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EIGENVALUE DISTRIBUTION OF CERTAIN RAY PATTERNS

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Abstract. In this paper, the eigenvalue distribution of complex matrices with certain ray patterns is investigated. Cyclically real ray patterns and ray patterns that are signature similar to real sign patterns are characterized, and their eigenvalue distribution is discussed. Among other results, the following classes of ray patterns are characterized: ray patterns that require eigenvalues along a fixed line in the complex plane, ray patterns that require eigenvalues symmetric about a fixed line, and ray patterns that require eigenvalues to be in a half-plane. Finally, some generalizations and open questions related to eigenvalue distribution are mentioned.

Keywords: eigenvalue distribution, ray patterns, sign patterns, sector patterns, cyclically real, signature similarity, cycles

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1. Introduction

Interest in qualitative matrix analysis was stimulated, in part, by the need to analyze certain dynamical systems for which only qualitative information was available. Such systems arise in economics, ecology, biology, chemistry, mechanics and energy planning when only the directions of certain effects are known. Much interest has been focused on qualitative matrix methods, since certain combinatorial results can be obtained from them.

Until recently, qualitative matrix analysis involved the study of properties referring to a real matrix, based strictly upon knowledge of the signs of the entries of the matrix. A matrix whose entries are from the set $\{+,-,0\}$ is called a (real) *sign pattern (matrix)*. However, both in theory and in many applications, it is often necessary to consider complex matrices. For example, linear dynamical systems with complex entries occur in quantum mechanics (see [6, Chapter 8]). Hence, it is useful to investigate properties of complex matrices based upon the patterns of their entries. Two recent papers ([3] and [8]) have begun an investigation of patterns of complex matrices.

The set of all nonzero complex numbers whose arguments are equal to α ($0 \leqslant \alpha < 2\pi$), is called a *ray*, denoted by $e^{i\alpha}$. If $\beta - \alpha = 2k\pi$, where k is an integer and $0 \leqslant \alpha < 2\pi$, we identify $e^{i\beta}$ with $e^{i\alpha}$. It should be clear from the context that $e^{i\alpha}$ means either the ray $e^{i\alpha}$ or the complex number $e^{i\alpha}$. A *ray pattern* $A = (a_{kj})$ is a matrix, each of whose entries is either 0 or a ray $e^{i\alpha_{kj}}$ (see [3] or [8]). It can be seen from this definition that ray pattern matrices are generalizations of real sign pattern matrices.

Associated with each n-by-n ray pattern $A = (a_{kj})$ is a natural class of complex matrices called the *ray pattern class* of A, defined by

$$\mathscr{R}(A) = \{B \in M_n(\mathbb{C}) \mid b_{kj} = 0 \text{ iff } a_{kj} = 0,\ \arg b_{kj} = \arg a_{kj} \text{ whenever } a_{kj} \neq 0\}.$$

Let P be a property referring to a complex matrix. Then a ray pattern A is said to *require* P if every $B \in \mathscr{R}(A)$ has property P, and A is said to *allow* P if there is some $B \in \mathscr{R}(A)$ that has property P.

The primary motivating issue for this paper is to locate the eigenvalue regions in the complex plane for certain ray patterns. As in many other eigenvalue classification problems in qualitiative matrix analysis, we use the fact that the eigenvalues depend continuously upon the entries of a matrix.

The usual definitions of reducible and irreducible matrices can be extended to ray patterns. Consequently, the Frobenius normal form of a reducible ray pattern A is a block upper (lower) triangular ray pattern, each of whose diagonal blocks A_{ii} is an n_i-by-n_i irreducible ray pattern called an irreducible component of A. If A is a reducible ray pattern in Frobenius normal form, and if $B \in \mathscr{R}(A)$, then it is well known that the *spectrum* (the set of all eigenvalues) of B is the union (including multiplicities) of the spectra of the irreducible components B_{ii}, where each $B_{ii} \in \mathscr{R}(A_{ii})$. Since the eigenvalues of a matrix are similarity invariants, we may assume, without loss of generality, that the ray pattern is in Frobenius normal form. Hence, if P is the property that all eigenvalues lie in a specific subset of the complex plane, then it is clear that a reducible ray pattern A requires P if and only if each irreducible component of A requires P. Thus, without loss of generality, we may assume that A is irreducible in the statements of our results concerning eigenvalue regions.

To describe our results, we define a *simple p-cycle* in a ray pattern $A = (a_{kj})$ to be a formal product $\gamma = a_{k_1k_2}a_{k_2k_3}\ldots a_{k_pk_1}$, where the indices $k_1, k_2, \ldots, k_p$ are distinct. We define a *composite p-cycle* to be a product of the form $\gamma = \gamma_1\gamma_2\ldots\gamma_m$, where each γ_k is a simple p_k-cycle, $\sum\limits_{k=1}^{m} p_k = p$, and the index sets of the γ_k's are mutually disjoint. By $\mathrm{ap}(\gamma)$, we mean the *actual product* of the entries in γ, where the multiplication is carried out in the usual way. For example, for the 2-cycle $\gamma = \mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\beta}$, then $\mathrm{ap}(\gamma) = \mathrm{e}^{\mathrm{i}\theta}$, where $\theta = \alpha + \beta$ (modulo 2π). We associate the rays $\mathrm{e}^{0\mathrm{i}}$, $\mathrm{e}^{\pi\mathrm{i}}$, $\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}$ and $\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}$ with $+$, $-$, i and $-$i, respectively. If $\mathrm{ap}(\gamma)$ equals $+$, $-$, i or $-$i, we say γ is a *positive, negative, positive pure imaginary* or *negative pure imaginary cycle*, respectively. In the remainder of this paper, when we say cycle we mean a nonzero cycle, that is, we mean a cycle that contains no zero entries. We say that a ray pattern A is *cyclically real* if for every cycle γ in A, $\mathrm{ap}(\gamma)$ is real, that is, $+$ or $-$.

In Section 2, we show that an irreducible ray pattern is cyclically real if and only if it is signature similar to a real sign pattern matrix, where a *signature pattern* is a diagonal ray pattern with nonzero diagonal entries. This characterization shows that cyclically real ray patterns generalize real sign patterns in the sense that the eigenvalue distribution of any irreducible cyclically real pattern is the same as the eigenvalue distribution of a real sign pattern. This is analogous to the generalization of nonnegative patterns to cyclically nonnegative patterns, see [3]. Cyclically real ray patterns A are also characterized in terms of the spectra of the matrices in $\mathscr{R}(A)$. Finally, a characterization is given for reducible ray patterns that are signature similar to real patterns.

In Section 3, we characterize the ray patterns that require all the eigenvalues of the matrices in $\mathscr{R}(A)$ to lie along a specified line in the complex plane or in a half-plane. In particular, we characterize the patterns that require all real eigenvalues, and the patterns that require all pure imaginary eigenvalues. In Section 4, we discuss the more general sector patterns (see [3]), and we give some open questions concerning ray patterns and sector patterns.

2. Cyclically real ray patterns

Considerable research has been done to characterize the eigenvalues of certain real sign pattern matrices (see, for example, [1], [2], or [4]). A natural question to consider is: What are the ray patterns that preserve the eigenvalue characterizations of real sign patterns?

If S is a signature pattern, by S^{-1}, we mean the ray pattern such that

$$S^{-1}S = SS^{-1} = \begin{pmatrix} + & & 0 \\ & \ddots & \\ 0 & & + \end{pmatrix}.$$

If A is an n-by-n ray pattern matrix, and S is a signature pattern of order n, then $S^{-1}AS$ is called a *signature similarity* of A, where matrix multiplication is carried out in the usual way. We use the fact that all signature similarities of a ray pattern A preserve the qualitative cycle structure of A. That is, if γ is a cycle in A, and if γ_s is the corresponding cycle in the signature similarity $S^{-1}AS$, then $\mathrm{ap}(\gamma) = \mathrm{ap}(\gamma_s)$.

In this section, among other results, we show that irreducible cyclically real ray patterns are signature similar to real sign patterns, and, therefore, all eigenvalue characterizations for real sign patterns carry over to these patterns. We begin by considering entrywise nonzero cyclically real patterns in Lemma 2.1.

Lemma 2.1. *If A is an entrywise nonzero cyclically real ray pattern matrix, then A is signature similar to an entrywise nonzero real sign pattern matrix.*

Proof. We use induction on the order n of the ray pattern A. Clearly, the result is true for $n = 1$. Now assume the result is true for entrywise nonzero cyclically real ray patterns of order $n-1$. Let A be an entrywise nonzero cyclically real ray pattern of order n. By the induction hypothesis, there exists a signature pattern S, such that

$$S^{-1}AS = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where A_1 is an entrywise nonzero real sign pattern matrix of order $n-1$, and where $A_4 = (a_{nn}) = \pm$ since A is cyclically real.

We know that A cyclically real implies that $S^{-1}AS$ is cyclically real. Consequently, the cycle

$$(S^{-1}AS)_{nk}(S^{-1}AS)_{kj}(S^{-1}AS)_{jn}$$

is real, for all indices $1 \leqslant k,\ j < n$. Since $(S^{-1}AS)_{kj}$ is an entry in the real sign pattern matrix A_1, $(S^{-1}AS)_{kj}$ is real, and it follows that $(S^{-1}AS)_{nk}(S^{-1}AS)_{jn}$ is real for all indices $1 \leqslant k,\ j < n$. Since $(S^{-1}AS)_{nk}$ is an entry in A_3, and $(S^{-1}AS)_{jn}$ is an entry in A_2 for all indices $1 \leqslant k, j < n$, it follows that each entry in A_2 equals $\pm x$, where $x = \mathrm{e}^{i\theta}$ for some θ, $0 \leqslant \theta < 2\pi$, and each entry in A_3 equals $\pm\bar{x}$. Now let $S_1 = \operatorname{diag}\left(+, \ldots, +, \mathrm{e}^{-i\theta}\right)$. Then it is easy to verify that $S_1^{-1}(S^{-1}AS)S_1$ is an entrywise nonzero real sign pattern matrix. □

Theorem 2.2. *An irreducible ray pattern A is cyclically real if and only if A is signature similar to a real sign pattern matrix.*

Proof. Let $A = (a_{kj})$ be an n-by-n irreducible cyclically real pattern. Let $P_1 = a_{kj_1}a_{j_1j_2}\ldots a_{j_pj}$ be a path from k to j. Since A is irreducible, there is at least one path from j to k, say, $P = a_{jm_1}a_{m_1m_2}\ldots a_{m_qk}$. Since P_1P is a product of simple cycles in A, it follows that $\operatorname{ap}(P_1P) = +$ or $-$. Thus, if $x = \operatorname{ap}(P_1)$, and if $y = \operatorname{ap}(P)$, then $y = \pm\bar{x}$.

Define the n-by-n matrix $A' = (a'_{kj})$ by

$$a'_{kj} = \begin{cases} a_{kj} & \text{if } a_{kj} \neq 0; \\ \operatorname{ap}(P_{kj}) & \text{if } a_{kj} = 0, \text{ where } P_{kj} \text{ is some path from } k \text{ to } j. \end{cases}$$

Then it can be seen that A' is entrywise nonzero and cyclically real. From Lemma 2.1, there exists a signature pattern S such that $S^{-1}A'S$ is real; hence, $S^{-1}AS$ is real.

Since the converse is obvious, we omit the proof. $\square$

Example 2.3. Let

$$A = \begin{pmatrix} 0 & \mathrm{e}^{-i\theta_1} & 0 & -\mathrm{e}^{-i\theta_1} \\ 0 & + & -\mathrm{e}^{-i\theta_2} & 0 \\ \mathrm{e}^{i(\theta_1+\theta_2)} & -\mathrm{e}^{i\theta_2} & 0 & -\mathrm{e}^{i\theta_2} \\ \mathrm{e}^{i\theta_1} & - & 0 & - \end{pmatrix}.$$

Then A is cyclically real, and A is signature similar to $\begin{pmatrix} 0 & + & 0 & + \\ 0 & + & - & 0 \\ + & - & 0 & + \\ - & + & 0 & - \end{pmatrix} = S^{-1}AS$,

where $S = \operatorname{diag}\left(\mathrm{e}^{-i\theta_1}, +, \mathrm{e}^{i\theta_2}, -\right)$.

An immediate consequence of Theorem 2.2 is that if A is an irreducible cyclically real ray pattern, then there exists a signature pattern S such that $S^{-1}AS = \hat{A}$, where $\hat{A}$ is a real sign pattern matrix. Thus,

$$\mathscr{R}(\hat{A}) = \{\hat{B} \in M_n(\mathbb{R}) \mid \hat{B} = D^{-1}BD,\ B \in \mathscr{R}(A)\}$$

where D is the unique complex matrix in $\mathscr{R}(S)$, whose diagonal entries have moduli equal to 1. Now let T be a subset of the complex numbers $\mathbb{C}$. Since eigenvalues are similarity invariants, it follows that A requires k eigenvalues in T if and only if $\hat{A}$ requires k eigenvalues in T, for some integer $1 \leqslant k \leqslant n$. With this in mind, we state Theorem 2.4 which follows from Theorem 1.1 in [4]. First, however, we extend two definitions to ray patterns that are needed to describe our results. A square ray pattern matrix A is *ray nonsingular* if every $B \in \mathscr{R}(A)$ is nonsingular. A ray pattern A is *bipartite* if the directed graph $D(A)$ is bipartite (see [4]).

Theorem 2.4. *An irreducible n-by-n cyclically real ray pattern A requires all nonreal eigenvalues if and only if A satisfies all of the following:*

i) *A is bipartite;*

ii) *all simple cycles in A are negative; and*

iii) *A is ray nonsingular.*

In Section 3, we characterize additional cyclically real ray patterns. Now, however, we turn our attention to a result that relates a cyclically real ray pattern A to the eigenvalues of the matrices B in $\mathscr{R}(A)$

Theorem 2.5. *An n-by-n irreducible ray pattern A is cyclically real if and only if A requires all of the nonreal eigenvalues (if any) to occur in complex conjugate pairs.*

Proof. First assume that A is an n-by-n irreducible cyclically real ray pattern. From Theorem 2.2, we know that there exists a nonsingular diagonal matrix D such that $D^{-1}BD$ is a real matrix for each $B \in \mathscr{R}(A)$. Consequently, the nonreal eigenvalues (if any) of every $B \in \mathscr{R}(A)$ occur in complex conjugate pairs, that is, A requires the desired property.

Conversely, assume that A requires all the nonreal eigenvalues (if any) to occur in complex conjugate pairs. For contradiction, assume A has a nonreal simple p-cycle $\gamma = a_{k_1k_2} \dots a_{k_pk_1}$. We *emphasize* the cycle γ in A, by choosing a matrix $B \in \mathscr{R}(A)$ such that the entries in B along γ are of modulus 1, and the other nonzero entries in B are of moduli less than or equal to a sufficiently small $\varepsilon > 0$ (see [4]). Then p eigenvalues of B are arbitrarily close to the p^{th} complex roots of some nonreal number $\mathrm{e}^{i\theta}$. Since the product of these p roots is $\pm\mathrm{e}^{i\theta}$, it follows that there is at least one nonreal root whose conjugate is not a root, otherwise, the product would be real. Thus B has at least one nonreal eigenvalue that does not occur in a conjugate pair, contradicting our assumption that A requires all nonreal eigenvalues to occur in conjugate pairs. Consequently, we conclude that A is cyclically real. $\square$

An alternative statement of Theorem 2.5 is that the spectrum of every $B \in \mathscr{R}(A)$ is symmetric with respect to the real axis in the complex plane if and only if A is cyclically real. In order to generalize this result, we let $L(\theta)$ be the line in the complex plane containing the ray $\mathrm{e}^{i\theta}$. If $A = (a_{kj})$ is a ray pattern, we define the ray pattern $\mathrm{e}^{i\alpha}A$ by $\mathrm{e}^{i\alpha}A = (\mathrm{e}^{i\alpha}a_{kj})$.

Corollary 2.6. *Let A be an n-by-n irreducible ray pattern. Then A requires the locations of the eigenvalues in the complex plane to be symmetric with respect to the line $L(\theta)$ if and only if $\mathrm{e}^{-i\theta}A$ is cyclically real.*

Proof. Suppose $\mathrm{e}^{-i\theta}A$ is cyclically real. From Theorem 2.5 and the comment immediately preceeding this corollary, we know that the spectrum of every matrix in the ray pattern class of $\mathrm{e}^{-i\theta}A$ is symmetric with respect to the real axis. If an eigenvalue of any matrix $\mathrm{e}^{-i\theta}B \in \mathscr{R}(\mathrm{e}^{-i\theta}A)$ is real, it lies on $L(0)$. Consequently, if λ is a real eigenvalue of $\mathrm{e}^{-i\theta}B$ then $\mathrm{e}^{i\theta}\lambda \in \sigma(B)$ lies on $L(\theta)$. Now suppose λ is any nonreal eigenvalue of $\mathrm{e}^{-i\theta}B$. Then by symmetry, $\bar{\lambda} \in \sigma(\mathrm{e}^{i\theta}B)$. Hence, $\mathrm{e}^{i\theta}\lambda$ and $\mathrm{e}^{i\theta}\bar{\lambda}$ are eigenvalues of B that are clearly symmetric with respect to $L(\theta)$. Thus A requires that the eigenvalues of each of the matrices in $\mathscr{R}(A)$ to be symmetric about the line $L(\theta)$.

Since the converse argument is similar, we omit the proof. $\square$

The conjugate of a ray is defined in the natural way, for example, the conjugate of $\mathrm{e}^{i\alpha}$ $(0 \leqslant \alpha < 2\pi)$ is $\mathrm{e}^{i(2\pi-\alpha)}$. Hence A^* is defined in the usual way. If $A = (a_{kj})$ is a cyclically real ray pattern, then all 2-cycles of A are real, that is, $a_{jk} = \pm\bar{a}_{kj}$ or 0. Hence if A is a cyclically real ray pattern (as in Theorem 2.7), then each entry of $A + A^*$ is a ray, a line through the origin, or 0. We define a *generalized ray pattern* to be a matrix, each of whose entries is either 0, a ray, or a line through the origin. Note that the notions such as "cycles" and "cyclically real" extend to generalized ray patterns. In particular, a cycle γ in a generalized ray pattern is real if $\mathrm{ap}(\gamma)$ is $+$, $-$, or $\#$, where $\#$ is the ambiguous sum $(+)+(-)$, that is, in the context of this paper, $\#$ is the same as $L(0)$. We also observe that Theorem 2.2 extends to generalized ray patterns, that is, an irreducible generalized ray pattern A is cyclically real if and only if A is signature similar to a *generalized real sign pattern* [7], each of whose entries is in the set $\{+, -, 0, \#\}$.

Since every cycle of a reducible matrix in Frobenius normal form occurs in some irreducible component, a ray pattern is cyclically real if and only if each irreducible component is cyclically real. It is also of interest to characterize reducible n-by-n ray patterns that are signature similar to real sign pattern matrices. In Theorem 2.2, the irreducible ray patterns that are signature similar to real sign patterns are characterized in terms of cycles. However, if a ray pattern A is reducible, then a nonzero entry of A is not necessarily on a cycle, and, hence, the cycle condition is not sufficient for A to be signature similar to a real sign pattern. For example, if

$$A = \begin{pmatrix} + & + & i \\ 0 & - & - \\ 0 & 0 & + \end{pmatrix},$$

then A is cyclically real, but A is not signature similar to a real sign pattern, as can be seen from the next theorem.

Theorem 2.7. *Let A be an n-by-n ray pattern. Then the following are equivalent:*

i) *A is signature similar to a real sign pattern;*

ii) *A is cyclically real and $A + A^*$ is signature similar to a generalized real sign pattern;*

iii) *A and $A + A^*$ are both cyclically real.*

Proof. i) $\Rightarrow$ ii). If $S^{-1}AS$ is real for some signature pattern S, then $S^{-1}A^*S$ is also real. Hence $S^{-1}(A + A^*)S$ is a generalized real sign pattern.

ii) $\Rightarrow$ iii) is clear.

iii) $\Rightarrow$ i). Recall that A is a cyclically real ray pattern implies that $A + A^*$ is a generalized ray pattern. Now suppose both A and $A + A^*$ are cyclically real. Since $(A + A^*)^* = A + A^*$, the Frobenius normal form of $A + A^*$ is given by

$$P^*(A + A^*)P = \begin{pmatrix} A_{11} & & & \\ & A_{22} & & \\ & & \ddots & \\ & & & A_{mm} \end{pmatrix},$$

for some permutation pattern P. Since each irreducible component A_{ii} is cyclically real, by the modified version of Theorem 2.2 mentioned above, there exists a signature matrix S_{ii} such that $S_{ii}^{-1}A_{ii}S_{ii}$ is a generalized real sign pattern matrix. Therefore if

$$S = \begin{pmatrix} S_{11} & & \\ & \ddots & \\ & & S_{mm} \end{pmatrix},$$

then $S^{-1}P^*(A + A^*)PS$ is a generalized real sign pattern, and hence, so is $PS^{-1}P^*(A+A^*)PSP^*$. Clearly, $\tilde{S} = PSP^*$ is a signature pattern, and $\tilde{S}^{-1}(A+A^*)\tilde{S}$ is a generalized real sign pattern. Notice that $\tilde{S}^{-1}(A + A^*)\tilde{S} = \tilde{S}^{-1}A\tilde{S} + \tilde{S}^{-1}A^*\tilde{S}$. Since the sum of two ray patterns is a generalized real sign pattern only when the two ray patterns are real, it follows that $\tilde{S}^{-1}A\tilde{S}$ is real, that is, A is signature similar to a real sign pattern. $\square$

Example 2.8. Let

$$A = \begin{pmatrix} 0 & e^{-i\theta_1} & 0 & e^{i\theta_2} & + \\ 0 & + & -e^{i\theta_1} & -e^{i(\theta_1+\theta_2)} & 0 \\ + & e^{-i\theta_1} & - & 0 & - \\ 0 & 0 & 0 & + & e^{-i\theta_2} \\ 0 & 0 & 0 & e^{i\theta_2} & 0 \end{pmatrix}.$$

Then since $A + A^*$ is cyclically real, it follows that A is signature similar to a real sign pattern. Indeed, $S^{-1}AS$ is a real sign pattern if $S = \operatorname{diag}\left(+, e^{i\theta_1}, +, e^{-i\theta_2}, +\right)$.

3. Patterns that require eigenvalues along a line or in a half-plane

We first consider ray patterns that require all eigenvalues distributed along a fixed line in the complex plane. We begin with the fundamental case where the line involved is the real axis.

Theorem 3.1. *Let A be an n-by-n irreducible ray pattern. Then A requires all real eigenvalues if and only if*

i) *each diagonal entry of A is in the set $\{0,+,-\}$,*
ii) *all simple 2-cycles in A are positive, and*
iii) *there are no simple cycles in A of length greater than 2.*

Proof. Suppose that A requires all real eigenvalues. Assume that A has a nonreal diagonal entry, that is, A has a nonreal 1-cycle γ. By emphasizing γ, we can find a complex matrix $B \in S(A)$ that has at least one nonreal eigenvalue, contradicting the assumption that A requires all real eigenvalues. Thus i) is true. To prove ii), assume that A has a simple nonpositive 2-cycle $\gamma = a_{k_1k_2}a_{k_2k_1}$. Then by emphasizing γ, we can find a matrix $B = (b_{kj}) \in S(A)$, such that $c = b_{k_1k_2}b_{k_2k_1} = \mathrm{e}^{i\theta}$ is not a positive number. It can be seen that B has two nonreal eigenvalues, arbitrarily close to $\pm\mathrm{e}^{i\frac{\theta}{2}}$, contradicting the assumption that A requires all real eigenvalues. Thus ii) holds. Finally, assume that A has a simple p-cycle γ, for some $p \geqslant 3$. Then by emphasizing γ, we can find a matrix $B = (b_{kj}) \in S(A)$, such that B has p eigenvalues arbitrarily close to the p-th complex roots of some $\mathrm{e}^{i\theta}$. At least $p-2$ of these eigenvalues would be nonreal, contradicting the assumption that A requires all real eigenvalues. Thus we have iii).

Conversely, suppose that conditions i)-iii) hold. Then it is easy to see that A is cyclically real. By Theorem 2.2, A is signature similar to a real sign pattern matrix (in which conditions ii) and iii) remain true). Since eigenvalue properties are invariant under signature similarity, it follows that A requires all real eigenvalues (see [4, Theorem 1.6]). □

By multiplying each entry of the ray patterns in Theorem 3.1 by a fixed ray $\mathrm{e}^{i\theta}$, clearly we get ray patterns that require all eigenvalues to be along the line $L(\theta)$. Therefore, we have the following two corollaries.

Corollary 3.2. *Let A be an n-by-n irreducible ray pattern. Then A requires all pure imaginary eigenvalues if and only if*

i) *each diagonal entry of A is in the set $\{0, i, -i\}$,*
ii) *all simple 2-cycles in A are negative, and*
iii) *there are no simple cycles in A of length greater than 2.*

P r o o f. A requires all pure imaginary eigenvalues if and only if iA requires all real eigenvalues. $\square$

Corollary 3.3. *Let A be an n-by-n irreducible ray pattern. Then A requires all eigenvalues on the line $L(\theta)$ if and only if*

i) *each diagonal entry of A lies on the line $L(\theta)$,*
ii) *all simple 2-cycles in A are of the form $(\mathrm{e}^{i\alpha})(\mathrm{e}^{i(2\theta-\alpha)})$, for some α, $0 \leqslant \alpha < 2\pi$, and*
iii) *there are no simple cycles in A of length greater than 2.*

P r o o f. A requires all eigenvalues on the line $L(\theta)$ if and only if $\mathrm{e}^{-i\theta}A$ requires all real eigenvalues. $\square$

Clearly, an n-by-n ray pattern A allows an eigenvalue not on the line $L(\theta)$ if and only if A does not satisfy at least one of the three conditions stated in the above corollary.

Another eigenvalue region of considerable interest is a half-plane. Recall that a matrix B is said to be *stable* if each eigenvalue of B has negative real part, that is, lies in the left half-plane. As in [3], we say a ray pattern A is *ray stable* if every $B \in \mathscr{R}(A)$ is stable. The following result is Theorem 5.2 in [3].

Theorem 3.4. *Let A be an n-by-n irreducible ray pattern, where all the diagonal entries of A are in the left half-plane. Then A is ray stable if and only if*

i) *all simple 2-cycles in A are negative, and*
ii) *there are no simple cycles in A of length greater than 2.*

Let $H(\theta)$ denote the open half-plane consisting of all rays that can be obtained by rotating $\mathrm{e}^{i\theta}$ counterclockwise through an angle γ, $0 < \gamma < \pi$. Note that $H(\theta)$ can be obtained by rotating the left half-plane counterclockwise through an angle $\theta - \dfrac{\pi}{2}$. Consequently, we obtain the following generalization of Theorem 3.4.

Corollary 3.5. *Let A be an n-by-n irreducible ray pattern, where all the diagonal entries of A are in $H(\theta)$. Then A requires all eigenvalues to be in $H(\theta)$ if and only if*

i) *all simple 2-cycles in A are of the form $(\mathrm{e}^{i\alpha})(\mathrm{e}^{i(2\theta-\alpha)})$, for some α, $0 \leqslant \alpha < 2\pi$, and*
ii) *there are no simple cycles in A of length greater than 2.*

P r o o f. A requires all eigenvalues to be in $H(\theta)$ if and only if $\mathrm{e}^{i(\pi/2-\theta)}A$ is ray stable. $\square$

We say that an n-by-n ray pattern A is *ray semistable* if, for each $B \in \mathscr{R}(A)$, $\mathrm{Re}(\lambda) \leqslant 0$ for all λ in the spectrum of B. Note that a matrix B is semistable if and only if $B - \varepsilon I$ is stable for every $\varepsilon > 0$. Thus the following corollary follows from Theorem 3.4.

Corollary 3.6. *Let A be an n-by-n irreducible ray pattern. Then A is ray semistable if and only if*

i) *all diagonal entries of A are in the closed left half-plane,*
ii) *all simple 2-cycles in A are negative, and*
iii) *there are no simple cycles in A of length greater than 2.*

Corollary 3.7. *Let A be an n-by-n irreducible ray pattern. Then A requires all eigenvalues to be in the closure of $H(\theta)$ if and only if*

i) *all diagonal entries of A are in the closure of $H(\theta)$,*
ii) *all simple 2-cycles in A are of the form $(\mathrm{e}^{i\alpha})(\mathrm{e}^{i(2\theta-\alpha)})$, for some α, $0 \leqslant \alpha < 2\pi$, and*
iii) *there are no simple cycles in A of length greater than 2.*

4. Generalizations and open questions

Recall that in [3], a *sector pattern* is defined to be a matrix where each entry is either 0 or an arbitrary sector (with vertex at the origin) in the complex plane. Associated with each sector pattern A is a *sector pattern class*, $\mathscr{S}(A)$, defined in an analogous manner to a ray pattern class. We note that the definition of a sector pattern is analogous to the definition of an interval matrix (see [9] for a recent reference).

Replacing real sign pattern classes with sector pattern classes, it is easy to see that the results on repeated eigenvalues in [5] hold for sector patterns. A key observation is that the proof of the lemma in [5] can be modified to show that the results of the lemma hold for sector patterns. Since all the other proofs of the results in [5] are entirely combinatorial and rely only on the zero/nonzero structure of the matrices, these proofs also hold for sector patterns.

If an irreducible sector pattern A (whose diagonal entries are rays) requires the eigenvalues to lie on a line or in a half-plane, then it can be seen that A has no simple cycle of length $\geqslant 3$, and each nonzero off-diagonal entry is on a simple 2-cyle whose actual product is a ray. Thus each nonzero off-diagonal entry is, in fact, a ray, and the sector pattern degenerates to a ray pattern. This is why we stated the results in Section 3 directly in terms of ray patterns, instead of sector patterns. However, we note that Theorem 3.4 and its corollaries could be stated in such a way that the diagonal entries are any sectors in the particular half-plane. The proofs remain the same since any complex matrix B in a sector pattern class is in a ray pattern class.

It should be noted that if the restriction on diagonal entries in Theorem 3.4 is relaxed, namely, to allow some diagonal entries to have real part 0, then the characterization of ray stable patterns remains open.

In Theorem 2.4, we characterized the n-by-n cyclically real ray patterns that require all nonreal eigenvalues. However, Theorem 2.4 is not valid for ray patterns that are not cyclically real. For example, $\begin{pmatrix} + & i \\ + & 0 \end{pmatrix}$ requires all nonreal eigenvalues, but does not satisfy conditions i) and ii) of Theorem 2.4. A natural research question is to identify the ray patterns that require all nonreal eigenvalues.

More generally, it is of interest to determine the ray patterns and sector patterns that require k nonreal eigenvalues, for some integer k between 1 and n. The following is an example of a sector pattern that requires 3 nonreal eigenvalues.

Example 4.1. Let

$$A = \begin{pmatrix} 0 & [e^{i0}, e^{i\pi/6}] & 0 & 0 & 0 \\ 0 & 0 & e^{i\pi/3} & 0 & 0 \\ [e^{i0}, e^{i\pi/6}] & 0 & 0 & [e^{i0}, e^{i\pi/5}] & 0 \\ 0 & 0 & 0 & 0 & e^{i\pi/5} \\ 0 & 0 & [e^{i0}, e^{i\pi/5}] & 0 & 0 \end{pmatrix}.$$

Then the characteristic polynomial of any $B \in \mathscr{S}(A)$ has the form $x^2(x^3 - c) = 0$, where the complex number c has a positive imaginary part. In fact, c is in $[e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}] + [e^{i\frac{\pi}{5}}, e^{i\frac{3\pi}{5}}] = (e^{i\frac{\pi}{5}}, e^{i\frac{2\pi}{3}})$, where the addition of the sectors is carried out in the usual way of adding two subsets. Hence, the three nonzero eigenvalues are in the union of the sectors $(e^{i\frac{\pi}{15}}, e^{i\frac{2\pi}{9}})$, $(e^{i\frac{11\pi}{15}}, e^{i\frac{8\pi}{9}})$, and $(e^{i\frac{21\pi}{15}}, e^{i\frac{14\pi}{9}})$. Further, each of these sectors contains precisely one eigenvalue of B. Clearly, A requires three nonreal eigenvalues and two zero eigenvalues. This type of pattern is called a C_3-cockade (see [7]).

In Section 3, the ray patterns that require all the eigenvalues lie along a line or in a half-plane were studied. It is known that all the simple cycles in such ray patterns have length 1 or 2. Note that a half-plane may be regarded as a sector with central angle π. If, however, a ray pattern A has a cycle γ of length $k \geqslant 3$, and requires that all the eigenvalues to lie in a sector Ω, then by emphasizing the cycle γ we can get a matrix $B \in \mathscr{R}(A)$ that has k eigenvalues arbitrarily close to the k-th roots of a complex number of modulus 1. Hence, the central angle of the sector Ω must be at least $\frac{k-1}{k}2\pi$ ($\geqslant 4\pi/3$). If a sector Ω with central angle $\theta \geqslant 4\pi/3$ is specified, how can one characterize the ray patterns (or sector patterns) that require all the eigenvalues to lie in Ω?

We now consider sector patterns that do not have simple cycles of length $\geqslant 4$. The following proposition can be proved by emphasizing the simple cyles of specified lengths in A.

Proposition 4.2. *Let $\Omega = (\mathrm{e}^{i\alpha}, \mathrm{e}^{i(2\pi-\alpha)})$, $\frac{\pi}{4} < \alpha \leqslant \frac{\pi}{3}$. Suppose a sector pattern A requires all eigenvalues to be in Ω. Then*

i) *each diagonal entry of A is 0 or is in Ω;*
ii) *each simple 2-cycle γ in A satisfies $\mathrm{ap}(\gamma) \subseteq (\mathrm{e}^{i2\alpha}, \mathrm{e}^{i(2\pi-2\alpha)})$;*
iii) *each simple 3-cycle γ in A satisfies $\mathrm{ap}(\gamma) \subseteq (\mathrm{e}^{i3\alpha}, \mathrm{e}^{i(2\pi-3\alpha)})$; and*
iv) *there is no simple cycle in A of length $k \geqslant 4$.*

In Proposition 4.2, for convenience of notation, we defined Ω so that its bisector is the negative real axis. However, Ω could be rotated, as in Section 3. A natural question is: How can the conditions in Proposition 4.2 be strengthened to guarantee sufficiency?

We now give a sector pattern that satisfies the conditions in Proposition 4.2, and that requires all eigenvalues to be in the specified sector.

Example 4.3. Let $\Omega = (\mathrm{e}^{i\frac{\pi}{3}}, \mathrm{e}^{i\frac{5\pi}{3}})$, and let

$$A = \begin{pmatrix} (\mathrm{e}^{i\frac{2\pi}{3}}, \mathrm{e}^{i\frac{4\pi}{3}}) & (\mathrm{e}^{i\frac{2\pi}{3}}, \mathrm{e}^{i\frac{4\pi}{3}}) \\ \\ + & (\mathrm{e}^{i\frac{2\pi}{3}}, \mathrm{e}^{i\frac{4\pi}{3}}) \end{pmatrix}.$$

Clearly, A and Ω satisfy the conditions in Proposition 4.2. We now show that A requires all eigenvalues to be in Ω. Let $\hat{A} = \begin{pmatrix} \mathrm{e}^{i\theta_1} & \mathrm{e}^{i\theta_2} \\ + & \mathrm{e}^{i\theta_3} \end{pmatrix}$, where $\frac{2\pi}{3} < \theta_1, \theta_2, \theta_3 < \frac{4\pi}{3}$. Then by Corollary 3.5, $\hat{A}$ requires all eigenvalues to be in $H(\theta_2/2)$. Since $\frac{2\pi}{3} < \theta_2 < \frac{4\pi}{3}$, we see that $H(\theta_2/2) \subseteq \Omega$. Since $\hat{A}$ is an arbitrary ray pattern such that $\mathscr{R}(\hat{A}) \subset \mathscr{S}(A)$, it follows that A requires all eigenvalues to be in Ω.

Another interesting open problem is to characterize the ray patterns and sector patterns that require nonsingularity (that is, A requires all the eigenvalues to be in the sector $\Omega = [\mathrm{e}^{0i}, \mathrm{e}^{2\pi i})$). Such patterns are known to have properties very different from real sign patterns that require nonsingularity. Some basic results can be found in [3] and [8].

In Theorem 2.1 of [3], it was shown that the determinantal region $\{\det B \mid B \in \mathscr{S}(A)\}$ is a sector for sign nonsingular complex patterns A of the form $A_1 + iA_2$, where A_1 and A_2 are n-by-n real sign patterns. Similarly, it can be proved that the determinantal region of any ray pattern or sector pattern that requires nonsingularity is again a sector. More generally, it is of interest to investigate the possible determinantal regions of ray patterns and sector patterns.

References

[1] *C. A. Eschenbach*: Sign patterns that require exactly one real eigenvalue. Linear and Multilinear Algebra *35* (1993), 213–224.

[2] *C. A. Eschenbach, F. J. Hall and Z. Li*: Eigenvalue frequency and consistent sign pattern matrices. Czechoslovak Math. J. *44(119)* (1994), 461–479.

[3] *C. A. Eschenbach, F. J. Hall and Z. Li*: From real to complex sign pattern matrices. Bull. Austral. Math. Soc. *57* (1998), 159–172.

[4] *C. A. Eschenbach and C. R. Johnson*: Sign patterns that require real, nonreal or pure imaginary eigenvalues. Linear and Multilinear Algebra *29* (1991), 299–311.

[5] *C. A. Eschenbach and C. R. Johnson*: Sign patterns that require repeated eigenvalues. Linear Algebra Appl. *190* (1993), 169–179.

[6] *R. Feynman, R. Leighton and M. Sands*: The Feynman Lectures on Physics, vol. III. Addison-Wesley, 1965.

[7] *Z. Li, F. J. Hall and C. A. Eschenbach*: On the period and base of a sign pattern matrix. Linear Algebra Appl. *212/213* (1994), 101–120.

[8] *J. McDonald, D. Olesky, M. Tsatsomeros and P. van den Driessche*: Ray patterns of matrices and nonsingularity. Linear Algebra Appl. *267* (1997), 359–373.

[9] *J. Rohn and G. Rex*: Interval P-matrices. SIAM J. Matrix Anal. Appl. *17* (1996), 1020–1024.

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