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QUASILINEAR ELLIPTIC PROBLEMS WITH MULTIVALUED TERMS

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Abstract. We study the quasilinear elliptic problem with multivalued terms.We consider the Dirichlet problem with a multivalued term appearing in the equation and a problem of Neumann type with a multivalued term appearing in the boundary condition. Our approach is based on Szulkin's critical point theory for lower semicontinuous energy functionals.

Keywords: subdifferential, critical point, Palais-Smale condition, Mountain Pass Theorem, Saddle Point Theorem, multivalued term, Dirichlet problem, Neumann problem, p-Laplacian, Rayleigh quotient

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1. Introduction

In this paper we consider quasilinear Dirichlet and Neumann problems with multivalued terms. Our approach is variational and uses the critical point theory and the related minimax principles for lower semicontinuous functions, as these were developed by Szulkin [22]. In the past quasilinear problems were studied by Del Pino-Elgueta-Manasevich [10], Guo [12], De Coster [9] for one dimensional problems (i.e. $N = 1$) and by Boccardo-Drábek-Giachetti-Kučera [6], Anane-Gossez [4], Arcoya-Calahorrano [5], Hachimi-Gossez [13] and Costa-Magalhaes [8] for multidimensional problems (i.e. $N > 1$). All these papers deal with problems that involve no multivalued terms and assume Dirichlet boundary conditions.

Here we allow for the presence of multivalued terms (either in the equation or in the boundary conditions). So the classical "smooth" critical point theory is not applicable here and we need to employ some suitable version of the nonsmooth critical point theory. For the problems that we study here the appropriate nonsmooth critical point theory, is that developed by Szulkin [22] which concerns energy functionals of the form $\Phi + \psi$ with Φ being C^1 and ψ being an $\overline{\mathbb{R}}$-valued proper, convex and lower semicontinuous functional. In the next section for the convenience of the reader we recall the basic aspects of Szulkin's theory, which we will need in the sequel. Full details can be found in the well-written paper of Szulkin [22].

2. Preliminaries

Let X be a Banach space. We will deal with $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$-valued functions. For such a function $\psi(\cdot)$, the effective domain of ψ is the set $\operatorname{dom}\psi = \{x \in X\colon\ \psi(x) < +\infty\}$. By $\Gamma_0(X)$ we denote the set of proper (i.e. $\operatorname{dom}\psi \neq \emptyset$), convex and lower semicontinuous functions. The subdifferential of $\psi \in \Gamma_0(X)$ at $x \in X$, is the set $\partial\psi(x) = \{x^* \in X^*\colon\ (x^*, y - x) \leqslant \psi(y) - \psi(x)$ for all $y \in \operatorname{dom}\psi\}$. Here $(\cdot,\cdot)$ denotes the duality brackets for the pair (X, X^*). The elements $x^* \in \partial\psi(x)$ are called "subgradients" of f at x. It is immediately clear from the definition that $\partial\psi(x)$ is always a w^*-closed and convex subset of X^*. It may be empty. The set of those x for which $\partial\psi(x) \neq \emptyset$ is called the domain of $\partial\psi$ and is denoted by $\operatorname{dom}(\partial\psi)$. We have that $\operatorname{dom}(\partial\psi) \subseteq \operatorname{dom}\psi$ and $\operatorname{intdom}\psi \subseteq \operatorname{dom}(\partial\psi)$. If $\psi(\cdot)$ is contininuous at $x_0 \in X$, then $\partial\psi(x_0) \neq \emptyset$. If ψ is Gateaux differentiable at $x_0 \in X$, then $\partial\psi(x_0) = \{\psi'(x_0)\}$. The set-valued map $\partial\psi\colon\ X \to 2^{X^*}$ is maximal monotone (in fact maximal cyclically monotone). Also, given $\psi_1, \psi_2 \in \Gamma_0(X)$, then for every $x \in \operatorname{dom}\psi_1 \cap \operatorname{dom}\psi_2$ we have

$$\partial\psi_1(x) + \partial\psi_2(x) \subseteq \partial(\psi_1 + \psi_2)(x).$$

If $\operatorname{intdom}\psi_1 \cap \operatorname{dom}\psi_2 \neq \emptyset$, then for all $x \in X$ we have

$$\partial(\psi_1 + \psi_2)(x) = \partial\psi_1(x) + \partial\psi_2(x).$$

We will deal with functionals of the form $R = \Phi + \psi$ with $\Phi \in C^1(X), \psi \in \Gamma_0(X)$ and X being a reflexive Banach space. A point $x \in \operatorname{dom}\psi$ is said to be a "critical point" of R if $-\Phi'(x) \in \partial\psi(x)$ or equivalently, if x satisfies the inequality

$$(\Phi'(x), y - x) + \psi(y) - \psi(x) \geqslant 0 \text{ for all } y \in \operatorname{dom}\psi.$$

It is easy to see that if $x \in X$ is a local minimum of $R = \Phi + \psi$, then it is a critical point. As in the classical (smooth) case, in order to have minimax principles, we need some kind of compactness condition known as the "Palais-Smale condition"((PS)-condition). For functionals $R = \Phi + \psi$, Szulkin formulated the (PS)-condition as follows:

Definition. We say that $R = \Phi + \psi$ satisfies the (PS)-condition, if any sequence $\{x_n\}_{n \geqslant 1} \subseteq X$ such that $R(x_n) \to c \in \mathbb{R}$ and $(\Phi'(x_n), y - x_n) + \psi(y) - \psi(x_n) \geqslant -\varepsilon_n\|y - x_n\|$ for all $y \in X$ with $\varepsilon_n \downarrow 0$ has a strongly convergent subsequence.

Remark. Szulkin proved that the (PS)-condition can be equivalently reformulated as follows: "every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$ such that $R(x_n) \to c \in \mathbb{R}$ and $(\Phi'(x_n), y - x_n) + \psi(y) - \psi(x_n) \geqslant (u_n, y - x_n)$ for all $y \in X$ where $u_n \to 0$ in X^* as $n \to \infty$, has a strongly convergent subsequence" (see Szulkin [22], Proposition 2, p. 80).

Szulkin proved the following results on the existence of critical points for functionals $R = \Phi + \psi$, which generalize the well-known "smooth" results of Ambrosetti-Rabinowitz [3] and Rabinowitz [19], [20]. So in the following three theorems X is a reflexive Banach space and $R = \Phi + \psi$ with $\Phi \in C^1(X)$ and $\psi \in \Gamma_0(X)$.

Theorem 1. *If $R = \Phi + \psi$ is bounded below and satisfies the (PS)-condition, then $R(\cdot)$ has a critical point $x \in X$ such that $R(x) = \inf[R(y)\colon\ y \in X]$.*

The next theorem extends to the present nonsmooth setting the well-known "Mountain Pass Theorem" of Ambrosetti-Rabinowitz [3].

Theorem 2. *If $R = \Phi + \psi$ satisfies the (PS)-condition and* (i) *$R(0) = 0$ and there exist $\alpha, \varrho > 0$ such that $R\big|_{\partial B_\varrho} \geqslant \alpha$ where $\partial B_\varrho = \{x \in X\colon\ \|x\| = \varrho\}$;* (ii) *$R(e) \leqslant 0$ for some $e \notin \overline{B_\varrho} = \{x \in X\colon\ \|x\| \leqslant \varrho\}$, then $R(\cdot)$ has a critical point x, with $c = R(x) \geqslant \alpha$ and c is characterized by*

$$c = \inf_f \sup_t \big[R(f(t))\colon\ f \in \Gamma, t \in [0,1]\big]$$

where $\Gamma = \big\{f \in C([0,1], X)\colon\ f(0) = 0, f(1) = e\big\}$.

The third theorem is the nonsmooth analog of the Saddle Point Theorem of Rabinowitz [19],[20].

Theorem 3. *If $R = \Phi + \psi$ satisfies the (PS)-condition, $X = X_1 \oplus X_2$ where* $\dim X_1 < +\infty$, (i) *there exist constants $\varrho > 0$ and α_1 such that $R\big|_{\partial B_\varrho \cap X_1} \leqslant \alpha_1$;* (ii) *there is a constant $\alpha_2 > \alpha_1$ such that $R\big|_{X_2} \geqslant \alpha_2$ then $R(\cdot)$ has a critical point $x \in X$, with $c = R(x) \geqslant \alpha_2$ and c is characterized by*

$$c = \inf_f \sup_x \big[R(f(x))\colon\ f \in \Gamma, x \in D\big]$$

where $D = \overline{B_\varrho} \cup X_1$ and $\Gamma = \big\{f \in C(D, X)\colon\ f\big|_{\partial D} = i\big|_{\partial D},\ i = \text{identity map}\big\}$.

In this section we study quasilinear equations with multivalued terms and Dirichlet boundary conditions. So let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^1-boundary Γ. We consider the nonlinear elliptic problem

$$\text{(1)} \qquad \begin{cases} -\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) + \beta(z,x(z)) \ni f(z,x(z)) \text{ a.e. on } Z \\ x\big|_\Gamma = 0,\ 2 \leqslant p < \infty. \end{cases}$$

Here $\beta\colon Z \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multifunction. For economy in the notation let $\Delta_p x = \operatorname{div}(\|Dx\|^{p-2}Dx)$(the p-Laplacian). Consider the first eigenvalue λ_1 of $(-\Delta_p, W_0^{1,p}(Z))$. From Lindqvist [18] we know that $\lambda_1 > 0$ is isolated and simple, that is any two solutions u, v of

$$\text{(2)} \qquad \left\{ \begin{array}{l} -\Delta_p u = -\operatorname{div}(\|Du\|^{p-2}Du) = \lambda_1 |u|^{p-2}u \text{ a.e. on } Z \\ \quad u\big|_\Gamma = 0,\ 2 \leqslant p < \infty \end{array} \right\}$$

satisfy $u = cv$ for some $c \in \mathbb{R}$. In addition, the λ_1-eigenfunctions do not change sign in Z. Finally, we have the following variational characterization of λ_1 (Rayleigh quotient):

$$\lambda_1 = \inf\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} \colon\ x \in W_0^{1,p}(Z),\ x \neq 0\right].$$

For our first existence theorem, we will need the following hypotheses on f and β.

$\mathbf{H}(f)_1$: $f\colon Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to f(z,x)$ is continuous;

(iii) for almost all $z \in Z$ and $x \in \mathbb{R}$ $|f(z,x)| \leqslant \alpha(z) + c|x|^{\mu-1}$, $2 \leqslant p < \mu < p^* = \frac{Np}{N-p}$, $\alpha \in L^{\mu'}(Z)$ $(\frac{1}{\mu} + \frac{1}{\mu'} = 1)$;

(iv) if $F(z,x) = \int_0^x f(z,r)\,dr$, then $\limsup\limits_{|x|\to\infty} \dfrac{pF(z,x)}{|x|^p} \leqslant \theta(z) \leqslant \lambda_1$ uniformly for almost all $z \in Z$ with $\theta \in L^\infty(Z)$ and the inequality $\theta(z) \leqslant \lambda_1$ is strict on a set of positive Lebesgue measure;

(v) there exist $\gamma_1 \in L^1(Z)$ and $M > 0$ such that for almost all $z \in Z$ and all $|x| \geqslant M$, $-f(z,x)x \geqslant \gamma_1(z)$.

$\mathbf{H}(\beta)_1$: $\beta(z,x) = \partial j(z,x)$ (the subdifferential is taken in the x-variable), where $j\colon Z \times \mathbb{R} \to \overline{\mathbb{R}_+} = \mathbb{R}_+ \cup \{+\infty\}$ is a normal convex integrand (i.e. $j(\cdot,\cdot)$ is jointly measurable and for all $z \in Z$, $j(z,\cdot) \in \Gamma_0(\mathbb{R})$ and for almost all $Z \in Z$, $j(z,0) = 0$).

Theorem 4. *If hypotheses $H(f)_1$ and $H(\beta_1)$ hold, then problem* (1) *has a solution.*

P r o o f. Let $\Phi, \psi\colon W_0^{1,p}(Z) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be the functionals defined by

$$\Phi(x) = -\int_Z \int_0^{x(z)} f(z,r)\,\mathrm{d}r\,\mathrm{d}z = -\int_Z F(z,x(z))\,\mathrm{d}z \text{ with } F(z,x) = \int_0^x f(z,r)\,\mathrm{d}r,$$

and

$$\psi(x) = \begin{cases} \frac{1}{p}\|Dx\|_p^p + \int_Z j(z,x(z))\,\mathrm{d}z & \text{if } j(\cdot,x(\cdot)) \in L^1(Z) \\ +\infty & \text{otherwise} \end{cases}$$

It is well-known that $\Phi \in C^1(W_0^{1,p}(Z))$. Also $\psi \in \Gamma_0(W_0^{1,p}(Z))$. Indeed, note that by virtue of hypothesis $H(\beta_1)$, $\psi \geqslant 0$ and $\psi \not\equiv \infty$. Clearly $\psi(\cdot)$ is convex. Finally, in order to prove the lower semicontinuity of $\psi(\cdot)$, we need to show that for every $\lambda \geqslant 0$ the sublevel set

$$L_\lambda = \left\{ x \in W_0^{1,p}(Z)\colon \frac{1}{p}\|Dx\|_p^p + \int_Z j(z,x(z))\,\mathrm{d}z \leqslant \lambda \right\}$$

is closed. So let $\{x_n\} \subseteq L_\lambda$ and assume that $x_n \to x$ in $W_0^{1,p}(Z)$ as $n \to \infty$. By passing to a subsequence if necessary, we may assume that $x_n(z) \to x(z)$ a.e. on Z. Using Fatou's lemma we conclude that

$$\int_Z j(z,x(z))\,\mathrm{d}z \leqslant \int_Z \liminf j(z,x_n(z))\,\mathrm{d}z \leqslant \liminf \int_Z j(z,x_n(z))\,\mathrm{d}z$$
$$\Rightarrow \psi(x) \leqslant \lambda, \text{ i.e. } x \in L_\lambda.$$

So indeed $\psi \in \Gamma_0(W_0^{1,p}(Z))$. Set $R(x) = \Phi(x) + \psi(x)$ with $x \in W_0^{1,p}(Z)$.

Claim 1. $R(\cdot)$ satisfies the (PS)-condition (in the sense of Szulkin, see Section 2).

To this end let $\{x_n\} \subseteq W_0^{1,p}(Z)$ be such that $R(x_n) \to c \in \mathbb{R}$ and

$$\langle \Phi'(x_n), y - x_n \rangle + \psi(y) - \psi(x_n) \geqslant -\varepsilon_n \|y - x_n\| \text{ for all } y \in W_0^{1,p}(Z)$$

with $\varepsilon_n \downarrow 0$, $\langle \cdot,\cdot \rangle$ being the duality brackets for the pair $\big(W_0^{1,p}(Z), W^{-1,q}(Z)\big)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ and $\|\cdot\|$ denoting the norm of the Sobolev space $W_0^{1,p}(Z)$. Take $y = 0$. We obtain

$$\langle \Phi'(x_n), x_n \rangle + \psi(x_n) \leqslant \varepsilon_n \|x_n\| \text{ (note that } \psi(0) = 0). \tag{3}$$

Also for every $n \geqslant 1$, we have

$$R(x_n) = \Phi(x_n) + \psi(x_n) \leqslant c_1. \tag{4}$$

Adding (3) and (4), we obtain

$$\tag{5} \begin{aligned} &\frac{p+1}{p}\|Dx_n\|_p^p + 2\int_Z j(z, x_n(z))\,\mathrm{d}z - \int_Z F(z, x_n(z))\,\mathrm{d}z + \langle \Phi'(x_n), x_n\rangle \\ &\qquad \leqslant c_1 + \varepsilon_n \|x_n\|. \end{aligned}$$

By virtue of hypothesis $H(f)_1$(iv), given $\varepsilon > 0$ we can find $M > 0$ such that for almost all $z \in Z$ and all $|x| \geqslant M$ we have

$$F(z, r) \leqslant \frac{|r|^p}{p}(\theta(z) + \varepsilon).$$

On the other hand, for $r \in [-M, M]$ we have $|F(z, x)| \leqslant \gamma(z)$ a.e. on Z with $\gamma \in L^1(Z)$ (see hypothesis $H(f)_1$(iii)). Hence for almost all $z \in Z$ and all $r \in \mathbb{R}$ we have

$$\tag{6} F(z, r) \leqslant \frac{|r|^p}{p}(\theta(z) + \varepsilon) + \gamma(z).$$

Moreover, there exists $\beta > 0$ such that $V(x) = \|Dx\|_p^p - \int_Z \theta(z)|x(z)|^p\,\mathrm{d}z \geqslant \beta$ for all $x \in W_0^{1,p}(Z)$ with $\|Dx\|_p = 1$. If not, then we can find $\{x_n\} \subseteq W_0^{1,p}(Z)$ with $\|Dx_n\|_p = 1$, $n \geqslant 1$, such that $V(x_n) \downarrow 0$ as $n \to \infty$ (note that $V \geqslant 0$; see the Rayleigh quotient in Section 2). Recall that the L^p - norm of the gradient is equivalent to the $W_0^{1,p}(Z)$-norm and so by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and since $W_0^{1,p}(Z)$ is embedded compactly in $L^p(Z)$, we also have that $x_n \to x$ in $L^p(Z)$ as $n \to \infty$. From the weak lower semicontinuity of the norm in a Banach space, we have $\|Dx\|_p^p \leqslant \liminf \|Dx_n\|_p^p$. Thus

$$\tag{7} \begin{aligned} 0 = \lim V(x_n) &= \lim \left[\|Dx_n\|_p^p - \int_Z \theta(z)|x_n(z)|^p\,\mathrm{d}z\right] \\ &\geqslant \liminf \|Dx_n\|_p^p - \limsup \int_Z \theta(z)|x_n(z)|^p\,\mathrm{d}z \\ &\geqslant \|Dx\|_p^p - \int_Z \theta(z)|x(z)|^p\,\mathrm{d}z \\ &\qquad \text{(since } x_n \to x \text{ in } L^p(Z) \text{ as } n \to \infty). \end{aligned}$$

From the variational characterization of λ_1 via the Rayleigh quotient (see Section 2) we have

$$\tag{8} \begin{aligned} &\lambda_1 \|x\|_p^p \leqslant \|Dx\|_p^p \\ &\Rightarrow \int_Z \theta(z)|x(z)|^p\,\mathrm{d}z \leqslant \int_Z \|Dx(z)\|^p\,\mathrm{d}z. \end{aligned}$$

From (7) and (8) we obtain that

$$\int_Z \|Dx(z)\|^p \,\mathrm{d}z = \int_Z \theta(z)|x(z)|^p \,\mathrm{d}z. \tag{9}$$

Recall that $V(x_n) = 1 - \int_Z \theta(z)|x_n(z)|^p \,\mathrm{d}z \to 0$ as $n \to \infty$. So in the limit we have $0 = 1 - \int_Z \theta(z)|x(z)|^p \,\mathrm{d}z \Rightarrow \int_Z \theta(z)|x(z)|^p \,\mathrm{d}z = 1$. Using this in (9), we have that $\|Dx\|_p^p = 1 \Rightarrow x \neq 0$. From this we obtain that

$$\lambda_1 \int_Z |x(z)|^p \,\mathrm{d}z = \int_Z \|Dx(z)\|^p \,\mathrm{d}z = \int_Z \theta(z)|x(z)|^p \,\mathrm{d}z < \lambda_1 \int_Z |x(z)|^p \,\mathrm{d}z,$$

a contradiction. So indeed $0 < \beta \leqslant V(x)$ for all $x \in W_0^{1,p}(Z)$ with $\|Dx\|_p^p = 1$.

Returning to (5) and using (6) we obtain

$$\begin{aligned}
&\frac{p+1}{p}\|Dx_n\|_p^p - \frac{2}{p}\int_Z \theta(z)|x_n(z)|^p \,\mathrm{d}z - \frac{2\varepsilon}{p}\int_Z |x_n(z)|^p \,\mathrm{d}z \\
&\qquad - \|\gamma\|_1 - \int_Z f(z, x_n(z))x_n(z)\,\mathrm{d}z \leqslant c_1 + \varepsilon_n\|x_n\| \quad (\text{since } j \geqslant 0) \\
&\qquad \Rightarrow (\beta_1 - \frac{2\varepsilon}{\lambda_1 p})\|Dx_n\|_p^p - \xi \leqslant c_1 + \sigma_n\|x_n\|. \qquad (10)
\end{aligned}$$

for some $\xi > 0$ (see hypothesis $H(f)(v)$) and some $\beta_1 > 0$.

Choose $\varepsilon > 0$ such that $\varepsilon < \frac{\lambda_1\beta_1 p}{2}$. Then (10) implies that $\{x_n\} \subseteq W_0^{1,p}(Z)$ is bounded.

By passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$, $x_n \to x$ in $L^p(Z)$ (since $W_0^{1,p}(Z)$ is embedded compactly in $L^p(Z)$) $x_n(z) \to x(z)$ a.e. on Z as $n \to \infty$ and $|x_n(z)| \leqslant k(z)$ a.e. on Z, $k \in L^p(Z)$.

Recall that

$$\langle \Phi'(x_n), y - x_n\rangle + \psi(y) - \psi(x_n) \geqslant -\varepsilon_n\|y - x_n\| \text{ for all } y \in W_0^{1,p}(Z).$$

Take $y = x$. We have

$$\begin{aligned}
&\langle \Phi'(x_n), x - x_n\rangle + \psi(x) - \psi(x_n) \geqslant -\varepsilon_n\|x - x_n\| \\
&\Rightarrow -\int_Z f(z, x_n(z))(x - x_n)(z)\,\mathrm{d}z + \frac{1}{p}(\|Dx\|_p^p - \|Dx_n\|_p^p) \qquad (11)\\
&\quad + \int_Z j(z, x(z))\,\mathrm{d}z - \int_Z j(z, x_n(z))\,\mathrm{d}z \geqslant -\varepsilon_n\|x - x_n\|.
\end{aligned}$$

Note that

$$\int_Z f(z, x_n(z))(x - x_n)(z)\,\mathrm{d}z \to 0 \text{ as } n \to \infty$$

and

$$\liminf \int_Z j(z, x_n(z))\,\mathrm{d}z \geqslant \int_Z j(z, x(z))\,\mathrm{d}z \text{ (by Fatou's lemma).}$$

Therefore (11) yields that

$$\|Dx\|_p \geqslant \limsup \|Dx_n\|_p$$

$$\Rightarrow \|Dx_n\|_p \to \|Dx\|_p.$$

Since $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$, we have $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$. Because $L^p(Z, \mathbb{R}^N)$ is uniformly convex, we have that $x_n \to x$ in $W_0^{1,p}(Z)$, i.e. $R(\cdot)$ satisfies the (PS)-condition.

Claim 2. $R(\cdot)$ is bounded below.

For every $x \in W_0^{1,p}(Z)$ we have

$$\begin{aligned}
(12)\qquad R(x) &= \frac{1}{p}\|Dx\|_p^p + \int_Z j(z, x(z))\,\mathrm{d}z - \int_Z F(z, x(z))\,\mathrm{d}z \\
&\geqslant \frac{1}{p}\|Dx\|_p^p - \int_Z F(z, x(z))\,\mathrm{d}z \text{ (since } j \geqslant 0) \\
&\geqslant \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \theta(z)|x(z)|^p\,\mathrm{d}z - \frac{\varepsilon}{\lambda_1 p}\|Dx\|_p^p - \|\gamma\|_1 \\
&\qquad \text{(using (6) and the Rayleigh quotient)} \\
&\geqslant \frac{1}{p}\Big(\beta - \frac{\varepsilon}{\lambda_1}\Big)\|Dx\|_p^p - \|\gamma\|_1.
\end{aligned}$$

Choose $\varepsilon < \lambda_1\beta$. From (12) we infer that $R(\cdot)$ is coercive, thus bounded from below.

Apply Theorem 1, to obtain $x \in W_0^{1,p}(Z)$ such that

$$-\Phi'(x) \in \partial\psi(x).$$

Let $A\colon W_0^{1,p}(Z) \to W^{-1,q}(Z)$ be defined by

$$\langle A(x), y\rangle = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbb{R}^N}\,\mathrm{d}z.$$

It is easy to see that $A(\cdot)$ is monotone and demicontinuous (thus maximal monotone).

Also let $\psi_1\colon L^\mu(Z) \to \overline{\mathbb{R}}$ be defined by

$$\psi_1(x) = \begin{cases} \psi(x) & \text{if } x \in \operatorname{dom}\psi \\ \infty & \text{otherwise.} \end{cases}$$

It is easy to check that $\psi_1 \in \Gamma_0(L^\mu(Z))$. Let $A_1\colon D_1 \subseteq L^\mu(Z) \to L^{\mu'}(Z)$ $(\frac{1}{\mu}+\frac{1}{\mu'}=1)$ be defined by $A_1(x)=A(x)$ for all $x \in D_1 = \{y \in W_0^{1,p}(Z)\colon A(y) \in L^{\mu'}(Z)\}$ and let $V_1\colon L^\mu(Z) \to \overline{\mathbb{R}}$ be defined by $V_1(x) = \int_Z j(z,x(z))\,\mathrm{d}z$ if $x \in W_0^{1,p}(Z), j(\cdot,x(\cdot)) \in L^1(Z)$ and $+\infty$ otherwise (note that from the choice of μ we have that $W_0^{1,p}(Z)$ is embedded continuously in $L^\mu(Z)$). From Proposition 5.2, pp. 194–195 of Showalter [21] we have that $x \in W_0^{1,p}(Z)$ is a critical point of $R(\cdot)$ if and only if $-\widehat{f}(x) \in \partial\psi_1(x) = A_1(x) + \partial V_1(x)$, where $\widehat{f}(x(\cdot)) = f(\cdot,x(\cdot))$. So we have

$$\begin{aligned}&\int_Z f(z,x(z))\varphi(z)\,\mathrm{d}z\\&\quad=\int_Z \|Dx(z)\|^{p-2}(Dx(z),D\varphi(z))_{\mathbb{R}^N}\,\mathrm{d}z + \int_Z u(z)\varphi(z)\,\mathrm{d}z \text{ for all } \varphi \in C_0^\infty(Z)\end{aligned}$$

with $u \in \partial V_1(x) \subseteq L^{\mu'}(Z)$. Since $f(\cdot,x(\cdot)) - u(\cdot) \in L^{\mu'}(Z)$, the definition of the distributional derivative yields

$$-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) + u(z) = f(z,x(z)) \text{ a.e. on } Z, x\big|_\Gamma = 0.$$

However $u \in \partial V_1(x)$ implies that $u(z) \in \partial j(z,x(z)) = \beta(z,x(z))$ a.e. on Z. So we have

$$\left\{\begin{array}{l}-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) + \beta(z,x(z)) \ni f(z,x(z)) \text{ a.e. on } Z\\ x\big|_\Gamma = 0,\ 2 \leqslant p < +\infty\end{array}\right\} \tag{13}$$

$\Rightarrow x(\cdot)$ is a solution of (1). $\square$

We can have another existence theorem for problem (1) under a different set of hypotheses on the data $f(z,x)$ and $\beta(z,x)$. So the hypotheses are now the following:

$\mathbf{H}(f)_2$: $f\colon Z\times\mathbb{R}\to\mathbb{R}$ is a function such that
(i) for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable;
(ii) for almost all $z \in Z$, $x \to f(z,x)$ is continuous;
(iii) for almost all $z \in Z$ and all $x \in \mathbb{R}$, $|f(z,x)| \leqslant \alpha(z) + c|x|^{\mu-1}$, $2 \leqslant p < \mu < p^*$, $\alpha \in L^\infty(Z)$;
(iv) there exist $\theta > p$ and $r_0 > 0$ such that for almost all $z \in Z$ and all $|x| \geqslant r_0$, $0 < \theta F(z,x) \leqslant f(z,x)x$;
(v) $\lim\limits_{x\to0}\dfrac{pF(z,x)}{|x|^p} < \lambda_1$ uniformly for almost all $z \in Z$.

Remark. Hypotheses $H(f)_2$(iv), (v) were first introduced by Ambrosetti-Rabinowitz [3] in the context of semilinear systems (i.e. $p=2$) with continuous $f(z,x)$ and with $\beta = 0$.

$\mathbf{H}(\beta)_2$: $\beta(z,x) = \partial j(z,x)$ where $j\colon Z \times \mathbb{R} \to \overline{\mathbb{R}_+} = \mathbb{R} \cup \{+\infty\}$ is a normal convex integrand, for almost all $z \in Z$, $j(z,0) = 0$, for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $v \in \partial j(z,x)$ we have $vx \leqslant \theta j(z,x)$ and if $u_1 \in W_0^{1,p}(Z)$ is the normalized eigenfunction to the simple first eigenvalue $\lambda_1 > 0$ of $(-\Delta_p, W_0^{1,p}(Z))$, then $\limsup\limits_{\xi\to\infty} \frac{1}{\xi^p} \int_Z j(z, \xi u_1(z))\,\mathrm{d}z < \infty$.

Theorem 5. *If hypotheses $H(f)_2$ and $H(\beta)_2$ hold, then problem* (1) *has a nontrivial solution.*

P r o o f. As in the proof of theorem 4, we consider the functionals Φ, ψ and $R = \Phi + \psi$.

Claim #1. $R(\cdot)$ satisfies the (PS)-condition (in the sense of Szulkin, see Section 2).

Let $\{x_n\} \subseteq W_0^{1,p}(Z)$ be such that $R(x_n) \to c$ as $n \to \infty$ and

$$\langle \Phi'(x_n), y - x_n\rangle + \psi(y) - \psi(x_n) \geqslant -\varepsilon_n \|y - x_n\|$$

for all $y \in W_0^{1,p}(Z)$ and with $\varepsilon_n \downarrow 0$. Let us divide $y = x_n + tx_n$, $t > 0$, by t and finally pass to the limit as $t \downarrow 0$. We obtain

$$-\int_Z f(z, x_n(z)) x_n(z)\,\mathrm{d}z + \psi'(x_n; x_n) \geqslant -\varepsilon_n \|x_n\|.$$

where $\psi'(x_n; x_n)$ is the directional derivative of $\psi(\cdot)$ at x_n in the direction x_n. Note that from the choice of the sequence $\{x_n\}$ we have that $\partial\psi(x_n) \neq \emptyset$ (see the remark in Section 2 from which it follows at once that $u_n - \Phi'(x_n) \in \partial\psi(x_n)$ for all $n \geqslant 1$). Moreover, if $\psi_1\colon L^\mu(Z) \to \overline{\mathbb{R}_+}$ is as in the proof of Theorem 4, using Lemma 5.2 and Proposition 5.2, pp. 194–195 of Showalter [21], we have that $\partial\psi(x_n) = \partial\psi_1(x_n) \subseteq L^{\mu'}(Z)$ $(\frac{1}{\mu} + \frac{1}{\mu'} = 1)$. Since $\psi'(x_n;\cdot)$ is the support function of $\partial\psi(x_n)$, given any $\delta > 0$ we can find $w_n \in \partial\psi(x_n) = \partial\psi_1(x_n)$, $w_n \in L^{\mu'}(Z)$ such that $(w_n, x_n)_{\mu,\mu'} + \delta \geqslant \psi'(x_n; x_n)$, where $(\cdot,\cdot)_{\mu,\mu'}$ denotes the duality brackets of $(L^\mu(Z), L^{\mu'}(Z))$. Recall that $\partial\psi_1 = A_1 + \partial V_1$ (see the proof of Theorem 4). So $w_n = A_1(x_n) + u_n = A(x_n) + u_n$ with $u_n \in \partial V_1(x_n)$. So we have

$$\int_Z f(z, x_n(z)) x_n(z)\,\mathrm{d}z - \|Dx_n\|_p^p - (u_n, x_n)_{\mu,\mu'} - \delta \leqslant \varepsilon_n \|x_n\|. \tag{14}$$

From the choice of the sequence $\{x_n\} \subseteq W_0^{1,p}(Z)$, we have that

$$\theta R(x_n) \leqslant M_1 \text{ for some } M_1 > 0. \tag{15}$$

Adding (14) and (15), we obtain

$$\Big(\frac{\theta}{p}-1\Big)\|Dx_n\|_p^p+\int_Z(f(z,x_n(z))x_n(z)-\theta F(z,x_n(z)))\,\mathrm{d}z$$
$$+\int_Z(\theta j(z,x_n(z))-u_n(z)x_n(z))\,\mathrm{d}z\leqslant\|Dx_n\|_p+M_2$$

for some $M_2>0$ (here we have used the gradient norm on $W_0^{1,p}(Z)$). Since $u_n\in\partial V_1(x_n)$, we have $u_n(z)\in\partial j(z,x_n(z))=\beta(z,x_n(z)))$ a.e. on Z (see for example Showalter [21], Example 8.B, p. 85). Then using hypotheses $H(f)_2(iv)$ and $H(\beta)_2$ we obtain

$$\int_Z(f(z,x_n(z))x_n(z)-\theta F(z,x_n(z)))\,\mathrm{d}z\geqslant 0$$

and

$$\int_Z(\theta j(z,x_n(z))-u_n(z)x_n(z))\,\mathrm{d}z\geqslant 0.$$

Therefore we can write

$$\Big(\frac{\theta}{p}-1\Big)\|Dx_n\|_p^p\leqslant\|Dx_n\|_p+M_2.$$

Since $\theta>p$, from the above inequality we infer that $\{Dx_n\}\subseteq L^p(Z,\mathbb{R}^N)$ is bounded,hence $\{x_n\}\subseteq W_0^{1,p}(Z)$ is bounded (Poincare's inequality). Then arguing as in the proof of theorem 4, we can extract a strongly convergent subsequence. So $R(\cdot)$ satisfies the (PS)-condition.

Next from hypotheses $H(f)_2(v)$, given $\varepsilon>0$ we can find $\gamma_\varepsilon>0$ such that for almost all $z\in Z$ and all $|x|\geqslant\gamma_\varepsilon$ we have

$$F(z,x)\leqslant\frac{\lambda_1-\varepsilon}{p}|x|^p.$$

On the other hand from hypothesis $H(f)_2$(iii), we have

$$|f(z,x)|\leqslant\alpha(z)+c|x|^{\mu-1}\text{ a.e. on }Z\text{ for all }|x|\geqslant\gamma_\varepsilon$$
$$\Rightarrow F(z,x)\leqslant\alpha(z)|x|+c_5|x|^\mu\text{ a.e. on }Z\text{ for all }|x|\geqslant\gamma_\varepsilon\text{ with }c_5>0.$$

Therefore we infer that for almost all $z\in Z$ and all $x\in\mathbb{R}$

$$F(z,x)\leqslant\frac{\lambda_1-\varepsilon}{p}|x|^p+c_6|x|^\mu. \tag{16}$$

Using (16), we have

$$R(x)=\frac{1}{p}\|Dx\|_p^p+\int_Z j(z,x(z))\,\mathrm{d}z-\int_Z F(z,x(z))\,\mathrm{d}z \tag{17}$$
$$\geqslant\frac{1}{p}\Big(1-\frac{\lambda_1-\varepsilon}{\lambda_1}\Big)\|Dx\|_p^p-c_7\|Dx\|_p^\mu\quad\text{for some }c_7>0.$$

(in the last inequality we have used the Rayleigh quotient and the fact that $W_0^{1,p}(Z)$ is embedded continuously in $L^{\mu}(Z)$). Since $p < \mu$ from (17) and with $\varepsilon > 0$ small enough we have that

$$R(x) \geqslant \eta > 0 \text{ for all } x \in W_0^{1,p}(Z) \text{ with } \|x\| = \varrho.$$

Also from hypothesis $H(f)_2$(iv), for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$F(z,x) \geqslant c_8|x|^{\theta} - c_9 \text{ for some } c_8, c_9 > 0 \tag{18}$$

(see Rabinowitz [20], remark 2.13 (ii), p. 9). Then for all $\xi > 0$ we have

$$\begin{aligned} R(\xi u_1) &= \frac{\xi^p}{p}\|Du_1\|_p^p + \int_Z j(z, \xi u_1(z))\,\mathrm{d}z - \int_Z F(z, \xi u_1(z))\,\mathrm{d}z \\ &\leqslant \frac{\xi^p}{p}\|Du_1\|_p^p + \int_Z j(z, \xi u_1(z))\,\mathrm{d}z + c_{10} - c_8\xi^{\theta}\|u_1\|_{\theta}^{\theta},\ c_{10} = c_9|Z| \text{ (see (18))} \\ &\leqslant \xi^p(c_{11} - c_{12}\xi^{\theta-p}) + \int_Z j(z, \xi u_1(z))\,\mathrm{d}z \text{ for some } c_{11},\ c_{12} > 0. \end{aligned}$$

By virtue of the last part of hypothesis $H(\beta)_2$, we see that for $\xi > \varrho$ large enough we will have $R(\xi u_1) \leqslant 0$. Hence we can apply Theorem 2 to deduce that $R(\cdot)$ has a critical point $x \in W_0^{1,p}(Z)$. So $-\Phi'(x) \in \partial\psi(x)$ and $R(x) \geqslant \eta > 0 = R(0)$, thus $x \neq 0$. Then as in the proof of Theorem 4 we can verify that $x \in W_0^{1,p}(Z)$ is a nontrivial solution of (1). $\square$

Remark. Problem (1) incorporates as a special case problems with monotone discontinuities. In this direction we should mention the important work of Chang [7], who studied semilinear problems with discontinuities (not necessarily monotone) using the subdifferential theory of locally Lipschitz functionals. Equations of the form (1) arise in physical problems, like in the study of a homogeneous gas flowing through a homogeneous porous medium (see for example Ames [2]).

4. Neumann problems

In this section we consider a quasilinear Neumann problem with multivalued boundary condition. More precisely, we study the following problem:

$$\left\{ \begin{array}{c} -\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) = f(z, x(z)) - h(z) \text{ a.e. on } Z \\ \dfrac{\partial x}{\partial n_p}(z) \in \beta(z, \tau(x)(z)) \text{ a.e. on } \Gamma,\ 2 \leqslant p < \infty \end{array} \right\}. \tag{19}$$

Here $\frac{\partial x}{\partial n_p}(z) = (\|Dx(z)\|^{p-2}Dx(z), n(z))_{\mathbb{R}^N}$ with $n(z)$ denoting the outward normal at $z \in \Gamma$ and τ is the trace operator on $W^{1,p}(Z)$. On Γ we consider the $(N-1)$-dimensional Hausdorff measure.

Our hypotheses on $f(z,x)$ and $\beta(z,x)$ are the following:

$\mathbf{H}(f)_3$: $f\colon Z \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) for almost all $z \in Z$ and all $x \in \mathbb{R}$, $|f(z,x)| \leqslant \alpha(z) + c|x|^{\theta-1}$ with $\alpha \in L^\infty(Z)$, $c > 0$, $1 \leqslant \theta < p$;

(ii) uniformly for almost all $z \in Z$ we have that

$$\frac{f(z,x)}{|x|^{\theta-2}x} \to f_+(z) \quad \text{as } |x| \to +\infty$$

where $f_+ \in L^1(Z)$, $f_+ \geqslant 0$ with strict inequality on a set of positive Lebesgue measure.

$\mathbf{H}(\beta)_3$: $\beta(z,x) = \partial j(z,x)$ where $j\colon Z \times \mathbb{R} \to \mathbb{R}_+$ is a Caratheodory convex integrand such that for almost all $z \in Z$ and all $x \in \mathbb{R}$, $|\beta(z,x)| = sup[|u|\colon u \in \beta(z,x)] \leqslant \alpha_1(z) + c_1|x|^\mu$, $0 \leqslant \mu < \theta - 1$ (θ the same as in $H(f)_3(i)$) with $\alpha_1 \in L^\infty$, $c_1 > 0$ and $j(\cdot,0) \in L^\infty(Z)$.

Theorem 6. *If hypotheses* $\mathbf{H}(f)_3$ *and* $\mathbf{H}(\beta)_3$ *hold, then problem* (19) *has a nontrivial solution.*

P r o o f. Let $\Phi\colon W^{1,p}(Z) \to \mathbb{R}$ and $\psi\colon W^{1,p}(Z) \to \mathbb{R}_+$ be defined by

$$\Phi(x) = -\int_Z F(z,x(z))\,\mathrm{d}z \text{ and } \psi(x) = \frac{1}{p}\|Dx\|_p^p + \int_\Gamma j(z,\tau(x)(z))\,\mathrm{d}\sigma.$$

In the definition of $\Phi(\cdot), F(z,x) = \int_0^x f(z,r)\,\mathrm{d}r$ (the potential of f), $\tau(\cdot)$ is the trace operator on $W^{1,p}(Z)$ and $\mathrm{d}\sigma$ is the $(N-1)$-dimensional Hausdorff measure. Clearly $\Phi \in C^1(W^{1,p}(Z))$, while as before we can check that $\psi \in \Gamma_0(W^{1,p}(Z))$. Set $R = \Phi + \psi$.

Claim 1. $R(\cdot)$ satisfies the (PS)-condition (in the sense of Szulkin, see Section 1).

Let $\{x_n\} \subseteq W^{1,p}(Z)$ and assume that $R(x_n) \to c$ as $n \to \infty$ and

$$\langle \Phi'(x_n), y - x_n\rangle + \psi(y) - \psi(x_n) \geqslant -\varepsilon_n\|y - x_n\|$$

for all $y \in W^{1,p}(Z)$, with $\varepsilon_n \downarrow 0$. Set $y = x_n - tx_n$, $t > 0$, divide by t and let $t \downarrow 0$. As in the proof of Theorem 5, in the limit we obtain

$$-\langle \Phi'(x_n), x_n\rangle - \|Dx_n\|_p^p - \int_\Gamma w_n(z)x_n(z)\,\mathrm{d}\sigma \geqslant -\varepsilon_n\|x_n\|$$

with $w_n \in \partial V_1(x_n)$ where $V_1\colon L^p(\Gamma) \to \mathbb{R}_+$ is defined by $V_1(x) = \int_\Gamma j(z, x(z))\,\mathrm{d}\sigma$.

Recall that $w_n(z) \in \partial j(z, \tau(x)(z)) = \beta(z, \tau(x_n)(z))$ a.e. on Γ (see Showalter [21], p. 85). So we have

$$\int_Z f(z, x_n(z))x_n(z)\,\mathrm{d}z - \|Dx_n\|_p^p - \int_\Gamma w_n(z)\tau(x_n)(z)\,\mathrm{d}\sigma \geqslant -\varepsilon_n\|x_n\|.$$

Suppose that $\{x_n\} \subseteq W^{1,p}(Z)$ is unbounded. Then (at least for a subsequence) we may assume that $\|x_n\| \to \infty$. Let $y = \frac{x_n}{\|x_n\|}$, $n \geqslant 1$. By passing to a subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(Z),\ y_n \to y \text{ in } L^p(Z),\ y_n(z) \to y(z) \text{ a.e. on } Z \text{ as } n \to \infty$$

and $|y_n(z)| \leqslant k(z)$ a.e. on Z with $k \in L^p(Z)$.

Recall that from the choice of the sequence $\{x_n\}$ we have $|R(x_n)| \leqslant M_1$ for some $M_1 > 0$ and all $n \geqslant 1$, which yields $\Rightarrow \frac{1}{p}\|Dx_n\|_p^p + \int_Z j(z, \tau(x_n)(z))\,\mathrm{d}\sigma - \int_Z F(z, x_n(z))\,\mathrm{d}z \leqslant M_1 \Rightarrow \frac{1}{p}\|Dx_n\|_p^p - \int_Z F(z, x_n(z))\,\mathrm{d}z \leqslant M_1$ (since $j \geqslant 0$).

Divide by $\|x_n\|^p$. We obtain

$$\frac{1}{p}\|Dy_n\|_p^p - \int_Z \frac{F(z, x_n(z))}{\|x_n\|^p}\,\mathrm{d}z \leqslant \frac{M_1}{\|x_n\|^p}. \tag{20}$$

We have

$$\begin{aligned}\left|\int_Z \frac{F(z, x_n(z))}{\|x_n\|^p}\,\mathrm{d}z\right| &\leqslant \frac{1}{\|x_n\|^p}\int_Z\int_0^{|x_n(z)|} |f(z, r)|\,\mathrm{d}r\,\mathrm{d}z \\ &\leqslant \frac{1}{\|x_n\|^p}\Big(\|\alpha\|_\infty\|x_n\| + \frac{c}{\theta}\|x_n\|^\theta\Big) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

So by passing to the limit as $n \to \infty$ in (20) we obtain

$$\begin{aligned}&\lim \frac{1}{p}\|Dy_n\|_p^p = 0 \\ &\Rightarrow \|Dy\|_p = 0 \ \ (\text{recall that } Dy_n \xrightarrow{w} Dy \text{ in } L^p(Z, \mathbb{R}^N) \text{ as } n \to \infty) \\ &\Rightarrow y = \xi \in \mathbb{R}.\end{aligned}$$

Note that $y_n \to \xi$ in $W_0^{1,p}(Z)$ and since $\|y_n\| = 1$, $n \geqslant 1$ we infer that $\xi \neq 0$. We deduce that $|x_n(z)| \to +\infty$ a.e. on Z as $n \to \infty$.

From the choice of the sequence $\{x_n\} \subseteq W^{1,p}(Z)$ we have

$$\int_Z f(z, x_n(z))x_n(z)\,\mathrm{d}z - \|Dx_n\|_p^p - \int_Z w_n(z)\tau(x_n)(z)\,\mathrm{d}z \geqslant -\varepsilon_n\|x_n\| \tag{21}$$

and

$$\|Dx_n\|_p^p + p\int_\Gamma j(z, \tau(x)(z))\,\mathrm{d}\sigma - p\int_Z F(z, x_n(z))\,\mathrm{d}z \geqslant -pM_1. \tag{22}$$

Adding (21) and (22), we obtain

$$\int_\Gamma (pj(z,\tau(x_n)(z)) - w_n(z)\tau(x_n)(z))\,\mathrm{d}\sigma + \int_Z (f(z,x_n(z))x_n(z) - pF(z,x_n(z)))\,\mathrm{d}z \geqslant -pM_1 - \varepsilon_n\|x_n\|.$$

Divide this inequality by $\|x_n\|^\theta$. We have

$$\begin{aligned}(23)\qquad & \int_Z \frac{f(z,x_n(z))}{\|x_n\|^{\theta-1}} y_n(z)\,\mathrm{d}z - \int_Z \frac{pF(z,x_n(z))}{\|x_n\|^\theta}\,\mathrm{d}z \\ & + \int_\Gamma \frac{pj(z,\tau(x_n)(z) - w_n(z)\tau(x_n)(z)}{\|x_n\|^\theta}\,\mathrm{d}\sigma \\ & \geqslant -\frac{1}{\|x_n\|^\theta} pM_1 - \frac{\varepsilon_n}{\|x_n\|^{\theta-1}}.\end{aligned}$$

Note that

$$\begin{aligned}& \int_Z \frac{f(z,x_n(z))}{\|x_n\|^{\theta-1}} y_n(z)\,\mathrm{d}z \\ & = \int_Z \frac{f(z,x_n(z))}{|x_n(z)|^{\theta-2}x_n(z)} |y_n(z)|^\theta\,\mathrm{d}z \to |\xi|^\theta \int_Z f_+(z)\,\mathrm{d}z \text{ as } n\to\infty.\end{aligned}$$

Also by virtue of hypothesis $H(f)_3$(ii), given $z \in Z\setminus N$, $|N| = 0$ ($|C|$ denotes the Lebesgue measure of a measurable set $C \subseteq Z$) and $\varepsilon > 0$, we can find $M_\varepsilon > 0$ such that for all $|r| \geqslant M_\varepsilon$ we have $|f_+(z) - \frac{f(z,r)}{|r|^{\theta-2}r}| \leqslant \varepsilon$. Then, if $x_n(z) \to +\infty$, we have

$$\begin{aligned}(24)\qquad \frac{1}{|x_n(z)|^\theta} F(z,x_n(z))\,\mathrm{d}z & \geqslant \frac{1}{|x_n(z)|^\theta} F(z,M_\varepsilon)\,\mathrm{d}z \\ & \quad + \frac{1}{|x_n(z)|^\theta} \int_{M_\varepsilon}^{x_n(z)} (f_+(z)|r|^{\theta-2}r - \varepsilon|r|^{\theta-2}r)\,\mathrm{d}r \\ & = \frac{1}{|x_n(z)|^\theta}\eta(z) + \frac{|x_n(z)|^\theta - M_\varepsilon^\theta}{\theta|x_n(z)|^\theta}(f_+(z)-\varepsilon) \\ & \quad \text{for some } \eta \in L^1(Z) \\ \Rightarrow \liminf_{n\to\infty} \frac{F(z,x_n(z))}{|x_n(z)|^\theta} & \geqslant \frac{1}{\theta}(f_+(z)-\varepsilon).\end{aligned}$$

Similarly we obtain that

$$(25)\qquad \limsup_{n\to\infty} \frac{F(z,x_n(z))}{|x_n(z)|^\theta} \leqslant \frac{1}{\theta}(f_+(z)+\varepsilon).$$

From (24) and (25) and since $\varepsilon > 0$ and $z \in Z \setminus N$ were arbitrary, we infer that

$$\frac{F(z, x_n(z))}{|x_n(z)|^\theta} \to \frac{1}{\theta} f_+(z) \text{ a.e. on } Z \text{ as } n \to \infty$$

$$(26) \Rightarrow \int_Z \frac{F(z, x_n(z))}{\|x_n\|^\theta} \,\mathrm{d}z = \int_Z \frac{F(z, x_n(z))}{|x_n(z)|^\theta} \frac{|x_n(z)|^\theta}{\|x_n\|^\theta} \,\mathrm{d}z$$
$$= \int_Z \frac{F(z, x_n(z))}{|x_n(z)|^\theta} |y_n(z)|^\theta \,\mathrm{d}z \to \xi^\theta \int_Z \frac{1}{\theta} f_+(z) \text{ as } n \to \infty.$$

Note that since $j(z, \cdot)$ is convex and continuous for almost all $z \in Z$, it is locally Lipschitz. So by Lebourg's mean value theorem, for almost all $z \in Z$ and all $x \in \mathbb{R}$ we can find $w \in \beta(z, \eta x)$, $0 < \eta < 1$ such that

$$|j(z, x) - j(z, 0)| = wx$$
$$\Rightarrow |j(z, x)| \leqslant |j(z, \cdot)| + |w||x| \leqslant \beta + |w||x| \text{ (since } j(\cdot, \cdot) \in L^\infty(Z)).$$

However by $H(\beta)_3$ we have

$$|w| \leqslant a_1(z) + c_1|x|^\mu$$
$$\Rightarrow |j(z, x)| \leqslant a_2 + c_2|x|^{\mu+1} \text{ for some } a_2,\ c_2 > 0.$$

So it is easy to see that

$$\int_\Gamma \frac{pj(z, \tau(x_n)(z)) - w_n(z)\tau(x_n)(z)}{\|x_n\|^\theta} \,\mathrm{d}\sigma \to 0 \text{ as } n \to \infty \text{ (recall } \mu + 1 < \theta).$$

Thus by passing to the limit in (23), we obtain

$$\Big(1 - \frac{p}{\theta}\Big)\xi^\theta \int_Z f_+(z) \geqslant 0,$$

a contradiction to hypothesis $H(f)_3$(ii) (recall that $p > \theta$). If $x_n(z) \to -\infty$, by similar arguments as above we show that

$$\int_Z \frac{F(z, x_n(z))}{\|x_n\|^\theta} \,\mathrm{d}z \to \xi^\theta \int_Z \frac{1}{\theta} f_+(z) \text{ as } n \to \infty$$

(note that $\int_0^{x_n(z)} f(z, r) \,\mathrm{d}r = -\int_{x_n(z)}^{o} f(z, r) \,\mathrm{d}r$). Therefore it follows that $\{x_n\} \subseteq W^{1,p}(Z)$ is bounded. Hence we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(Z), x_n \to x$ in $L^p(Z)$, $x_n(z) \to x(z)$ a.e. on Z as $n \to \infty$ and $|x_n(z)| \leqslant k(z)$ a.e. on Z with $k \in L^p(Z)$. We have

$$\langle \Phi'(x_n), y - x_n \rangle + \psi(y) - \psi(x_n) \geqslant -\varepsilon_n \|y - x_n\| \text{ for all } y \in W^{1,p}(Z).$$

Take $y = x$. We obtain

$$
\begin{aligned}
(27) \qquad & \langle \Phi'(x_n), x - x_n \rangle + \frac{1}{p}(\|Dx\|_p^p - \|Dx_n\|_p^p) \\
& + \int_\Gamma (j(z, \tau(x)(z)) - j(z, \tau(x_n)(z)) \, \mathrm{d}\sigma \geqslant -\varepsilon_n \|x - x_n\|.
\end{aligned}
$$

Note that

$$
(28) \qquad \langle \Phi'(x_n), x - x_n \rangle \to 0 \text{ as } n \to \infty.
$$

From the compactness of the trace operator (see Kufner-John-Fučik [17], Theorem 6.10.5, p. 344) we have that $\tau(x_n) \to \tau(x)$ in $L^p(\Gamma)$ as $n \to \infty$ and by passing to a subsequence if necessary, we may also assume that $\tau(x_n)(z) \to \tau(x)(z)$ a.e. on Γ as $n \to \infty$. So by Fatou's lemma we have

$$
\limsup_{n \to \infty} \int_\Gamma -j(z, \tau(x_n)(z)) \, \mathrm{d}\sigma \leqslant \int_\Gamma -j(z, \tau(x)(z)) \, \mathrm{d}\sigma
$$

and by passing to the limit in (27), we obtain

$$
\limsup \|Dx_n\|_p^p \leqslant \|Dx\|_p^p.
$$

However from the weak lower semicontinuity of the norm functional we also have that

$$
\begin{aligned}
& \liminf \|Dx_n\|_p^p \geqslant \|Dx\|_p^p \\
\Rightarrow\ & \|Dx_n\|_p \to \|Dx\|_p \text{ as } n \to \infty.
\end{aligned}
$$

Since $Dx_n \xrightarrow{w} Dx$ in $L^p(Z)$ and the latter is uniformly convex, we conclude that $x_n \to x$ in $W^{1,p}(Z)$ as $n \to \infty$, which proves that $R(\cdot)$ satisfies the (PS)-condition.

Now let $W^{1,p}(Z) = X_1 \oplus X_2$ with $X_1 = \mathbb{R}$ and $X_2 = \{y \in W^{1,p}(Z) \colon \int_Z y(z) \, \mathrm{d}z = 0\}$. For every $\xi \in X_1$ we have

$$
\begin{aligned}
R(\xi) = \Phi(\xi) + \psi(\xi) &= \int_Z j(z, \xi) \, \mathrm{d}\sigma - \int_Z F(z, \xi) \, \mathrm{d}z \\
&\leqslant \|\alpha_1\|_\infty |\xi| |\Gamma| + \frac{c_1}{\mu} |\xi|^\mu |\Gamma| - \int_Z F(z, \xi) \, \mathrm{d}z \text{ (see hypothesis } \mathbf{H}(\beta)_3) \\
\Rightarrow \frac{1}{|\xi|^\mu} R(\xi) &\leqslant \frac{1}{|\xi|^{\mu-1}} \|\alpha_1\|_\infty |\Gamma| + \frac{c}{\mu} |\Gamma| - \frac{1}{|\xi|^\mu} \int_Z F(z, \xi) \, \mathrm{d}z.
\end{aligned}
$$

By virtue of hypothesis $\mathbf{H}(f)_3$(ii) we conclude that $R(\xi) \to -\infty$ as $|\xi| \to \infty$. On the other hand for $y \in X_2$, we have

$$
\begin{aligned}
R(y) &\geqslant \frac{1}{p} \|Dy\|_p^p - \int_Z F(z, y(z)) \, \mathrm{d}z \text{ (since } j \geqslant 0) \\
&\geqslant \frac{1}{p} \|Dy\|_p^p - c_2 \|y\|_p - c_3 \|y\|_p^\theta \text{ for some } c_2,\ c_3 > 0 \text{ (since } \theta < p, \text{ see } \mathbf{H}(f)_3\text{(i))}.
\end{aligned}
$$

From the Poincaré-Wirtinger inequality we know that $\|Dy\|_p$ is an equivalent norm on X_2. So we have

$$R(y) \geqslant \frac{1}{p}\|Dy\|_p^p - c_4\|Dy\|_p - c_5\|Dy\|_p^\theta \ \text{ for some } c_4, c_5 > 0,$$

which implies that $R(\cdot)$ is coercive on X_2 (recall that $\theta < p$), hence bounded below on X_2.

Apply Theorem 3 to produce $x \in W^{1,p}(Z)$, $x \neq 0$ such that $-\Phi'(x) \in \partial\psi(x)$. If $\psi_1\colon L^p(Z) \to \overline{\mathbb{R}}$ is defined by

$$\psi_1(x) = \begin{cases} \psi(x) & \text{if } x \in \operatorname{dom}\psi = W^{1,p}(Z) \\ \infty & \text{otherwise} \end{cases}$$

then from the Proposition 5.2, pp. 194–195 of Showalter [21] we have that $-\widehat{f}(x) \in \partial\psi_1(x)$ with $\widehat{f}(x)(\cdot) = f(\cdot, x(\cdot))$ (see the proof of Theorem 4).

Let $V\colon W^{1,p}(Z) \to \mathbb{R}$ be the convex integrand functional defined by

$$V(x) = \int_\Gamma j(z, \tau(x)(z))\,\mathrm{d}\sigma.$$

Clearly $V \in C(W^{1,p}(Z))$ (see $\mathbf{H}(\beta)_3$). Let $A\colon W^{1,p}(Z) \to W^{1,p}(Z)^*$ be defined by

$$\langle A(x), y\rangle = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbb{R}^N}\,\mathrm{d}z$$

Recall that $A(\cdot)$ is monotone, demicontinuous, hence maximal monotone. Then $K = A + \partial V\colon W^{1,p}(Z) \to 2^{W^{1,p}(Z)^*}$ is maximal monotone (see Zeidler [23], Theorem 32.I, p. 888). Let $D = \{x \in W^{1,p}(Z)\colon K(x) \cap L^q(Z) \neq \emptyset\}$ ($\frac{1}{p} + \frac{1}{q} = 1$) and define $K_1\colon D \subseteq L^q(Z) \to 2^{L^q(Z)}$ by $K_1(x) = K(x) \cap L^q(Z)$.

Claim 2. $K_1(\cdot)$ is maximal monotone.

Let $C\colon L^p(Z) \to L^q(Z)$ be defined by $C(x)(\cdot) = |x(\cdot)|^{p-2}x(\cdot)$. First note that if $v \in \partial V(x)$, then $\langle v, \eta\rangle \leqslant V(x+\eta) - V(x) = 0$ for all $\eta \in C_0^\infty(Z)$ and so $v = 0$ in the sense of distributions. Hence $K_1(\cdot)$ is single valued and $K_1(x) = A(x)$ in the sense of distributions. Now we will show that $R(K_1 + C) = L^q(Z)$. Note that $C(\cdot)$ is continuous monotone, hence $K + C$ is maximal monotone. Moreover, for every $u \in K(x)$, $\langle u + C(x), x\rangle \geqslant \langle A(x), x\rangle + \langle C(x), x\rangle = \|Dx\|_p^p + \|x\|_p^p = \|x\|^p$. Thus $K + C$ is coercive, therefore surjective (see Zeidler [23], Corollary 32.35, pp. 887). So if $g \in L^q(Z)$, we can find $(x, u) \in GrK$ such that $u + C(x) = g \Rightarrow u = g - C(x) \in L^q(Z) \Rightarrow u \in K(x) \cap L^q(Z) = K_1(x)$, i.e. $u = K_1(x)$. Since $g \in L^q(Z)$ was arbitrary, we deduce that $R(K_1+C)=L^q(Z)$. Next we will show that this surjectivity implies the maximality of K_1. Indeed, suppose that for a pair $(u,v)\in L^p(Z)\times L^q(Z)$ we have $(K_1(x)-v,x-u)_{pq}\geqslant 0$ for all $x\in D$. Take $x\in D$ such that $K_1(x)+C(x)=v+C(u)$ (recall $R(K_1+C)=L^q(Z)$) and $v+C(u)\in L^q(Z)$. So $v=K_1(x)+C(x)-C(u)$ and we have $(K_1(x)-K_1(x)-C(x)+C(u),x-u)_{pq}=(C(u)-C(x),x-u)_{pq}\geqslant 0$. But $C(\cdot)$ is strictly monotone. So $x=u$ and $K_1(x)=v$, i.e. $K_1(\cdot)$ is maximal monotone.

Because $K_1\subseteq\partial\psi_1$, using Claim 2, we infer that $K_1=\partial\psi_1$.

Now let $L\colon W^{\frac{1}{q},p}(\Gamma)\to\mathbb{R}$ be defined by $L(u)=\int_\Gamma j(z,u(z))\,\mathrm{d}\sigma$. By virtue of hypothesis $\mathbf{H}(\beta)_3$, we see that $L(\cdot)$ is continuous and convex. Also $L\circ\tau=V$.

Claim 3. If $g\in L^q(Z)$ and $g=K_1(x)$ for some $x\in D$, then we have $-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z))=g(z)$ a.e. on Z and $\frac{\partial x}{\partial n_p}\in\beta(z,\tau(x)(z))$ a.e. on Γ.

Since $x\in D\subseteq W^{1,p}(Z)$ by virtue of the representation theorem for the elements of $W^{-1,q}(Z)$ (see Adams [1], Theorem 3.10, p. 50), we have that $\operatorname{div}(\|Dx\|^{p-2}Dx)\in W^{-1,q}(Z)$. Recall that for all $x\in D$, $K_1(x)=A(x)$ in the sense of distributions. So for any $\eta\in C_0^\infty(Z)$ we have

$$
\begin{aligned}
(g,\eta)_{pq} &= \langle A(x),\eta\rangle=\int_Z\|Dx(z)\|^{p-2}(Dx(z),D\eta(z))_{\mathbb{R}^N}\,\mathrm{d}z\\
&= \big\langle -\operatorname{div}(\|Dx\|^{p-2}Dx,\eta\big\rangle\\
&\Rightarrow -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)=g(z)\text{ a.e. on } Z
\end{aligned}
\tag{29}
$$

(since $C_0^\infty(Z)$ is dense in $W_0^{1,p}(Z)$ and $W^{-1,q}(Z)=W_0^{1,p}(Z)^*$). Because

$$\operatorname{div}(\|Dx\|^{p-2}Dx)\in L^q(Z)\quad\text{and}\quad\|Dx\|^{p-2}Dx\in L^q(Z,\mathbb{R}^N),$$

from Proposition 1.4 of Kenmochi [16] we have that $\frac{\partial x}{\partial n_p}\in W^{-\frac{1}{q},q}(\Gamma)=W^{\frac{1}{q},p}(\Gamma)^*$ and

$$
\begin{aligned}
&\int_Z\operatorname{div}(\|Dx\|^{p-2}Dx)y(z)\,\mathrm{d}z+\int_Z\|Dx\|^{p-2}(Dx,Dy)_{\mathbb{R}^N}\,\mathrm{d}z\\
&\quad=\left\langle\frac{\partial x}{\partial n_p},\tau(y)\right\rangle_\Gamma\ \text{ for all } y\in W^{1,p}(Z)
\end{aligned}
\tag{30}
$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality brackets for the pair $(W^{\frac{1}{q},p}(\Gamma),W^{-\frac{1}{q},q}(\Gamma))$. For every $w\in G(\tau(x))=\{u\in L^q(\Gamma)\colon\ u(z)\in\beta(z,\tau(x)(z))\text{ a.e. on }\Gamma\}$, we also have

$$
\begin{aligned}
V(y)-V(x) &= \int_\Gamma(j(z,\tau(y)(z))-j(z,\tau(x)(z))\,\mathrm{d}\sigma\\
&\geqslant \int_\Gamma w(z)(\tau(y)(z)-\tau(x)(z))\,\mathrm{d}\sigma\\
&= \langle w,\tau(y-x)\rangle_\Gamma=\langle\tau^*w,y-x\rangle\ \text{ for all } y\in W^{1,p}(Z)\\
&\qquad\Rightarrow\tau^*w\in\partial V(x).
\end{aligned}
$$

So we have that $(\tau^* G \tau)(x) \subseteq \partial V(x)$. Note that $G(\tau(x)) = \partial L(\tau(x))$ (see Hu-Papageorgiou [14], Example 4.28(c), p. 349). Since $V = L \circ \tau$ and using Theorem 2, p. 201 of Ioffe-Tichomirov [15], we have $(\tau^* G \tau)(x) = \partial V(x) \Rightarrow \tau^* w = g - A(x)$ for some $w \in \partial L(\tau(x)) = G(\tau(x)) \subseteq L^q(Z)$. Hence for all $y \in W^{1,p}(Z)$, we have

$$
\begin{aligned}
\langle \tau^* w, y \rangle &= \langle g - A(x), y \rangle \\
&= -\int_Z \operatorname{div}(\|Dx\|^{p-2} Dx) y \, \mathrm{d}z - \int_Z \|Dx\|^{p-2} (Dx, Dy)_{\mathbb{R}^N} \, \mathrm{d}z \text{ (see (28))} \\
\Rightarrow \langle w, \tau(y) \rangle_\Gamma &= -\int_Z \operatorname{div}(\|Dx\|^{p-2} Dx) y \, \mathrm{d}z - \int_Z \|Dx\|^{p-2} (Dx, Dy)_{\mathbb{R}^N} \, \mathrm{d}z \\
\Rightarrow \langle w, \tau(y) \rangle_\Gamma &= \left\langle -\frac{\partial x}{\partial n_p}, \tau(y) \right\rangle_\Gamma \quad \text{for all } y \in W^{1,p}(Z) \text{ (see (29))}.
\end{aligned}
$$

Since $\operatorname{Range}(\tau) = W^{\frac{1}{q},p}(\Gamma)$, from the above equality we infer that

$$w(z) = -\frac{\partial x}{\partial n_p}(z) \quad \text{a.e. on } \Gamma.$$

Because $w \in \partial L(\tau(x)) = G(\tau(x))$, we conclude that $-\frac{\partial x}{\partial n_p}(z) \in \beta(z, \tau(x)(z))$ a.e. on Γ.

Now recall that $-\widehat{f}(x) \in \partial \psi_1(x) = K_1(x)$ and $\widehat{f}(x) \in L^q(Z)$ (see hypothesis $\mathbf{H}(f)_3$(i)). Thus according to claim #3, $x(\cdot)$ is a nontrivial solution of (18). $\square$

Remark. Problems like (19) are of physical interest and arise in the theory of heat transfer between solids and gases (see Friedman [11]). In this respect it will be interesting to have Theorem 6 without the condition that $\operatorname{dom} \beta(z, \cdot) = \mathbb{R}$ a.e. on Z, which is the case here (see hypothesis $\mathbf{H}(\beta)_3$).

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References

[1] *R. Adams*: Sobolev Spaces. Academic Press, New York, 1975.

[2] *W. F. Ames*: Nonlinear Partial Differential Equations in Engineering. Academic Press, New York, 1965.

[3] *A. Ambrosetti and P. H. Rabinowitz*: Dual variational methods in critical point theory and applications. J. Funct. Anal. *14* (1973), 349–381.

[4] *A. Anane and J. P. Gossez*: Strongly nonlinear elliptic problems near resonance: a variational approach. Comm. Partial Differential Equations *15* (1990), 1141–1159.

[5] *D. Arcoya and M. Calahorrano*: Some discontinuous problems with a quasilinear operator. J. Math. Anal. Appl. *187* (1994), 1059–1072.

[6] *L. Boccardo, P. Drábek, D. Giachetti and M. Kučera*: Generalization of Fredholm alternative for nonlinear differential operators. Nonlinear Anal. TMA *10* (1986), 1083–1103.

[7] *K. C. Chang*: Variational methods for nondifferentiable functionals and their applications to partial differential equations. J. Math. Anal. Appl. *80* (1981), 102–129.
[8] *D. Costa and C. Magalhaes*: Existence results for perturbations of the p-Laplacian. Nonlinear Anal. TMA *24* (1995), 409–418.
[9] *C. De Coster*: Pairs of positive solutions for the one-dimensional p-Laplacian. Nonlinear Anal. TMA *23* (1994), 669–681.
[10] *M. Del Pino, M. Elgueta and R. Manasevich*: A homotopic deformation along p of a Leray-Shauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$. J. Differential Equations *80* (1989), 1–13.
[11] *A. Friedman*: Generalized heat transfer between solids and gases under nonlinear boundary conditions. J. Math. Mech. *8* (1959), 161–184.
[12] *Z. Guo*: Boundary value problems for a class of quasilinear ordinary differential equations. Differential Integral Equations *6* (1993), 705–719.
[13] *A. El. Hachimi, J.-P. Gossez*: A note on a nonresonance condition for a quasilinear elliptic problem. Nonlinear Anal. TMA *22* (1994), 229–236.
[14] *S. Hu and N. S. Papageorgiou*: Handbook of Multivalued Analysis Volume I: Theory. Kluwer Academic Publishers, Dordrecht, 1997.
[15] *A. Ioffe and V. Tichomirov*: Theory of Extremal Problems. North Holland, Amsterdam, 1979.
[16] *N. Kenmochi*: Pseudomonotone operators and nonlinear elliptic boundary value problems. J. Math. Soc. Japan *27* (1975), 121–149.
[17] *A. Kufner, O. John and S. Fučík*: Function Spaces. Noordhoff, Leyden, The Netherlands, 1977.
[18] *P. Lindqvist*: On the equation $\operatorname{div}(|Dx|^{p-2}Dx) + \lambda|x|^{p-2}x = 0$. Proc. AMS. vol. 109, 1991, pp. 157–164.
[19] *P. H. Rabinowitz*: Some minimax theorems and applications to nonlinear partial differential equations. Nonlinear Analysis: A collection of papers of E. Rothe (L. Cesari, R. Kannan, H. F. Weinberger, eds.). Acad. Press, New York, 1978, pp. 161–177.
[20] *P. H. Rabinowitz*: Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS, Regional Conference Series in Math, No 65, AMS, Providence, R. J., 1986.
[21] *R. Showalter*: Monotone Operators in Banach Space and Nonlinear Partial Differential Equations. Math. Surveys, vol. 49, AMS, Providence, R. I., 1997.
[22] *A. Szulkin*: Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems. Ann. Inst. H. Poincare Anal. Non Linéaire *3* (1986), 77–109.
[23] *E. Zeidler*: Nonlinear Functional Analysis and its Applications II. Springer Verlag, New York, 1990.

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