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ON SOME CLASSES OF MODULES

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Abstract. The aim of this paper is to investigate quasi-corational, comonoform, copolyform and α-(co)atomic modules. It is proved that for an ordinal α a right R-module M is α-atomic if and only if it is α-coatomic. And it is also shown that an α-atomic module M is quasi-projective if and only if M is quasi-corationally complete. Some other results are developed.

Keywords: quasi-corational module, copolyform module, α-coatomic module

MSC 2000: 16D10, 16D99

1. Introduction

Throughout the paper all rings will have identities and all modules will be unital right modules. Let R be a ring and M an R-module. We write $\mathrm{Rad}(M)$ and $E(M)$ for the radical and injective hull of M, respectively, and $J(R)$ for the Jacobson radical of R. We write $N \leqslant M$ for N a submodule of M and $N \ll M$ for $N \leqslant M$ and N small in M, equivalently $M = N + K$ for some $K \leqslant M$ implies $K = M$.

Let M be a module and N a proper submodule of M. We call M a quasi-corational extension of N in the case $\mathrm{Hom}(M, N/K) = 0$ for each submodule K of N. M is called quasi-corationally complete if for each proper submodule N of M and for any $V \leqslant N$ with $\mathrm{Hom}(M, V/K) = 0$ for all $K \leqslant V$, any homomorphism from M to N/V lifts to a homomorphism from M to N.

Let $\mathbb{Z}, \mathbb{Q}$ denote the integers and rational numbers, respectively. $\mathbb{Q}$ is a quasi-corational extension of $\mathbb{Z}$ as a $\mathbb{Z}$-module since $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}/K) = 0$ for all $K \leqslant \mathbb{Z}$.

A module M is called coatomic whenever, provided $\mathrm{Rad}(M/N) = M/N$ for $N \leqslant M$, we have $M/N = 0$ (see for example Exer.9, Page 239 in[4]). It is easy to check that M is coatomic if and only if each submodule of M is contained in a maximal submodule. Any homomorphic image of a coatomic module is coatomic. Every ring R is a coatomic right R-module.

We say that M is comonoform (copolyform resp.) if M is a quasi-corational extension of every(small) submodule N with $N \neq M$. A homomorphic image of any comonoform module is comonoform, and since an inverse image of a small module need not be small, a homomorphic image of a copolyform module is not always copolyform. Every comonoform module is copolyform.

Let M denote the $\mathbb{Z}$-module $\mathbb{Z}$. Since the only small submodule of M is zero, then M is copolyform but M is not comonoform since $\mathrm{Hom}(\mathbb{Z}, 2\mathbb{Z}/4\mathbb{Z}) \neq 0$.

2. Results

Lemma 1. *Let M be a quasi-corational extension of a submodule N. Then N is small in M.*

P r o o f. Let K be a submodule of M such that $M = K + N$. Then $M/K \cong N/N \cap K$ and so there is a homomorphism f from M onto $N/N \cap K$. Since M is a quasi-corational extension of N we have $f = 0$. Hence $N = N \cap K \leqslant K$ and $K = M$. Thus N is small in M. □

Let $N \leqslant M$. If for all proper submodules V of N, N/V is not small in M/V then N is called a coclosed submodule of M [see for example [7]]. If $M = K + N$ and $K \cap N$ is small in N for some submodule K of M then N is called a supplement of K in M. M is called amply supplemented if for any submodules A, B of M with $M = A + B$, A has a supplement in B, that is, there exists a submodule C of B such that $M = A + C$ and $A \cap C$ is small in C. Cf. [10] and [6] in which amply supplemented is called supplemented.

Lemma 2. *Let M be a module. Assume M is a quasi-corational extension of some submodule N. Then N is not coclosed in M.*

P r o o f. Let M be a quasi-corational extension of some submodule N. Assume N is coclosed in M. Then we can find a nonzero submodule K of N such that $N/K + L/K = M/K$ for some $L \leqslant M$ and $L/K \neq M/K$. Then there exists a homomorphism f from M onto $N/N \cap L$. By assumption $f = 0$, and so $N = N \cap L \leqslant L$. Thus $L/K = M/K$. This is a contradiction. □

Lemma 3. *Let M be an amply supplemented module. A submodule N of M is coclosed in M if and only if N is a supplement in M.*

P r o o f. Assume N is a coclosed submodule of M. Since $M = N + M$ and M is amply supplemented, N has a supplement L in M and L has a supplement K in N.

Then it is easily checked that N/K is small in M/K. By assumption $N/K = 0$, and so N is a supplement of L in M. Conversely let U be a submodule of M such that $M = U + N$ and $U \cap N$ is small in N. By hypothesis N has a supplement T in U or $M = T + N$, $T \cap N$ is small in T and $T \leqslant U$. Let $V \leqslant N, V \neq N$. Then $M \neq V + T$ and $M = N + T + V$. Hence $M/V = N/V + (T + V)/V$, and so N/V is not small in M/V. Thus N is coclosed in M. □

Lemma 4. *Let M be a module and V a submodule in M. Assume V is a coatomic module. Then the following are equivalent:*

(1) *V is coclosed in M.*
(2) *For every maximal submodule X of V, V/X is a direct summand of M/X.*

Proof. (1) ⇒ (2): Let X be a maximal submodule of V. By (1) V is coclosed and so V/X is not small in M/X or $M/X = V/X + L/X$ for some $L \lneqq M$. Since V/X is simple we have $(V/X) \cap (L/X) = 0$. Hence V/X is a direct summand of M/X.

(2) ⇒ (1): Let X be a nonzero submodule of V such that V/X is small in M/X. Since V is coatomic, then V/X is coatomic and so V/X contains a maximal submodule Y/X. By (2) $(V/Y) \oplus (L/Y) = M/Y$ for some submodule L of M. Consider the map $f\colon M/X \to M/Y$ defined by $f(m + X) = m + Y (m \in M)$. Then $f(V/X) = V/Y$. Since V/X is small in M/V and any homomorphic image of a small module is small, V/Y is small in M/Y. Hence $L/Y = M/Y$ and so $V = Y$. This is a contradiction since Y is a maximal submodule of V. It follows that V/X is not small for all proper submodules X of V. Hence V is coclosed. □

A module M is called hollow whenever every submodule N of M with $N \neq M$ is small in M, that is, for any submodule K of M, $M = N + K$ implies $K = M$.

Lemma 5. *Let M be a comonoform module. Then M is hollow.*

Proof. Let N be a submodule of a comonoform module M with $N \neq M$. Assume $M = N + L$ for some submodule L of M. Then there exists a homomorphism f from M onto $N/N \cap L$. By hypothesis $f = 0$, and so $N/N \cap L = 0$. Hence $L = M$. Thus M is hollow. □

There are submodules of comonoform modules which are not comonoform.

Example 6. Let M denote the Prüfer p-group $\mathbb{Z}(p^\infty)$ for some prime integer p. It is known that for any submodule N with $N \neq M$, $M/N \cong M$. Let N be a submodule with $N \neq M$ and L any submodule of N and $f \in \operatorname{Hom}(M, N/L)$. Set $K = \operatorname{Ker}(f)$. Assume $f \neq 0$. Then M/K is isomorphic to a submodule of N/L which is Noetherian. This is a contradiction since $M \cong M/K$. Then M is comonoform. Let $N_t = (1/p^t + \mathbb{Z})\mathbb{Z}$ denote the submodule of M such that $p^t N_t = 0$ where t is a positive integer with $t \geqslant 4$. Let m and n be positive integers such that $m < n < t$. Then there exists a nonzero homomorphism f from N_t to N_n/N_m defined by $f(a/p^t + \mathbb{Z}) = a/p^n + N_m$ where $a/p^t + \mathbb{Z} \in N_t$. Hence N_t is not comonoform.

Lemma 7. *Let M be a comonoform module and N a submodule of M with $N \neq M$. If for any submodules K, L of N with $K \leqslant L$, L/K is M-injective then N is comonoform.*

Proof. Let K, L be submodules of N such that $K \leqslant L$ and $L \neq N$ and $f \in \operatorname{Hom}(N, L/K)$. Since L/K is M-injective f extends to a homomorphism $g \in \operatorname{Hom}(M, L/K)$. By hypothesis $g = 0$. Then N is comonoform. □

Lemma 8. *Let M be a hollow and copolyform module. Then M is comonoform.*

Proof. Let N be a proper submodule of M. Then N is small in M, and so N/K is small in M/K for all $K \leqslant N$. Since M is copolyform we have $\operatorname{Hom}(M, N/K) = 0$. Hence M is comonoform. □

Lemma 9. *Let M be a module. Then M is copolyform if for all submodules N of M, $\operatorname{Im}(f)$ is coclosed in M/N for all $f \in \operatorname{Hom}(M, M/N)$ with $\operatorname{Im}(f) \neq M/N$.*

Proof. Assume M is not copolyform. Then there exists a nonzero homomorphism f in $\operatorname{Hom}(M, N/K)$ for some small submodule N in M and some submodule K of N. Then N/K and so $\operatorname{Im}(f) = L/K$ is small in M/K as a submodule of N/K. Let L_1/K be any submodule of L/K. Then L/L_1 is small in M/L_1. Hence $\operatorname{Im}(f)$ is not coclosed. □

Lemma 10. *Let M be a module. Then the following are equivalent:*

(1) *M is comonoform.*
(2) *For any nonzero submodule N of M, every nonzero homomorphism f from M to M/N is onto.*

Proof. (1) ⇒ (2): Let N be a nonzero submodule of M and $f\colon M \to M/N$ a nonzero homomorphism. Set $\operatorname{Im}(f) = L/N$. If $L \neq M$, then $f \in \operatorname{Hom}(M, L/N)$ and so $f = 0$ by (1). Hence f must be onto.

(2) ⇒ (1): Let K and N be submodules of M such that $K \leqslant N$, $N \neq M$ and $f \in \operatorname{Hom}(M, N/K)$. Then by (2) we have $f = 0$ or f is onto. It follows that M is comonoform. □

Lemma 11. *Let R be a commutative ring and M a local module with* $\operatorname{Rad}(M)$ *a small submodule of M. Then M is not copolyform.*

P r o o f. Let M be a local module over a commutative ring R having $\operatorname{Rad}(M) \neq 0$ as a small submodule. Then $M = mR$ for some $m \in M$. Let $0 \neq x \in \operatorname{Rad}(M)$. Define $f\colon M \to \operatorname{Rad}(M)$ by $f(mr) = xr (r \in R)$. It is clear that f is a nonzero homomorphism from M to $\operatorname{Rad}(M)$. Since M is local and so hollow and $\operatorname{Rad}(M)$ is small, hence M is not copolyform. □

Example 12. Let n be a positive integer. Since the only small submodule of $\mathbb{Z}$ is 0, then $\mathbb{Z}$ is a copolyform $\mathbb{Z}$-module. But by Lemma 11 we have $\mathbb{Z}/n\mathbb{Z}$, which is a homomorphic image of $\mathbb{Z}$ as a $\mathbb{Z}$-module is not copolyform.

It is clear that every projective module is quasi-corationally complete. We prove the converse for comonoform modules.

Lemma 13. *Let M be a comonoform quasi-corationally complete module. Then M is a quasi-projective module and* $\operatorname{End}(M)$ *is a division ring.*

P r o o f. Suppose that M is a comonoform quasi-corationally complete module. Let N be a proper submodule of M and $f\colon M \to M/N$ a homomorphism. By hypothesis $\operatorname{Hom}(M, N/K) = 0$ for all $K \leqslant N$, and then f lifts to a homomorphism g from M to M. Hence M is quasi-projective. For the last part let $0 \neq f \in \operatorname{End}(M)$. Since M is comonoform hence by Lemma 10 f is epic. Since M is quasi-projective then we can find an $h \in \operatorname{End}(M)$ such that $fh = 1$. Since M is comonoform, h is also epic, and then there exists $g \in \operatorname{End}(M)$ such that $gf = 1$. Hence $g = h$ and f has an inverse. Thus $\operatorname{End}(M)$ is a division ring. □

Note that there are quasi-projective modules which are not comonoform.

Example 14. Let m and n be distinct positive integers and let the function $f\colon \mathbb{Z} \to m\mathbb{Z}/mn\mathbb{Z}$ be defined by $f(t) = mt + mn\mathbb{Z} (t \in \mathbb{Z})$. Then f is a nonzero homomorphism. Hence $\mathbb{Z}$ is not comonoform as a $\mathbb{Z}$-module. Since $\mathbb{Z}$ is a (quasi)-projective $\mathbb{Z}$-module, $\mathbb{Z}$ is quasi corationally complete.

Corollary 15. *Let R be a ring such that R is a comonoform R-module. Then R is a division ring.*

P r o o f. Since every quasi-projective module is quasi-corationally complete, Corollary follows from Lemma 13. □

Definition 16. Let P be an ideal of a ring R. If R/P is a comonoform right R-module we call P a cocritical right ideal.

Theorem 17. *Let R be a ring and P an ideal. Then the followings are equivalent:*

(1) *P is a cocritical right ideal.*

(2) *R/P is a division ring.*

Proof. (1) $\Rightarrow$ (2): Let $\bar{x}$ be a nonzero element in R/P. Then $x \notin P$ and define $f\colon R/P \to (xR+P)/P$ by $f(\bar{r}) = xr + P$ where $\bar{r} \in R/P$. By (1) $f = 0$ and then $x \in P$. This is a contradiction. Hence $R/P = \bar{x}(R/P)$ for $\bar{0} \neq \bar{x} \in R/P$. Thus R/P is a division ring.

(2) $\Rightarrow$ (1): Assume that R/P is a division ring. Let $L/P \leqslant K/P \lneqq R/P$ be submodules and let $0 \neq f \in \operatorname{Hom}(R/P, K/L)$. Let $x \in K$ be such that $f(\bar{1}) = f(1+P) = x + L$. Then $x \notin L$ and $(x+P)(y+P) = 1 + P$ for some $y \in R$. Hence $xy - 1 \in P \leqslant L$ and $f(\bar{1})y = f(\bar{y}) = xy + L = 1 + L \in K/L$. Thus $1 \in K$ and so $K = R$. This is a contradiction. It follows that $\operatorname{Hom}(R/P, K/L) = 0$ for all submodules K and L of R with $L/P \leqslant K/P \lneqq R/P$ and then R/P is comonoform and P is a cocritical right ideal. □

Theorem 18. *Let R be a ring such that each R-module has no quasi-corational extension. Then:*

(1) *Each R-module has a proper radical.*

(2) *Each R-module is coatomic.*

Proof. (1): Let M be a module and $0 \neq m \in M$. Let H be a maximal submodule in M with respect to $m \notin H$. Let T be the intersection of proper submodules of M containing H properly. Then $m \in T$ and T/H is a simple module. By hypothesis M is not a quasi-corational extension of T. We claim $\operatorname{Hom}(M, T/H) \neq 0$. Otherwise, $\operatorname{Hom}(M, T/H) = 0$. Then for all submodules X of H, $\operatorname{Hom}(M, T/X) = 0$, and so $\operatorname{Hom}(M, H/X) = 0$. Hence M is a quasi-corational extension of H. This contradicts the hypothesis. Let f be a nonzero element of $\operatorname{Hom}(M, T/H)$. Then $\operatorname{Ker}(f)$ is a maximal submodule of M. This proves (1).

(2): Let M be a module and N a submodule of M. By (1), M/N has a proper radical. Hence M/N has a maximal submodule, and so N is contained in a maximal submodule of M. □

Let M be a module. $k^0(M)$ will stand for the dual Krull dimension of M as defined in (for example) [1, 5, 8]. M is called α-atomic for some ordinal α if $k^0(M) = \alpha$ and for any proper submodule N of M, $k^0(N) < \alpha$. M is a Noetherian module if and only if $k^0(M) \leqslant 0$ [1]. We call M α-coatomic if M/N is α-atomic for all proper submodules N of M for some ordinal α. It is clear from the definitions that 0-coatomic modules and 1-coatomic modules are coatomic modules.

As an easy reference we record

Lemma 19. *(see [1]) Let $0 \to N \to M \to K \to 0$ be a short exact sequence of R-modules. Then $k^0(M) = \max\{k^0(N), k^0(K)\}$.*

Lemma 20. *Let M be a module. Then for some ordinal α, M is α-atomic if and only if M is α-coatomic.*

Proof. Suppose that M is α-atomic. Then $k^0(M) = \alpha$ and $k^0(N) < \alpha$ for all submodules N with $N \neq M$. Let $N \subsetneqq M$. Since $k^0(M) = \max\{k^0(N), k^0(M/N)\}$, then $k^0(M/N) = \alpha$. Let $N \leqslant L \subsetneqq M$. Then $k^0(L/N) \leqslant k^0(L) < \alpha$. Hence M is α-coatomic. Conversely, suppose that M is α-coatomic. Then $k^0(M/N) = \alpha$ and $k^0(L/N) < \alpha$ for all $N \leqslant L \subsetneqq M$. For $N = 0$, we have $k^0(M/N) = k^0(M) = \alpha$, and for any $L \subsetneqq M$, $k^0(L/N) = k^0(L) < \alpha$. Hence M is α-atomic. □

Theorem 21. *Let M be an α-atomic module. Then M is comonoform.*

Proof. Let N be a proper submodule of M and let $0 \neq f \in \mathrm{Hom}(M, N/K)$ for some $K \leqslant N$. Then $k^0(M) = \alpha$ and $k^0(N) < \alpha$ and $f(M) = L/K$ for some $L \leqslant N$ with $K \leqslant L \leqslant N$. Since $f(M) \cong M/\mathrm{Ker}(f)$ we have by Lemma 19 $k^0(M) = \max\{k^0(f(M)), k^0(\mathrm{Ker}(f))\} = k^0(f(M)) \leqslant k^0(N/K) \leqslant k^0(N) < \alpha$. It is a contradiction. Hence $f = 0$ and M is comonoform. □

Combining Lemma 13 with Theorem 21 we get

Theorem 22. *Let M be an α-atomic module. Then M is quasi-projective if and only if M is quasi-corationally complete.*

An R-module M is called quasi-rationally complete if for any submodule N of M and a submodule K of N such that $\mathrm{Hom}(L/K, M) = 0$ for every $L/K \leqslant N/K$, any homomorphism from K to M can be extended to a homomorphism from N to M. Every quasi-injective module is quasi-rationally complete. By modifying the proof of Lemma 1.2 in [9], M is quasi-rationally complete if and only if for any $N \leqslant M$ and $K \leqslant N$, $\mathrm{Hom}(N/K, E(M)) = 0$ implies that any homomorphism from K to M can be extended to a homomorphism from N to M.

Theorem 23. *Let M be a module. Suppose that for any $N \leqslant M$, $\mathrm{Hom}(N/K, M) = 0$ for all $0 \neq K \leqslant N \leqslant M$. Then M is quasi-injective if and only if M is quasi-rationally complete.*

Proof. Suppose that M is a quasi-rationally complete module. Let $N \leqslant M$ and $f \in \mathrm{Hom}(N, M)$. Assume that $\mathrm{Hom}(M/N, E(M)) = 0$. Then $\mathrm{Hom}(M/N, M) = 0$. Since M is quasi-rationally complete then f extends to a homomorphism from M to M. If $\mathrm{Hom}(M/N, E(M) \neq 0$, let h be a nonzero element of $\mathrm{Hom}(M/N, E(M))$

and set $L = h(M/N) \cap M$. Then $h^{-1}(L) = K/N$ for some $K \leqslant M$ and h induces an element t of $\mathrm{Hom}(M/N, M)$ which is zero by hypothesis. Hence $L = 0$ and then $h(M/N) = 0$. This is a contradiction. Thus $\mathrm{Hom}(M/N, E(M)) = 0$. This completes the proof. □

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