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ASYMPTOTIC PROPERTIES OF DIFFERENTIAL EQUATIONS  
WITH ADVANCED ARGUMENT

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*Abstract.* The paper discusses the asymptotic properties of solutions of the scalar functional differential equation

$$y'(x) = ay(\tau(x)) + by(x), \quad x \in [x_0, \infty)$$

of the advanced type. We show that, given a specific asymptotic behaviour, there is a (unique) solution  $y(x)$  which behaves in this way.

*Keywords:* functional differential equation, functional (nondifferential) equation, advanced argument, asymptotic behaviour

*MSC 2000:* 34K15, 34K25; secondary 39B22

## 1. INTRODUCTION AND PRELIMINARIES

In this paper we consider the linear functional differential equation

$$(1.1) \quad y'(x) = ay(\tau(x)) + by(x), \quad x \in I = [x_0, \infty)$$

with an advanced argument  $\tau(x)$  and nonzero real coefficients  $a, b$ .

The qualitative properties of solutions of equation (1.1) are usually studied under the hypothesis that the function  $r(x) = \tau(x) - x$  is constant or at least bounded (for references see, e.g., [3] or [8]). For a discussion of the asymptotic properties of solutions of advanced type equations (1.1) with an unbounded  $r(x)$  we refer to papers [6] and [4], where the deviating arguments  $\tau(x) = \lambda x$ ,  $\lambda > 1$  and  $\tau(x) = x^\alpha$ ,  $\alpha > 1$ ,

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respectively, have been considered. We use the method of proof similar to [6], [4] and the results of the theory of functional (nondifferential) equations to unify and extend the results obtained in [6] and [4].

Some asymptotic results for equations (1.1) with an unbounded delayed argument can be obtained by using the transformation approach developed by M.L. Heard [5] and F. Neuman [9], [10]. These results relate the asymptotic behaviour of all solutions of (1.1) to the behaviour of a solution of the functional (nondifferential) equation

$$(1.2) \quad a\psi(\tau(x)) + b\psi(x) = 0, \quad x \in I$$

(see, e.g., [5] and also [1]). It is interesting that this resemblance between the behaviour of solutions of (1.1) and (1.2) remains preserved (with certain modifications) also for advanced type equations.

Throughout the paper we assume that  $\tau(x)$  is an increasing differentiable function defined on  $I$  such that  $\tau(x) > x$  for every  $x \in I$ . Nevertheless, our results are valid also for equations (1.1) with  $\tau(x_0) = x_0$  (the proofs require only small modifications).

Let  $d \leq \infty$ . By a solution of (1.1) we understand a function  $y(x)$  differentiable on  $[x_0, d)$ , satisfying (1.1) for each  $x \in [x_0, d)$  and extended to the right as far as it is possible. We note that if we are given a solution  $y(x)$  on a finite subinterval, we can carry out this extension to all  $x \geq x_0$  only if  $y(x)$  and  $\tau(x)$  satisfy certain conditions of differentiability. Therefore instead of discussing the asymptotic behaviour of all solutions (as has been done for delay equation (1.1)) we show that, being prescribed a specific asymptotic behaviour by the use of (1.2), there is a (unique) solution  $y(x)$  of (1.1) which exhibits this.

In the sequel, by the symbol  $\tau^n$ , where  $n \in \mathbb{Z}$ , we mean the  $n$ -th iterate of  $\tau$  (for  $n > 0$ ) or the  $-n$ -th iterate of the inverse function  $\tau^{-1}$  (for  $n < 0$ ) and put  $\tau^0 = \text{id}$ .

## 2. MAIN RESULTS

We introduce a parameter  $\lambda$  as

$$(2.1) \quad \lambda = \inf \{ \tau'(x) : x \in I \}$$

and consider the Schröder functional equation

$$(2.2) \quad \varphi(\tau(x)) = \lambda\varphi(x), \quad x \in I,$$

where  $\tau(x)$  is known,  $\lambda$  is defined by (2.1),  $\varphi(x)$  is unknown. The survey of results concerning this equation can be found in the monograph [7]. We state the following result.

**Proposition 1.** Let  $\tau(x) \in C^r(I)$ ,  $r \geq 1$ , be such that  $\tau'(x) > 0$  for every  $x \in I$ . Further, let  $\varphi_0(x) \in C^r(I_0)$ , where  $I_0 = [x_0, \tau(x_0)]$ , be a positive function with a positive derivative on  $I_0$  satisfying

$$(\varphi_0 \circ \tau)^{(k)}(x_0) = \lambda \varphi_0^{(k)}(x_0), \quad k = 1, \dots, r.$$

Then there exists a unique positive solution  $\varphi(x) \in C^r(I)$  of (2.2) such that  $\varphi'(x)$  is positive and bounded and  $\varphi(x) = \varphi_0(x)$  for every  $x \in I_0$ . This solution is given by

$$(2.3) \quad \varphi(x) = \lambda^n \varphi_0(\tau^{-n}(x)), \quad \tau^n(x_0) \leq x \leq \tau^{n+1}(x_0), \quad n = 0, 1, 2, \dots$$

Moreover, if  $r, s$  are real constants,  $s > 0$ , then

$$(2.4) \quad \int_x^\infty \left( (\varphi(t))^r e^{-st} \right) dt \leq \frac{1}{s} \left( 1 + \frac{K}{\varphi(x)} \right) (\varphi(x))^r e^{-sx}, \quad x \in I,$$

where the constant  $K$  depends on  $\varphi(x)$ ,  $r$ ,  $s$ .

*Proof.* The existence and uniqueness of such a solution  $\varphi(x)$  with a positive derivative  $\varphi'(x)$  is proved by the step method (cf. [9, Theorem 1]). We show that  $\varphi'(x)$  is bounded. By differentiating (2.2) we obtain the equation

$$\varphi'(\tau(x)) = \frac{\lambda}{\tau'(x)} \varphi'(x), \quad x \in I.$$

Then the boundedness of  $\varphi'(x)$  can be proved inductively by virtue of the inequality  $\frac{\lambda}{\tau'(x)} \leq 1$ .

Further, integrating by parts we have

$$\begin{aligned} \int_x^\infty \left( (\varphi(t))^r e^{-st} \right) dt &= \frac{1}{s} e^{-sx} (\varphi(x))^r + \frac{r}{s} \int_x^\infty \left( e^{-st} (\varphi(t))^{r-1} \varphi'(t) \right) dt \\ &\leq \frac{1}{s} e^{-sx} (\varphi(x))^r + \frac{rL}{s} \int_x^\infty \left( e^{-st} (\varphi(t))^{r-1} \right) dt \leq \frac{1}{s} \left( 1 + \frac{K}{\varphi(x)} \right) (\varphi(x))^r e^{-sx}. \end{aligned}$$

□

In the sequel we are going to discuss the existence and uniqueness of a solution  $y(x)$  of (1.1) having a specific asymptotic behaviour. We start off with the case  $a \neq 0$ ,  $b > 0$ .

**Theorem 1.** Let  $a \neq 0$ ,  $b > 0$  be scalars and let  $\tau(x) \in C^1(I)$  be such that  $\lambda > 1$ . Further, let  $g(x)$  be a periodic function of period  $\log \lambda$  which is Hölder continuous with exponent  $\theta$ ,  $0 < \theta \leq 1$ , let  $\varphi(x)$  be a solution of (2.2) given by (2.3), let  $\alpha$  be

such that  $a\lambda^\alpha + b = 0$  and put  $\alpha_r = \operatorname{Re} \alpha$ . Then there is a unique solution  $y(x)$  of (1.1) satisfying the asymptotic relation

$$(2.5) \quad y(x) = (\varphi(x))^\alpha g(\log \varphi(x)) + O\{(\varphi(x))^{\alpha_r - \theta}\} \quad \text{as } x \rightarrow \infty.$$

**Remark.** Notice that the function  $(\varphi(x))^\alpha g(\log \varphi(x))$  is a solution of (1.2). Hence, Theorem 1 essentially says that there exists a solution of (1.1) which approaches a prescribed solution of (1.2).

**Proof.** First we show the uniqueness. Suppose that  $y_1(x), y_2(x)$  are solutions of (1.1) having asymptotic behaviour (2.5) with the same  $g(x)$ . Then the function  $y(x) = y_1(x) - y_2(x)$  is a solution of (1.1) such that

$$y(x) = O\{(\varphi(x))^{\alpha_r - \theta}\} \quad \text{as } x \rightarrow \infty.$$

Integrating (1.1) we have

$$e^{-bx}y(x) = -a \int_x^\infty \left( e^{-bt}y(\tau(t)) \right) dt.$$

Choose  $u \in I$  and put  $w(u) = \sup \{v(x) : x \geq u\}$ , where  $v(x) = (\varphi(x))^{-\alpha_r}|y(x)|$ . Due to our assumptions  $w(u)$  tends to zero as  $u \rightarrow \infty$ . We wish to show that  $w(u)$  is identically zero.

Using the previous relation and (2.4) (with  $r = \alpha_r$  and  $s = b$ ) we obtain for each  $x \geq u$

$$\begin{aligned} e^{-bx}|y(x)| &= e^{-bx}(\varphi(x))^{\alpha_r}v(x) = |a| \int_x^\infty \left( e^{-bt}(\varphi(\tau(t)))^{\alpha_r}v(\tau(t)) \right) dt \\ &\leq bw(\tau(x)) \int_x^\infty \left( (\varphi(t))^{\alpha_r}e^{-bt} \right) dt \leq w(\tau(x))(\varphi(x))^{\alpha_r}e^{-bx} \left( 1 + \frac{K}{\varphi(u)} \right), \end{aligned}$$

i.e.,

$$w(u) \leq w(\tau(u)) \left( 1 + \frac{K}{\varphi(u)} \right).$$

Repetition leads to

$$w(u) \leq w(\tau^{n+1}(u)) \prod_{j=0}^n \left( 1 + \frac{K}{\lambda^j \varphi(u)} \right).$$

Letting  $n \rightarrow \infty$  and taking into account the convergence of the infinite product we can see that  $w(u)$  (and  $y(x)$  as well) is identically zero on  $I$ .

To dispose with the existence of the required solution we define inductively

$$u_1(x) = be^{bx} \int_x^\infty \left( e^{-bt} (\varphi(t))^\alpha g(\log \varphi(t)) \right) dt - (\varphi(x))^\alpha g(\log \varphi(x)),$$

$$u_{n+1}(x) = -ae^{bx} \int_x^\infty \left( e^{-bt} u_n(\tau(t)) \right) dt, \quad n = 1, 2, \dots$$

and put  $u(x) = \sum_{n=1}^\infty u_n(x)$ . Our aim is to prove that this series is absolutely and uniformly convergent on  $I$ ,

$$u(x) = O\{(\varphi(x))^{\alpha_r - \theta}\} \quad \text{as } x \rightarrow \infty$$

and the function

$$y(x) = (\varphi(x))^\alpha g(\log \varphi(x)) + u(x), \quad x \in I$$

defines a solution of (1.1).

We show by induction that the relation

$$(2.6) \quad u_n(x) = O\left\{ \lambda^{-(n-1)\theta} (\varphi(x))^{\alpha_r - \theta} e^{\frac{\lambda K}{(\lambda-1)\varphi(x)}} \right\} \quad \text{as } x \rightarrow \infty$$

is true for  $n = 1, 2, \dots$ , where the  $O$ -term is uniform in  $n$  and  $K$  is the same as in (2.4) with  $r = \alpha_r - \theta$  and  $s = b$ . First we rewrite  $u_1(x)$  as

$$(2.7) \quad u_1(x) = be^{bx} \int_x^\infty \left( e^{-bt} [(\varphi(t))^\alpha g(\log \varphi(t)) - (\varphi(x))^\alpha g(\log \varphi(x))] \right) dt$$

$$= be^{bx} \int_x^\infty \left( e^{-bt} [((\varphi(t))^\alpha - (\varphi(x))^\alpha) g(\log \varphi(x)) + (\varphi(t))^\alpha (g(\log \varphi(t)) - g(\log \varphi(x)))] \right) dt$$

$$= bg(\log \varphi(x)) \int_x^\infty \left( e^{-bt} e^{bx} [(\varphi(t))^\alpha - (\varphi(x))^\alpha] \right) dt$$

$$+ be^{bx} \int_x^\infty \left( e^{-bt} (\varphi(t))^\alpha (g(\log \varphi(t)) - g(\log \varphi(x))) \right) dt.$$

Now we estimate both terms occurring in (2.7). Integrating by parts and using (2.4) (with  $r = \alpha_r - 1$  and  $s = b$ ) we have

$$\left| bg(\log \varphi(x)) \int_x^\infty \left( e^{-bt} e^{bx} [(\varphi(t))^\alpha - (\varphi(x))^\alpha] \right) dt \right|$$

$$= |\alpha_r g(\log \varphi(x))| e^{bx} \int_x^\infty \left( e^{-bt} (\varphi(t))^{\alpha_r - 1} \varphi'(t) \right) dt$$

$$\leq L_1 e^{bx} \int_x^\infty \left( e^{-bt} (\varphi(t))^{\alpha_r - 1} \right) dt \leq \frac{L_1}{b} \left( 1 + \frac{K}{\varphi(x)} \right) (\varphi(x))^{\alpha_r - 1} \leq M_1 (\varphi(x))^{\alpha_r - 1},$$

$M_1 > 0$  being a suitable constant. Further,

$$\begin{aligned}
 & \left| be^{bx} \int_x^\infty \left( e^{-bt} (\varphi(t))^{\alpha_r} (g(\log \varphi(t)) - g(\log \varphi(x))) \right) dt \right| \\
 & \leq L_2 be^{bx} \int_x^\infty \left( e^{-bt} (\varphi(t))^{\alpha_r} (\log \varphi(t) - \log \varphi(x))^\theta \right) dt \\
 & = L_2 be^{bx} \int_x^\infty \left( e^{-bt} (\varphi(t))^{\alpha_r} \left( \log \left( 1 + \frac{\varphi(t) - \varphi(x)}{\varphi(x)} \right) \right)^\theta \right) dt \\
 & \leq L_2 be^{bx} \int_x^\infty \left( e^{-bt} (\varphi(t))^{\alpha_r} \left( \frac{\varphi(t) - \varphi(x)}{\varphi(x)} \right)^\theta \right) dt \\
 & \leq L_2 be^{bx} (\varphi(x))^{-\theta} \int_x^\infty \left( e^{-bt} (\varphi(t))^{\alpha_r} (1 + \varphi(t) - \varphi(x)) \right) dt.
 \end{aligned}$$

The integration by parts and application of (2.4) yield

$$L_2 be^{bx} (\varphi(x))^{-\theta} \int_x^\infty \left( e^{-bt} (\varphi(t))^{\alpha_r} (1 + \varphi(t) - \varphi(x)) \right) dt \leq M_2 (\varphi(x))^{\alpha_r - \theta}.$$

Substituting these estimates into (2.7) we obtain that (2.6) is true for  $n = 1$ . Assuming that

$$|u_n(x)| \leq M \lambda^{-(n-1)\theta} (\varphi(x))^{\alpha_r - \theta} e^{\frac{\lambda K}{(\lambda-1)\varphi(x)}}$$

for each  $x \in I$ , we obtain by virtue of (2.4)

$$\begin{aligned}
 |u_{n+1}(x)| & \leq |a| \int_x^\infty \left( e^{-b(t-x)} |u_n(\tau(t))| \right) dt \\
 & \leq M |a| \lambda^{-(n-1)\theta} \int_x^\infty \left( e^{-b(t-x)} (\varphi(\tau(t)))^{\alpha_r - \theta} e^{\frac{\lambda K}{(\lambda-1)\varphi(\tau(t))}} \right) dt \\
 & = M |a| \lambda^{\alpha_r} \lambda^{-n\theta} \int_x^\infty \left( e^{-b(t-x)} (\varphi(t))^{\alpha_r - \theta} e^{\frac{K}{(\lambda-1)\varphi(t)}} \right) dt \\
 & \leq Mb \lambda^{-n\theta} e^{\frac{K}{(\lambda-1)\varphi(x)}} \int_x^\infty \left( e^{-b(t-x)} (\varphi(t))^{\alpha_r - \theta} \right) dt \\
 & \leq M \lambda^{-n\theta} (\varphi(x))^{\alpha_r - \theta} e^{\frac{K}{(\lambda-1)\varphi(x)}} \left( 1 + \frac{K}{\varphi(x)} \right) \\
 & \leq M \lambda^{-n\theta} (\varphi(x))^{\alpha_r - \theta} e^{\frac{K}{(\lambda-1)\varphi(x)}} e^{\frac{K}{\varphi(x)}} \\
 & = M \lambda^{-n\theta} (\varphi(x))^{\alpha_r - \theta} e^{\frac{\lambda K}{(\lambda-1)\varphi(x)}}.
 \end{aligned}$$

Relation (2.6) implies the absolute and uniform convergence of the series  $u(x) = \sum_{n=1}^\infty u_n(x)$  and, moreover,

$$|u(x)| \leq \sum_{n=1}^\infty |u_n(x)| \leq \frac{M}{1 - \lambda^{-\theta}} (\varphi(x))^{\alpha_r - \theta} e^{\frac{\lambda K}{(\lambda-1)\varphi(x)}},$$

i.e.,

$$u(x) = O\{(\varphi(x))^{\alpha r - \theta}\} \quad \text{as } x \rightarrow \infty.$$

Finally, noting that the function  $(\varphi(x))^\alpha g(\log \varphi(x))$  is a solution of (1.2), it is not difficult to verify that

$$y(x) = (\varphi(x))^\alpha g(\log \varphi(x)) + u(x)$$

satisfies (1.1). □

Now we consider the case  $a \neq 0$ ,  $b < 0$ . The asymptotic form of solutions as well as the method of proof are almost identical (except the uniqueness property) with those in the previous case. Therefore we point out only some modifications.

The analogue of Proposition 1 is

**Proposition 2.** *Let  $\tau(x)$  and  $\varphi(x)$  be the same as in Proposition 1. If  $r, s$  are real constants,  $s > 0$ , then*

$$(2.8) \quad \int_{\varrho}^x \left( (\varphi(t))^r e^{st} \right) dt \leq \frac{1}{s} \left( 1 + \frac{K}{\varphi(x)} \right) (\varphi(x))^r e^{sx}, \quad x \geq \varrho \geq x_0,$$

where  $K$  depends on  $\varphi(x)$ ,  $r$ ,  $s$ ,  $\varrho$ .

*P r o o f.* The proof is similar to that of Proposition 1 and is therefore omitted. □

**Lemma.** *Assume that  $a \neq 0$ ,  $b < 0$ ,  $\tau(x) \in C^1(I)$  and  $\lambda > 1$ . Then there exists a unique solution  $y^*(x)$  of (1.1) such that  $e^{-bx}y^*(x)$  tends to unity as  $x \rightarrow \infty$ . Further, let  $\varphi(x)$  be a solution of (2.2) given by (2.3) and let  $\alpha_r = \frac{\log |\frac{-a}{b}|}{\log \lambda^{-1}}$ . If  $y(x)$  is any solution of (1.1) such that*

$$(2.9) \quad y(x) = o\{(\varphi(x))^{\alpha_r}\} \quad \text{as } x \rightarrow \infty,$$

then  $y(x) = Ly^*(x)$  for a real constant  $L$  and all  $x \geq x_0$ .

*P r o o f.* We introduce the change of variables  $z(x) = e^{-bx}y(x)$  in (1.1) to obtain

$$(2.10) \quad z'(x) = p(x)z(\tau(x)), \quad x \in I,$$

where  $p(x) = a e^{b(\tau(x)-x)}$ . It is known that if equation (2.10) is of advanced type,  $\int_{x_0}^{\infty} |p(s)| ds < \infty$  and  $x_1 > x_0$  is large enough, then there exists a unique solution  $z^*(x)$  of (2.10) defined on  $[x_1, \infty)$  and fulfilling the terminal condition  $\lim_{x \rightarrow \infty} z^*(x) = 1$



(see, e.g., [11]). Since  $\int_{x_0}^{\infty} |p(s)| ds$  obviously converges, we can define  $y^*(x) = e^{bx} z^*(x)$  for  $x \geq x_1$  and extend  $y^*(x)$  onto  $[x_0, \infty)$  to obtain the required solution of (1.1). This proves the first part of the assertion.

Now let  $y(x)$  be any solution of (1.1) such that (2.9) holds. To show  $y(x) = L y^*(x)$  on  $I$  we first prove that there exists  $\delta > 0$  such that

$$(2.11) \quad y(x) = O\{e^{-\delta x}\} \quad \text{as } x \rightarrow \infty.$$

Integrating (1.1) we obtain

$$(2.12) \quad y(x) = y(\varrho) e^{b(x-\varrho)} + a \int_{\varrho}^x \left( e^{b(x-t)} y(\tau(t)) \right) dt, \quad x \geq \varrho \geq x_0.$$

Set  $v(x) = (\varphi(x))^{-\alpha_r} |y(x)|$  and  $w(u) = \sup \{v(x) : x \geq u\}$ ,  $x, u \in I$ . Then  $w(u)$  decreases to zero as  $u \rightarrow \infty$  and (2.12) gives

$$\begin{aligned} v(x)(\varphi(x))^{\alpha_r} &\leq w(\varrho)(\varphi(\varrho))^{\alpha_r} e^{b(x-\varrho)} \\ &\quad + |a| \lambda^{\alpha_r} w(\tau(\varrho)) \int_{\varrho}^x \left( e^{b(x-t)} (\varphi(t))^{\alpha_r} \right) dt, \quad x \geq \varrho \geq x_0, \end{aligned}$$

i.e.,

$$v(x) \leq w(\varrho)(\varphi(\varrho))^{\alpha_r} (\varphi(x))^{-\alpha_r} e^{b(x-\varrho)} + w(\tau(\varrho)) \left( 1 + \frac{K}{\varphi(x)} \right), \quad x \geq \varrho \geq x_0,$$

by virtue of Proposition 2. From here we get

$$v(x) \leq C_1 w(\varrho)(\varphi(\varrho))^{\alpha_r} e^{b'x - b\varrho} + w(\tau(\varrho)) \left( 1 + \frac{K}{\varphi(x)} \right), \quad x \geq \varrho \geq x_0,$$

where  $C_1 > 0$  and  $b' < 0$ ,  $b' > b$  may be chosen arbitrarily close to  $b$  (we can choose  $b' = b$  when  $\alpha_r \geq 0$ ). Then take  $\sigma \geq \varrho$  and sup for  $x \geq \sigma$  to obtain

$$w(\sigma) \leq C_1 w(\varrho)(\varphi(\varrho))^{\alpha_r} e^{b'\sigma - b\varrho} + w(\tau(\varrho)) \left( 1 + \frac{K}{\varphi(\sigma)} \right), \quad \sigma \geq \varrho.$$

Now put  $\varrho = \tau^{-\frac{1}{2}}(\sigma)$ . Let us remark that the  $u$ -th iteration  $\tau^u$  of  $\tau$  can be defined for real  $u$  as  $\tau^u(x) = \varphi^{-1}(\lambda^u \varphi(x))$ ,  $x \in I$  (for definitions, results and references concerning continuous iterations see [7]).

Then taking into account the boundedness of  $w$  we have

$$w(\sigma) \leq C_2 (\varphi(\sigma))^{\alpha_r} \exp(b'\sigma - b\tau^{-\frac{1}{2}}(\sigma)) + w(\tau^{\frac{1}{2}}(\sigma)) \left( 1 + \frac{K}{\varphi(\sigma)} \right), \quad \sigma \geq \tau^{\frac{1}{2}}(x_0).$$

Since  $b'$  may be arbitrarily close to  $b$ , we get

$$b'\sigma - b\tau^{-\frac{1}{2}}(\sigma) < 0.$$

Then there exists  $\delta > 0$  such that

$$C_2(\varphi(\sigma))^{\alpha r} \exp(b'\sigma - b\tau^{-\frac{1}{2}}(\sigma)) \leq C_3 e^{-2\delta\sigma}, \quad \sigma \geq \tau^{\frac{1}{2}}(x_0),$$

i.e.,

$$w(\sigma) \leq C_3 e^{-2\delta\sigma} + w(\tau^{\frac{1}{2}}(\sigma)) \left(1 + \frac{K}{\varphi(\sigma)}\right), \quad \sigma \geq \tau^{\frac{1}{2}}(x_0).$$

Repeated use of the last inequality gives

$$\begin{aligned} w(\sigma) &\leq C_3 e^{-2\delta\sigma} + C_3 \sum_{n=1}^N \left( \exp(-2\delta\tau^{\frac{n}{2}}(\sigma)) \prod_{m=0}^{n-1} \left(1 + \frac{K}{\lambda^{\frac{m}{2}} \varphi(\sigma)}\right) \right) \\ &\quad + w(\tau^{\frac{N+1}{2}}(\sigma)) \prod_{m=0}^N \left(1 + \frac{K}{\lambda^{\frac{m}{2}} \varphi(\sigma)}\right), \quad \sigma \geq \tau^{\frac{1}{2}}(x_0), \quad N = 1, 2, \dots \end{aligned}$$

Letting  $N \rightarrow \infty$  we can see that the infinite products converge and  $w(\tau^{\frac{N+1}{2}}(\sigma))$  decreases to zero. Hence,

$$w(\sigma) \leq C_4 \sum_{n=0}^{\infty} \exp(-2\delta\tau^{\frac{n}{2}}(\sigma)), \quad \sigma \geq \tau^{\frac{1}{2}}(x_0),$$

which implies (2.11).

Now by (2.12) and (2.11) there exists  $M > 0$  such that

$$|y(x)e^{-bx} - y(\varrho)e^{-b\varrho}| \leq M \int_{\varrho}^x \exp(-bt - \delta\tau(t)) dt, \quad x \geq \varrho \geq x_0.$$

Since the integral on the right converges as  $x \rightarrow \infty$  it follows that

$$\lim_{x \rightarrow \infty} e^{-bx} y(x) = L \in \mathbb{R}.$$

Further, set  $\bar{y}(x) = y(x) - L y^*(x)$ ,  $x \in I$ . Then  $\bar{y}(x)$  is a solution of (1.1) such that

$$\bar{y}(x) = o\{e^{bx}\} \quad \text{as } x \rightarrow \infty.$$

Integrating (1.1) we obtain

$$e^{-bx} \bar{y}(x) = -a \int_x^{\infty} \left( e^{-bt} \bar{y}(\tau(t)) \right) dt.$$

Let  $u \in I$  and put  $q(u) = \sup \{e^{-bx}|\bar{y}(x)| : x \geq u\}$ . Then for each  $x \geq u$ ,

$$e^{-bx}|\bar{y}(x)| \leq |a| \int_x^\infty \left( e^{b(\tau(t)-t)} e^{-b\tau(t)} \bar{y}(\tau(t)) \right) dt \leq \frac{|a|}{-b(\lambda-1)} e^{b(\tau(x)-x)} q(\tau(u)),$$

i.e.,

$$q(u) \leq \frac{|a|}{-b(\lambda-1)} e^{b(\tau(u)-u)} q(\tau(u)).$$

Repeated application of this inequality yields

$$q(u) \leq \frac{|a|^n}{(-b)^n(\lambda-1)^n} e^{b(\tau^n(u)-u)} q(\tau^n(u)).$$

If we let  $n \rightarrow \infty$  we can deduce that  $q(u)$  is identically zero on  $I$ , i.e.,  $y(x) = L y^*(x)$  for all  $x \in I$ .  $\square$

**Theorem 2.** *Let  $a \neq 0$ ,  $b < 0$  be scalars, let  $\tau(x) \in C^1(I)$  be such that  $\lambda > 1$  and let  $y^*(x)$  be given by Lemma. Further, let  $g(x)$ ,  $\varphi(x)$ ,  $\alpha$  and  $\alpha_r$  be as in Theorem 1. Then there exists a solution  $y(x)$  of (1.1) satisfying asymptotic relation (2.5). Furthermore,  $y(x)$  is unique up to addition of a constant multiple of  $y^*(x)$ .*

*Proof.* First we deal with the existence of a solution  $y(x)$  of (1.1) having the prescribed asymptotic behaviour (2.5). Define inductively

$$u_1(x) = -be^{bx} \int_\varrho^x \left( e^{-bt} (\varphi(t))^\alpha g(\log \varphi(t)) \right) dt - (\varphi(x))^\alpha g(\log \varphi(x)),$$

$$u_{n+1}(x) = ae^{bx} \int_\varrho^x \left( e^{-bt} u_n(\tau(t)) \right) dt, \quad n = 1, 2, \dots,$$

where  $x \geq \varrho$ ,  $\varrho \in I$  being a constant large enough.

We verify that the series  $u(x) = \sum_{n=1}^\infty u_n(x)$  absolutely and uniformly converges on  $I$ ,

$$u(x) = O\{(\varphi(x))^{\alpha_r - \theta}\} \quad \text{as } x \rightarrow \infty$$

and the function

$$y(x) = (\varphi(x))^\alpha g(\log \varphi(x)) + u(x), \quad x \in I$$

satisfies (1.1). Similarly as in the proof of Theorem 1 (using (2.8) instead of (2.4)) we can estimate  $u_n(x)$  as

$$u_n(x) = O\left\{ \lambda^{-(n-1)\theta} (\varphi(x))^{\alpha_r - \theta} \left( 1 + \frac{K}{\varrho} \right)^{n-1} \right\} \quad \text{as } x \rightarrow \infty,$$

where the  $O$ -term is uniform in  $n$  and  $K$  is the same as in (2.8) with  $r = \alpha_r - \theta$ ,  $s = -b$ .

Once  $\varrho$  has been chosen large enough, we obtain

$$|u_n(x)| \leq M(\varphi(x))^{\alpha_r - \theta} L^{n-1},$$

where  $L = \lambda^{-\theta}(1 + \frac{K}{\varrho}) < 1$ .

Summarizing this we obtain that the series  $u(x) = \sum_{n=1}^{\infty} u_n(x)$  is absolutely and uniformly convergent on  $I$ . The remaining parts are easy to check.

Now let  $y_1(x)$ ,  $y_2(x)$  be solutions of (1.1) satisfying (2.5) with the same  $g(x)$ . If we put  $y(x) = y_1(x) - y_2(x)$  on  $I$  then  $y(x)$  fulfils (2.9) and Lemma yields

$$y_1(x) - y_2(x) = L y^*(x), \quad x \in I.$$

□

### 3. APPLICATIONS

**Example 1.** First we consider the equation

$$(3.1) \quad y'(x) = a y(\lambda x) + b y(x), \quad x \in [0, \infty),$$

where  $a$ ,  $b$ ,  $\lambda$  are constants,  $a$ ,  $b \neq 0$ ,  $\lambda > 1$ . The asymptotic behaviour of solutions of (3.1) has been deeply investigated in [6]. Applying our previous results to this equation we note that the deviation  $\tau(x) = \lambda x$  fulfils all the required assumptions except  $\tau(x) > x$  for each  $x \in [0, \infty)$ . Nevertheless, using a small modification in Proposition 1 and Proposition 2 we get that the results of the previous sections are valid also for deviations  $\tau(x)$  intersecting the identity function at the initial point. Schröder equation (2.2) then becomes

$$\varphi(\lambda x) = \lambda \varphi(x), \quad x \in [0, \infty)$$

and admits the identity  $\varphi(x) = x$  as the required solution. Substituting this  $\varphi(x)$  into (2.5) we obtain the coincidence between our asymptotic results and the corresponding results of [6].

**Example 2.** Now we discuss the asymptotic behaviour of solutions of the equation

$$(3.2) \quad y'(x) = a y(x^\gamma) + b y(x), \quad x \in [1, \infty),$$

where  $a, b, \gamma$  are constants,  $a, b \neq 0, \gamma > 1$ . Schröder equation (2.2) with  $\tau(x) = x^\gamma$  has the form

$$\varphi(x^\gamma) = \gamma\varphi(x), \quad x \in [1, \infty)$$

with  $\varphi(x) = \log x$  as the required solution.

The asymptotic properties of solutions of equation (3.2) have been studied in [4]. We note that our previous results with  $\varphi(x) = \log x$  are just Theorem 3 and Theorem 4 of the cited paper.

**Example 3.** We investigate the asymptotic properties of solutions of the equation

$$(3.3) \quad y'(x) = a y(x^{\log x}) + b y(x), \quad x \in [e, \infty),$$

where  $a, b \neq 0$ . Since  $\tau(x) = x^{\log x}$ , we have

$$\lambda = \inf\{\tau'(x): x \geq e\} = \inf\{2x^{\log x - 1} \log x: x \geq e\} = 2.$$

Substituting this into (2.2) we obtain the functional equation

$$(3.4) \quad \varphi(x^{\log x}) = 2\varphi(x), \quad x \in [e, \infty).$$

It is easy to check that the function  $\varphi(x) = \log \log x$  is a solution of (3.4) with a positive and bounded derivative on  $[e, \infty)$ . We apply conclusions of Theorem 1 and Theorem 2 to equation (3.4). Assume that we are given a periodic function  $g(x)$  of period  $\log 2$  which is Hölder continuous with exponent  $\theta, 0 < \theta \leq 1$ . If  $b > 0$ , then there exists a unique solution  $y(x)$  of (3.4) satisfying the asymptotic relation

$$(3.5) \quad y(x) = (\log \log x)^\alpha g(\log \log \log x) + O\{(\log \log x)^{\alpha_r - \theta}\} \quad \text{as } x \rightarrow \infty,$$

where  $\alpha$  is a root of the equation  $a 2^\alpha + b = 0$  and  $\alpha_r = \operatorname{Re} \alpha$ .

If  $b < 0$ , then there exists a unique solution  $y^*(x)$  of (3.4) asymptotic to  $e^{bx}$  and a solution  $y(x)$  of (3.4) satisfying (3.5). Moreover, this solution  $y(x)$  is unique up to the addition of a constant multiple of  $y^*(x)$ .

**Example 4.** Consider the equation

$$(3.6) \quad y'(x) = b(y(x) - y(\tau(x))), \quad x \in I,$$

where  $b$  is a nonzero constant and assume that  $\lambda = \inf\{\tau'(x): x \in I\} > 1$ . Since obviously  $\alpha = \alpha_r = 0$  it is easy to reformulate relation (2.5) and the related results into the corresponding simplified form.

Let, e.g.,  $b > 0$ . If  $g(x)$  is a periodic function of period  $\log \lambda$  which is Hölder continuous with exponent  $\theta$ ,  $0 < \theta \leq 1$ , then there is a unique solution  $y(x)$  of (3.6) such that

$$y(x) = g(\log \varphi(x)) + O\{(\varphi(x))^{-\theta}\} \quad \text{as } x \rightarrow \infty,$$

where  $\varphi(x)$  is a solution of (2.2) given by (2.3). The case  $b < 0$  can be dealt with quite similarly.

We note that the problem of asymptotic behaviour of solutions of (3.6) has been studied, under special hypotheses, in many papers (for some results and references see, e.g., [2]). As remarked above, most authors have preferably studied equations with a delay or with a bounded function  $r(x) = \tau(x) - x$ . Our previous results extend the validity of some asymptotic formulas concerning equation (3.6) also to equations (3.6) of the advanced type with an unbounded  $r(x)$ .

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