a topology $\varrho \supset \tau$ on X such that (X, ϱ) has no dense Urysohn (resp. completely Hausdorff) subspace.*

P r o o f. The assumptions of the theorem imply that X is infinite. We denote by $\mathscr{D}$ the set of all dense subspaces of (X, τ) with the property that for each $U \in \tau^*$, the set $D \cap U$ is not the union of finitely many Urysohn (completely Hausdorff) subspaces. The family $\mathscr{D}$ is non-empty, since for each $x \in X$, we have $X \setminus \{x\} \in \mathscr{D}$. A *filter* in $\mathscr{D}$ is a subset $\mathscr{F}$ of $\mathscr{D}$ which is closed under finite intersections and supersets. Since the union of a chain of filters in $\mathscr{D}$ is a filter in $\mathscr{D}$, it is a simple consequence of Zorn's lemma that each filter in $\mathscr{D}$ is contained in a maximal filter in $\mathscr{D}$ which we will call an *ultrafilter* in $\mathscr{D}$. Let $\mathscr{U}$ be an ultrafilter in $\mathscr{D}$ and let ϱ be the topology on X generated by $\tau \cup \mathscr{U}$. We will prove that (X, ϱ) has no dense Urysohn (completely Hausdorff) subspaces.

To this end, suppose that D is dense in (X, ϱ); then D is dense in (X, τ) and we claim that D is not the finite union of Urysohn (completely Hausdorff) subspaces of (X, τ); that (D, τ) is not Urysohn (completely Hausdorff) is then immediate. To prove the claim, we suppose to the contrary that D is a finite union of Urysohn (completely Hausdorff) subspaces; then if $U \in \tau^*$, it follows that $U \not\subset D$, and hence $X \setminus D$ is dense in (X, τ). Now since $\mathscr{U}$ is an ultrafilter in $\mathscr{D}$, it follows that either $X \setminus D \in \mathscr{U}$, which implies that D is not dense in (X, ϱ), a contradiction, or there exist some $U \in \mathscr{U}$ and $W \in \tau^*$ such that $U \cap W \cap (X \setminus D)$ is a finite union of Urysohn (completely Hausdorff) subspaces, say

$$U \cap W \cap (X \setminus D) \subset \bigcup \{C_k \colon 1 \leqslant k \leqslant n\},$$

where for each k, C_k is a Urysohn (completely Hausdorff) subspace of (X, τ). But then

$$U \cap W \subset D \cup \bigcup \{C_k \colon 1 \leqslant k \leqslant n\},$$

that is to say, $U \cap W$ is a finite union of Urysohn (completely Hausdorff) subspaces, contradicting the properties of the ultrafilter $\mathscr{U}$.

It remains to show that $(D, \varrho|_D)$ is not Urysohn (completely Hausdorff). Observe that the topology ϱ has a base of sets of the form $U \cap V$ where $U \in \mathscr{U}$ and $V \in \tau$, and hence, since D is dense in (X, ϱ), D meets each such set $U \cap V$ so that $D \cap U$ is dense in (X, τ) (and, *a fortiori*, in $(D, \tau|_D)$) for each $U \in \mathscr{U}$. It now follows that each open set in $(D, \varrho|_D)$ is dense in some open set in $(D, \tau|_D)$ so the ϱ-closure of any ϱ-open set coincides with its τ-closure. Hence, since the topology τ is not Urysohn, neither is $(D, \varrho|_D)$.

Now if X is countable, then, by the remark following Theorem 2.9, the subspace D is completely Hausdorff iff the quasicomponents of D are singletons. However, the spaces $(D, \tau|_D)$ and $(D, \varrho|_D)$ have the same clopen sets and therefore the same quasicomponents. The space $(D, \tau|_D)$ being not completely Hausdorff, we have that $(D, \varrho|_D)$ is not completely Hausdorff either and our theorem is proved. $\square$

Thus to construct a Hausdorff space with no dense Urysohn subspace, we need first to construct a Hausdorff space in which no open set is a finite union of Urysohn subspaces. We give two examples, the second of which is countable, thus showing that Theorem 2.9 cannot be proved for π-weight $\mathfrak{c}$. In what follows, B will denote Bing's countable connected Hausdorff space [B]. Every pair of non-empty regular open sets in B have non-empty intersection; however, this space is the union of two Urysohn subspaces.

2.17. Example. Let $X = B^\omega$. Since each open set in X contains a copy of X, it suffices to show that X is not a union of finitely many Urysohn subspaces. To this end, we suppose to the contrary that $X = \bigcup \{S_k \colon 1 \leqslant k \leqslant n\}$ where for each k, the set S_k is a Urysohn subspace of X. By a theorem of Tkačenko [T], there is some $Y \subset X$ homeomorphic to X such that for some $m \leqslant n$ the set $S_m \cap Y$ fills all finite faces of Y. It is clear, that without loss of generality we can identify X and Y and assume that $S = S_m$ covers all finite faces of X. We claim that S is not Urysohn.

Proof. Suppose that $x, y \in S$; since X is Hausdorff we can find disjoint basic open sets U, V in X with $x \in U$ and $y \in V$; say $U = \bigcap\{{p_k}^{-1}(U_j)\colon 1 \leqslant j \leqslant l\}$ and $V = \bigcap\{{p_k}^{-1}(V_k)\colon 1 \leqslant k \leqslant m\}$, where for all j, k, the sets U_j and V_k are open in B and p_k is the projection onto k-th coordinate; without loss of generality, in the sequel we will assume that $l = m$. Now let $C = X^m$ be the finite face determined by the coordinates $1, \ldots, m$ and let p_C denote the projection from X onto C; since S fills all finite faces of X, $p_C\colon S \to C$ is an open surjection, and hence $p_C(U)$ and $p_C(V)$ are open subsets of C. Now, since in B all pairs of regular closed sets have non-empty intersection, it is clear that $\mathrm{cl}(p_C(U)) \cap \mathrm{cl}(p_C(V)) \neq \emptyset$; then since $\mathrm{cl}(U) = {p_C}^{-1}[p_C(\mathrm{cl}(U))]$ and $\mathrm{cl}(V) = {p_C}^{-1}[p_C(\mathrm{cl}(V))]$ and S fills the finite face C, it follows that $\mathrm{cl}(U) \cap \mathrm{cl}(V) \cap S \neq \emptyset$. □

Before proceeding with the construction of a countable example, we need the following lemma:

2.18. Lemma. *Let Y be the σ-product of countably many copies of some space B. If $Y = \bigcup\{M_i\colon 1 \leqslant i \leqslant n\}$, then for some j, the set M_j fills all finite faces of Y.*

Proof. Suppose to the contrary that Y is a union of n subspaces, each of which fails to fill all finite faces. Without loss of generality, we will assume that n is minimal with respect to this property. Now M_1 fails to fill some finite face B^J of the σ-product, and hence there is some $x \in Y$ such that $p_J(x) \notin p_J(M_1)$. Thus $Y \cap {p_J}^{-1}(p_J(x)) \subset \bigcup\{M_i\colon 2 \leqslant i \leqslant n\}$. However, $Y \cap {p_J}^{-1}(p_J(x))$ is homeomorphic to Y which contradicts the minimality of n. □

2.19. Example. Again, taking B to be Bing's countable connected space, the space Y of the previous lemma is a countable Hausdorff space; we claim that no open subset of Y is a finite union of Urysohn subspaces.

Proof. To prove this, since each open subset of Y contains a copy of Y, it clearly suffices to show that Y is not the union of finitely many Urysohn subspaces. To this end, we suppose to the contrary that $Y = \bigcup\{M_i\colon 1 \leqslant i \leqslant n\}$ where each subspace M_i is Urysohn. By the previous lemma, one of these subspaces, say M_j fills all finite faces of Y. An argument similar to that given in Example 2.17 now shows that M_j is not Urysohn. □

The method given by Theorem 2.16 makes it possible to construct countable Urysohn spaces without dense completely Hausdorff subspaces. Before producing the example, let us develop first the necessary technique. It is clear that all we need is a Urysohn countable space no open non-empty subspace of which is a union of finitely many completely Hausdorff subspaces.

2.20. Proposition. *Suppose that X is any (countable) space, which can not be represented as a union of finitely many completely Hausdorff subspaces. Then the*

σ-product S of X^ω is a (countable) space, no non-empty open subspace of which is a union of finitely many completely Hausdorff subspaces.

P r o o f. This is evident, because X embeds in every open subset of S. $\square$

The following proposition is obvious.

2.21. Proposition. *Suppose that for every $n \in \omega$ we have a (countable) space X_n which is not representable as a union of n completely Hausdorff subspaces. Then the discrete union $X = \bigoplus\{X_n\colon\ n \in \omega\}$ is a (countable) space which can't be represented as a finite union of completely Hausdorff subspaces.*

The following method of producing connected spaces is a generalization of the construction used in [RO].

2.22. Theorem. *For $n \geqslant 1$ let X be a countable Urysohn space, no non-empty open subset of which is a union of n completely Hausdorff subspaces. Choose countably many disjoint σ-products $\{S_j\colon\ j \in \omega\}$ from X^ω and let $a \notin \bigcup\{S_j\colon\ j \in \omega\}$. Introduce a topology τ on the set $S(X) = \{a\} \cup \{S_j\colon\ j \in \omega\}$ in the following way:*

(1) if j is even and $x \in S_j$, then the basic neighbourhoods of x are given by $E(U) = U \cap S_j$, where $x \in U$ and U is an open set in X^ω;

(2) if j is odd and $x \in S_j$, then the basic neighbourhoods of x are given by $O(U) = U \cap (S_{j-1} \cup S_j \cup S_{j+1})$, where U is an open neighbourhood of x in X^ω;

(3) the basic neighbourhoods of a are the sets $\{a\} \cup \bigcup\{S_j\colon\ j \geqslant m, m \in \omega\}$.

Then $(S(X), \tau)$ is a Urysohn space which can not be represented as a union of $\leqslant (n+1)$ completely Hausdorff subspaces.

P r o o f. Suppose that $S(X) = C_1 \cup \ldots \cup C_{n+1}$, where all C_i's are completely Hausdorff. Then for each $k = 1, \ldots, n+1$ and for all $i \in \omega$ the set $C_k \cap S_j$ is dense in S_j because otherwise some open set of S_j would be covered with $\leqslant n$ completely Hausdorff subspaces, while any open non-empty subspace of S_j contains subspaces homeomorphic to X.

If $a \in C_m$, then we claim that C_m is connected contradicting the fact that it is completely Hausdorff and countable.

The proof of connectedness of C_m runs essentially as in [RO]: if a clopen set U of C_m contains a then it contains all $C_m \cap S_j$ for all $j \geqslant p$, where $p \in \omega$. If p is odd, then by openness, the set U has to contain some open dense subset of $C_m \cap S_{p-1}$. The set U is also closed, so it contains the whole set $C_m \cap S_{p-1}$. If $p > 0$ is even, then U contains the closure of $C_m \cap S_p$ which is $C_m \cap C_{p-1}$ and therefore $C_m \cap S_{p-1} \subset U$. Going on like this we infer that $U = C_m$. $\square$

2.23. Corollary. *There exists a countable Urysohn space without dense completely Hausdorff subspaces.*

P r o o f. There exists a countable connected (and hence not completely Hausdorff) Urysohn space X_1 (see [RO]). Suppose that we have countable Urysohn spaces $X_1, \ldots, X_n$ such that X_i is not a union of i completely Hausdorff subspaces.

Let T be a σ-product of the spaces $X_n, n \in \omega$ and $X_{n+1} = S(T)$, where $S(T)$ is defined as in Theorem 2.21. Using Theorem 2.21 we conclude that X_{n+1} is not a union of $(n+1)$ completely Hausdorff subspaces. Applying Propositions 2.20 and 2.21 we see that there exists a space in which no open non-empty subset can be represented as a finite union of completely Hausdorff subspaces. Now apply Theorem 2.16. $\square$

2.24. Theorem. *There exists a dense subspace X of the Tychonoff cube $I^{\mathfrak{c}}$ ($I = [0,1]$) with the following properties:*

(1) *X is connected and $|X| = \mathfrak{c}$;*
(2) *X is a union of countably many closed discrete subspaces;*
(3) *if $A \subset X$ and $|A| < \mathfrak{c}$, then A is discrete and closed in X;*
(4) *no dense subspace of X is normal.*

P r o o f. We follow the main idea of E.A. Reznichenko's paper [RE]. Let T be a set of power $\mathfrak{c}$. Represent T as a disjoint union of subsets $T_\alpha,\ \alpha < \mathfrak{c}$, such that $|T_\alpha| = \mathfrak{c}$ for every $\alpha < \mathfrak{c}$. Enumerate as $P = \{p_\alpha\colon\ \alpha < \mathfrak{c}\}$ the elements of the disjoint union of all finite faces of I^T in such a way that every point be enumerated $\mathfrak{c}$ times. If p_α is one of those points, let $S_\alpha \subset T$ be the finite set of coordinates which defines the face p_α belongs to.

For every $\alpha < \mathfrak{c}$ define the point x_α in the following way:

$$x_\alpha(t) = \begin{cases} 0, \text{ if } t \in T \setminus (T_\alpha \cup S_\alpha) \\ p_\alpha(t), \text{ if } t \in S_\alpha \\ 1, \text{ if } t \in T_\alpha \setminus S_\alpha. \end{cases}$$

Let $X = \{x_\alpha\colon\ \alpha < \mathfrak{c}\}$. It is clear that X is dense in I^T. To establish that it is connected, assume the contrary. Then there is a continuous surjective function from X onto the discrete two-point space. Since X is dense in I^T, this function depends only on countably many coordinates [A1] and hence there is a countable $D \subset T$ such that $\pi_D(X)$ is not connected (where $\pi_D\colon\ I^T \to I^D$ is the natural projection). We will prove that $\pi_D(X)$ contains the σ-product σ_D of I^D thus obtaining a contradiction, because σ_D is connected and dense in I^D.

Let S be a finite subset of D and take any $b \in I^S$. There are only finitely many $\alpha < \mathfrak{c}$ with $T_\alpha \cap S \neq \emptyset$. Denote by B the set of those α. The set of ordinals $\beta < \mathfrak{c}$ for which $S_\beta = S$ and $p_\beta = b$ has the power $\mathfrak{c}$, so pick a $\beta \notin B$ with $S_\beta = S$ and $p_\beta = b$. It is clear that $\pi_S(x_\beta) = b$ and $x_\beta(t) = 0$ for each $t \in D \setminus S$, so we are done.

To prove (2) consider for every $n \in \omega$ the set $F_n = \{x_\alpha \in X\colon\ |S_\alpha| = n\}$. Let us establish that F_n is closed and discrete in X for all n.

Indeed, take any $\beta < \mathfrak{c}$. Pick different points $t_1, \dots, t_{n+1}$ from $T_\beta \setminus S_\beta$ and consider the set

$$W = \{x \in X \colon \; x(t_i) > \tfrac{1}{2} \text{ for all } i = 1, \dots, n+1\}.$$

It is immediate, that W is an open neighbourhood of x_β in X and that W does not contain more than one point from F_n. The latter being true for any β we infer that F_n is closed and discrete in X. Of course, $\bigcup\{F_n \colon \; n \in \omega\} = X$, so (2) is proved.

If $A \subset X$ and $|A| < \mathfrak{c}$, then for any $\beta < \mathfrak{c}$ there is a point $t \in T_\beta$ such that if $x_\alpha \in A$, then $t \notin S_\alpha$. Now the set $W = \{x \in X \colon \; x(t) > \frac{1}{2}\}$ is an open neighbourhood of x_β which intersects at most one element of A and therefore (3) is established.

Finally, if Y is dense in X, then in view of (1) and (3) the cardinality of Y is $\mathfrak{c}$. It follows from (2) that there is a $Z \subset Y$ which is closed and discrete in X and with $|Z| = \mathfrak{c}$. If Y were normal, then every subset of Z could be functionally separated from its complement, so there would be at least $2^{\mathfrak{c}}$ distinct real-valued continuous functions on Y. But it is false, because each function on Y is a composition of a projection onto a countable face and some continuous function from this face to the reals [A1]. This shows that there are only $\mathfrak{c}$ many continuous functions on Y and this contradiction proves (4) and the theorem. □

There are much simpler examples to illustrate the absence of dense subspaces with higher axioms of separation.

2.25. Proposition. *There exist*

(1) *a normal space without dense hereditarily normal subspaces;*

(2) *a hereditarily normal space without dense perfectly normal subspaces.*

Proof. (1) Let X be the σ-product lying in $2^{\mathfrak{c}}$ with the base point all of whose coordinates are zeros. The following facts about X are well known (and easy to prove):

(i) X is σ-compact and hence normal;

(ii) X is σ-discrete;

(iii) any dense subset of X has power $\mathfrak{c}$.

Now if Y is a dense subset of X, then by (ii) and (iii) it has a discrete subset Z of power $\mathfrak{c}$. Then the set $H = Y \setminus (\overline{Z} \setminus Z)$ is a dense subset of $2^{\mathfrak{c}}$ and Z is closed and discrete in H. Reasoning as in the final part of Theorem 2.24 we conclude that H is not normal and hence Y is not hereditarily normal.

(2) Take any linearly ordered space X without points of first countability (this could be, for example the set 2^{ω_1} with the lexicographic order). The space X is hereditarily normal [E, Problem 2.7.5(c)]. If X had a dense perfectly normal subspace Y, then this subspace would have countable pseudocharacter and hence countable character. Therefore every point of Y would have countable character in X and this is a contradiction. □

3. Unsolved problems

Below we list some problems we failed to solve in the process of preparation of this paper. Some of them might be difficult and some not, but solutions to all of them seem to require methods different from those used above. It is worth mentioning as well that the topic in question is so general and natural that it would not be surprising at all if some of them had already been asked (or even published) elsewhere. For this reason we make no claim as to originality.

3.1. Problem. *Suppose $X \times Y$ has a dense regular subspace. Do X and Y have such a subspace?*

3.2. Problem. *Does every Hausdorff space have a dense hereditarily disconnected subspace?*

3.3. Problem. *Does every hereditarily disconnected Hausdorff space have a dense completely separated subspace?*

3.4. Problem. *Is it true in ZFC that every countable Hausdorff space of weight less than $\mathfrak{b}$ has a dense regular subspace?*

3.5. Problem. *Does there exist in ZFC a countable Hausdorff space of weight $\mathfrak{p}$ with no dense regular subspace?*

3.6. Problem. *Is it true in ZFC that every countable Hausdorff space of π-weight less than $\mathfrak{c}$ has a dense completely Hausdorff (or Urysohn) subspace?*

3.7. Problem. *Is it true in ZFC that every countable Hausdorff space of π-weight less than $\mathfrak{b}$ has a dense completely Hausdorff (or Urysohn) subspace?*

3.8. Problem. *Does there exist in ZFC a countable Hausdorff space of π-weight $\mathfrak{p}$ (or of π-weight $\mathfrak{b}$) with no dense Urysohn subspace?*

3.9. Problem. *Does every regular space have a dense Tychonoff subspace?*

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