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DR-IRREDUCIBILITY OF CONNECTED MONOUNARY ALGEBRAS WITH A CYCLE

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For a monounary algebra A let $R(A)$ be the class of all monounary algebras which are isomorphic to a retract of A.

In [4] the notion of irreducibility of a monounary algebra in a given class $\mathscr{K}$ was defined. The corresponding definition is as follows. Let $\mathscr{K}$ be a class of monounary algebras. A monounary algebra A is said to be retract irreducible in $\mathscr{K}$ if, whenever $A \in R\Big(\prod\limits_{i\in I} B_i\Big)$ and $B_i \in \mathscr{K}$ for each $i \in I$, then there is $j \in I$ such that $A \in R(B_j)$.

An analogous definition can be applied also for other classes of algebraic structures.

Let A be a connected monounary algebra. Irreducibility of A in the class of all connected monounary algebras $\mathscr{U}_c$ was dealt with in [2], [3], and in the class of all monounary algebras $\mathscr{U}$ it was investigated in [4]. The case when A is not connected and $\mathscr{K} = \mathscr{U}$ was studied in [5].

Duffus and Rival [1] solved some problems concerning retract irreduciblity of a poset P; they considered retract irreducibility in the class $R(P)$.

The aim of this paper is to describe all connected monounary algebras A with a cycle which are retract irreducible in the class $R(A)$ (Theorem 2.9). Such algebras will be called retract irreducible in the sense of Duffus and Rival, or, more shortly, DR-irreducible.

1. Auxiliary results

We will use the notion of the degree of an element $x \in B$, where (B, f) is a monounary algebra; for this notion cf. e.g. [7], [6] and [2]. The degree of x is an ordinal or the symbol ∞ and is denoted by $s_f(x)$.

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According to [2], 1.3 we obtain

(Thm) Let $n \in \mathbb{N}$ and let (B, f) be a monounary algebra such that if a connected component (B, f) contains a cycle C, then $\operatorname{card} C = n$. Suppose that (M, f) is a subalgebra of (B, f) such that (M, f) contains a cycle. Then M is a retract of (B, f) if and only if the following condition is satisfied:

(1) if $y \in f^{-1}(M)$, then there is $z \in M$ such that $f(y) = f(z)$ and $s_f(y) \leqslant s_f(z)$.

In what follows let A be a connected monounary algebra with a cycle C, $\operatorname{card} C = n$.

For a connected monounary algebra D possessing a cycle let $V_0(D)$ be the set of all elements of the cycle of D; further, if $k \in \mathbb{N}$, then put

$$V_k(D) = \{x \in D\colon\ x \notin V_l(D) \text{ for } l \in \mathbb{N} \cup \{0\}, l < k, f(x) \in V_{k-1}(D)\}.$$

1.1. Lemma. *Suppose that* $\operatorname{card} C = 1$, $\operatorname{card} V_1(A) > 1$. *Then* A *is* DR-*reducible.*

P r o o f. By [2], A is retract reducible in the class $\mathscr{U}_c$. There exist connected monounary algebras B_i, $i \in I$, such that

$$A \in R\Big(\prod_{i \in I} B_i\Big),$$
$$A \notin R(B_i) \text{ for each } i \in I.$$

The algebras B_i (for each $i \in I$) used in this construction (cf. the proof of 3.7, [2]) are such that $B_i \in R(A)$, hence A is DR-reducible. □

1.2. Lemma. *Suppose that* $\operatorname{card} C = n > 1$ *and* $\operatorname{card} V_1(A) > 1$. *Then* A *is* DR-*reducible.*

P r o o f. Let $C = \{c_1, \dots, c_n\}$, $f(c_1) = c_2, \dots, f(c_n) = c_1$. Further let
$V_1(A) = \{a_i\colon\ i \in I\}$;
the assumption yields that $\operatorname{card} I > 1$. If $i \in I$, then denote
$A_i = \{x \in A\colon\ (\exists k \in \mathbb{N} \cup \{0\})(f^k(x) = a_i)\}$,
$B_i = C \cup A_i$.
Then B_i is a subalgebra of A and it is obvious that

(1) $B_i \in R(A)$ for each $i \in I$,
(2) $A \notin R(B_i)$ for each $i \in I$.

Put

$$B = \prod_{i \in I} B_i.$$

Let $\overline{c}_1, \ldots, \overline{c}_n \in B$ be such that $\overline{c}_1(i) = c_1, \ldots, \overline{c}_n(i) = c_n$ for each $i \in I$. We can suppose that $0 \notin I$. Denote

$$T_0 = \{\overline{c}_1, \ldots, \overline{c}_n\}.$$

If $i \in I$, $f(a_i) = c_l, l \in \{1, \ldots, n\}$, then let T_i be the set of all elements $b \in B$ such that

(a) $b(i) \in A_i$, i.e., $b(i) \in f^{-m}(a_i)$ for $m \in \mathbb{N} \cup \{0\}$,

(b) if $j \in I - \{i\}$, then $b(j) = c_k$, where $k \in \{1, \ldots, n\}$ is such that $k \equiv l - m - 1 \pmod{n}$.

Put

$$T = \bigcup_{i \in I \cup \{0\}} T_i.$$

Notice that $T_i \cap T_j = \emptyset$ for each $i, j \in I \cup \{0\}$, $i \neq j$. Define a mapping $\nu \colon T \to A$ as follows: if $x \in T_i$ for some $i \in I \cup \{0\}$, then $\nu(x) = x(i)$. It can be verified that ν is an isomorphism, thus

(3) $A \cong T$.

To complete the proof we have to show that T is a retract of B. By (Thm), it suffices to prove

(4) if $y \in f^{-1}(T)$, then there is $z \in T$ with $f(y) = f(z)$ and $s_f(y) \leqslant s_f(z)$.

Let $y \in f^{-1}(T)$, $y \notin T$, $f(y) = b$. If $b \in T_0$, then $b = \overline{c_j}$ for some $j \in \{1, \ldots, n\}$ and there is $z \in T_0$ with $f(z) = b$. Since $s_f(z) = \infty$, we have $s_f(y) \leqslant s_f(z)$.

Now suppose that $b \in T_i$ for some $i \in I$. Then (a) and (b) are valid. Let $k' \in \{1, \ldots, n\}$ be such that $k' \equiv k - 1 \pmod{n}$. There exists $z \in B$ such that

(a') $z(i) = y(i)$,

(b') $z(j) = c_{k'}$ for each $j \in I - \{i\}$.

We have

$$f(z(i)) = f(y(i)) = b(i) \in A_i,$$

thus, by (a),

(a") $z(i) \in A_i$, $z(i) \in f^{-m-1}(a_i)$, $m \in \mathbb{N} \cup \{0\}$.

Further, (b) implies that if $j \in I - \{i\}$, then

$$k' \equiv k - 1 \equiv (l - m - 1) - 1 \equiv l - m - 2,$$

hence $z \in T_i$. The relation $f(z) = b = f(y)$ is valid since, if $j \in I - \{i\}$,

$$(f(z))(j) = f(c_{k'}) = c_k = b(j).$$

By the definition of z we have $s_f(z(j)) = \infty$ for each $j \in I - \{i\}$, thus

$$s_f(y) \leqslant s_f(y(i)) = s_f(z(i)) = s_f(z),$$

which completes the proof. $\square$

1.3. Corollary. *If* $\operatorname{card} V_1(A) > 1$, *then* A *is* DR-*reducible.*

1.4. Notation. For $k \in \mathbb{N}$ denote

$$M_k(A) = \{x \in V_k(A) \colon \operatorname{card} f^{-1}(x) > 2\}.$$

If $M_k(A) \neq \emptyset$, then let

$$S_k(A) = \{x \in M_k(A) \colon \max\{s_f(y) \colon y \in f^{-1}\} \text{ exists}\}.$$

1.5. Lemma. *Let* $k \in \mathbb{N}$ *and suppose that* $M_k(A) \neq \emptyset$, $S_k(A) \neq \emptyset$. *Then* A *is* DR-*reducible.*

Proof. For each $x \in S_k(A)$ take a fixed $y^x \in f^{-1}(x)$ with $s_f(y^x) = \max\{s_f(y) \colon y \in f^{-1}(x)\}$. Denote

$$\{a_i \colon i \in I\} = \{y \in f^{-1}(x) - \{y^x\} \colon x \in S_k(A)\},$$
$$A_i = \bigcup_{m \in \mathbb{N} \cup \{0\}} f^{-m}(a_i) \text{ for each } i \in I,$$
$$E = A - \bigcup_{i \in I} A_i.$$

If $i \in I$, then let a_i^* be such that $a_i^* = y^x$, where $f(a_i^*) = x$. Since $s_f(a_i^*) \geqslant s_f(a_i)$, there exists an endomorphism ψ_i of A such that $\psi_i(a_i) = a_i^*$ and $\psi_i(z) = z$ for each $z \in A - A_i$. Put

$$B_i = E \cup A_i.$$

Then B_i is a subalgebra of A and, by (Thm),

(1) B_i is a retract of A for each $i \in I$.

Let $M_k(B_i)$ and $S_k(B_i)$ be defined analogously to $M_k(A)$ and $S_k(A)$. If $x \in M_k(B_i)$, then $\operatorname{card} f^{-1}(x) > 2$ in B_i, thus the construction of B_i implies that $\max\{s_f(y) \colon y \in f^{-1}(x)\}$ does not exist, thus $S_k(B_i) = \emptyset$. Hence A is not isomorphic to any subalgebra of B_i, therefore

(2) $A \notin R(B_i)$ for each $i \in I$.

Let

$$B = \prod_{i \in I} B_i.$$

If $e \in E$, then denote $\overline{e} \in B$ such that $\overline{e}(i) = e$ for each $i \in I$. Put

$$T_0 = \{\overline{e}\colon\ e \in E\}.$$

If $i \in I$, then let

$$T_i = \{b \in B\colon\ b(i) \in A_i, b(j) = \psi_i(b(i)) \text{ for each } j \in I - \{i\}\}.$$

Further denote

$$T = \bigcup_{i \in I \cup \{0\}} T_i.$$

We obtain

(3) $A \cong T$.

Let us show that T is a retract of B. We will apply (Thm); it suffices to prove

(4) if $y \in f^{-1}(T)$, then there is $z \in T$ with $f(y) = f(z)$ and $s_f(y) \leqslant s_f(z)$.

The case $y \in T$ is trivial. Let $y \in f^{-1}(T) - T$. We have

$$s_f(y) \leqslant \min\{s_f(y(i))\colon\ i \in I\}$$

and there is $i_0 \in I$ with $\min\{s_f(y(i))\colon\ i \in I\} = s_f(y(i_0))$. If $y(i_0) \in E$, then there is $\overline{y(i_0)} \in T$ and we have

(5.1) $s_f(y) \leqslant s_f(\overline{y(i_0)})$, $\overline{y(i_0)} \in T$, $f(\overline{y(i_0)}) = f(y)$.

If $y(i_0) \notin E$, take $z \in B$ with

$$z(j) = \begin{cases} y(i_0) & \text{if } j = i_0, \\ \psi_{i_0}(y(i_0)) & \text{if } j \in I - \{i_0\}. \end{cases}$$

Then $z \in T_{i_0}$ and we have

$$s_f(z) = \min\{s_f(y(i_0)), s_f(\psi_{i_0}(y(i_0)))\}.$$

The mapping ψ_i is a homomorphism, thus

$$s_f(y(i_0)) \leqslant s_f(\psi_{i_0}(y(i_0))),$$

hence

(5.2) $s_f(y) \leqslant s_f(z), z \in T, f(y) = f(z)$.

Therefore T is a retract of B and (1)–(3) imply that A is DR-reducible. $\square$

1.6. Lemma. *Let $k \in \mathbb{N}$ and suppose that $M_k(A) \neq \emptyset$, $S_k(A) = \emptyset$. Then A is* DR*-reducible.*

Proof. Let the assumption hold. There exists a system $\{\alpha_i \colon i \in I\} \neq \emptyset$ of ordinals such that

(1) if $i, j \in I$, $i \neq j$, then $\alpha_i \neq \alpha_j$,

(2) $\{\alpha_i \colon i \in I\} = \{s_f(y) \colon y \in f^{-1}(x), x \in M_k(A)\}$.

We have

(3) if $x \in M_k(A)$, then $\max\{s_f(y) \colon y \in f^{-1}(x)\}$ does not exist.

For $i \in I$ let U_i be the set of all $z \in \bigcup_{j \in \mathbb{N} \cup \{0\}} f^{-j}(y)$, where $y \in f^{-1}(M_k(A))$ and $s_f(y) = \alpha_i$. Further put

$$B_i = A - U_i$$

and let

$$B = \prod_{i \in I} B_i.$$

According to (Thm), the definition of B_i implies

(4) $B_i \in R(A)$.

Further, if $i \in I$, then

$$\{y \in f^{-1}(M_n(B_i)) \colon s_f(y) = \alpha_i\} = \emptyset,$$
$$\{y \in f^{-1}(M_n(A)) \colon s_f(y) = \alpha_i\} \neq \emptyset,$$

thus A is not isomorphic to any subalgebra of B_i, hence

(5) $A \notin R(B_i)$ for each $i \in I$.

For each $y \in f^{-1}(M_k(A))$ with $s_f(y) = \alpha_i$ take a fixed $y' \in f^{-1}(f(y))$ and $\alpha_i' > \alpha_i$ such that $s_f(y') = \alpha_i'$ (it exists by (3)). Then there exists an endomorphism ψ_y of A such that $\psi_y(y) = y'$ and $\psi_y(z) = z$ for each $z \in A - \bigcup_{j \in \mathbb{N} \cup \{0\}} f^{-j}(y)$.

Now let us define a mapping $\nu \colon A \to B$ as follows. Let $a \in A$. If $a \in A - \bigcup_{i \in I} U_i$, then put $\nu(a) = \overline{a}$, where $\overline{a}(i) = a$ for each $i \in I$. If $a \in U_i$ for some $i \in I$, then $a \in f^{-m}(y)$, $y \in f^{-1}(M)$, $m \in \mathbb{N} \cup \{0\}$, $s_f(y) = \alpha_i$; we set $\nu(a) = b$, where

$$b(j) = \begin{cases} a & \text{if } j \in I - \{i\}, \\ \psi_y(a) & \text{if } j = i. \end{cases}$$

Denote $T = \nu(A)$. It is a formal matter to prove that ν is an isomorphism,

(6) $T \cong A$.

To complete the proof, it suffices to show

(7) if $b \in f^{-1}(T)$, then there is $d \in T$ with $f(d) = f(b)$ and $s_f(b) \leqslant s_f(d)$.

Let $b \in f^{-1}(T)$. Then there is $a \in A$ such that either

(a) $a \in A - \bigcup_{i \in I} U_i$, $f(b) = \overline{a}$,

or

(b) $a \in f^{-m}(y)$, $y \in f^{-1}(M_k(A))$, $m \in \mathbb{N} \cup \{0\}$, $s_f(y) = \alpha_i$ and

$$(f(b))(j) = \begin{cases} a & \text{if } j \in I - \{i\}, \\ \psi_y(a) & \text{if } j = i. \end{cases}$$

We have $s_f(b) = \min\{s_f(b(i))\colon\ i \in I\}$, thus there is $i_0 \in I$ with

(8) $s_f(b) = s_f(b(i_0))$.

Let (a) hold. Take $d \in B$ such that $d(j) = b(i_0)$ for each $j \in I$. We have

$$b(i_0) \in f^{-1}(\overline{a}(i_0)) = f^{-1}(a),$$

thus (a) implies

$$b(i_0) \in A - \bigcup_{i \in I} U_i,$$

hence

(9) $d = \overline{b(i_0)} \in T$.

If $j \in I$, then we obtain

$$f(b(j)) = a = f(b(i_0)) = f(d(j)),$$

i.e.,

(10) $f(b) = f(d)$.

According to (8),

$$s_f(b) = s_f(b(i_0)) = s_f(\overline{d}),$$

hence (9) and (10) yield that if (a) is valid, then (7) holds.

Suppose that (b) is valid. There is $i_1 \in I - \{i\}$ such that

$$\min\{s_f(b(j))\colon\ j \in J - \{i\}\} = s_f(b(i_1)).$$

Then

(11) $s_f(b) = \min\{s_f(b(j))\colon\ j \in J\} \leqslant s_f(b(i_1))$.

We have $f(b(i_1)) = a$, hence $b(i_1) \in U_i$. Let $d \in B$ be such that

$$d(j) = \begin{cases} b(i_1) & \text{if } j \in I - \{i\}, \\ \psi_y(b(i_1)) & \text{if } j = i. \end{cases}$$

Then $d \in T$ and if $j \in I - \{i\}$,

$$f(d(j)) = f(b(i_1)) = a = f(b(j)),$$
$$f(d(i)) = f(\psi_y(b(i_1)) = \psi_y(f(b(i_1))) = \psi_y(a) = f(b(i)).$$

Thus

(12) $f(d) = f(b)$, $d \in T$.

Further, according to (11),

$$s_f(b) \leqslant s_f(b(i_1)) \leqslant \min\{s_f(b(i_1)), s_f(\psi_y(b(i_1)))\} = s_f(d),$$

which implies that (7) is valid, which completes the proof. □

1.7. Corollary. *If A is* DR*-irreducible, $k \in \mathbb{N}$, $x \in V_k(A)$, then* $\operatorname{card} f^{-1}(x) \leqslant 2$.

1.8. Corollary. *If A is* DR*-irreducible and $x \in A$, then* $\operatorname{card} f^{-1}(x) \leqslant 2$.

Proof. The assertion follows from 1.7 and 1.3. □

2. Chains

In 2.1–2.8 we suppose that $\operatorname{card} V_1(A) \leqslant 1$ and that $\operatorname{card} f^{-1}(x) \leqslant 2$ for each $x \in A$.

2.1.1. Definition. Let $a \in A$. An indexed system $\{a_i \colon i \in \mathbb{N}\}$ of elements of A will be called an infinite a-chain, if

(1) $a_i \notin C$ for each $i \in \mathbb{N}$,
(2) $a_1 \in f^{-1}(a)$ and $s_f(a_1) \geqslant s_f(x)$ for each $x \in f^{-1}(a)$,
(3) if $i \in \mathbb{N}$, $i > 1$, then $a_i \in f^{-1}(a_{i-1})$ and $s_f(a_i) \geqslant s_f(x)$ for each $x \in f^{-1}(a_{i-1})$.

2.1.2. Definition. Let $a \in A$, $m \in \mathbb{N}$. An indexed system $\{a_1, a_2, \ldots, a_m\}$ of elements of A will be called an m-element a-chain, if (1), (2) of 2.1.1 are valid and

(4) if $i \in \{1, \ldots, m\}$, $i > 1$, then $a_1 \in f^{-1}(a_{i-1})$ and $s_f(a_i) \geqslant s_f(x)$ for each $x \in f^{-1}(a_{i-1})$,
(5) $f^{-1}(a_m) = \emptyset$.

2.1.3. Definition. Let $a \in A$. By an a-chain we will understand either an infinite a-chain or an m-element a-chain for $m \in \mathbb{N}$. The set of all a-chains will be denoted by $Ch(a)$.

2.2. Lemma. (a) *$Ch(a) \neq \emptyset$ for each $a \in A - C$.*

(b) *If $A \neq C$, then there exists exactly one element $c_0 \in C$ such that $Ch(c_0) \neq \emptyset$.*

Proof. The relations $\operatorname{card} V_1(A) \leqslant 1$ and $\operatorname{card} f^{-1}(x) \leqslant 2$ for each $x \in A$ imply the required assertions. □

2.3. Lemma. *Suppose that $A \neq C$ and that D is a c_0-chain, $c_0 \in C$. Let $\operatorname{card}(f^{-1}(D) - D) \geqslant 2$. Then A is* DR-*reducible.*

Proof. Let the assumption hold. Then

$$f^{-1}(D) - D = \{v_i \colon i \in I\}, \quad \operatorname{card} I \geqslant 2.$$

For $i \in I$ let

$$A_i = \bigcup_{k \in \mathbb{N} \cup \{0\}} f^{-k}(v_i),$$
$$B_i = C \cup D \cup A_i.$$

Obviously, B_i is a subalgebra of A and B_i is a retract of A for each $i \in I$.

Let $i \in I$. There is $j \in I - \{i\}$. Denote $u = f(v_j)$. If $f(v_i) = u$, then

$$\begin{aligned} &\operatorname{card} f^{-1}(u) \geqslant 3 \text{ in } A, \\ &\operatorname{card} f^{-1}(u) = 2 \text{ in } B_i. \end{aligned} \tag{1.1}$$

If $f(v_i) \neq u$, then

$$\begin{aligned} &\operatorname{card} f^{-1}(u) \geqslant 2 \text{ in } A, \\ &\operatorname{card} f^{-1}(u) = 1 \text{ in } B_i. \end{aligned} \tag{1.2}$$

Therefore A is not isomorphic to any subalgebra of B_i, hence

(2) $A \notin R(B_i)$ for each $i \in I$.

Denote

$$B = \prod_{i \in I} B_i.$$

If $i \in I$, then there is an endomorphism γ_i of A such that $\gamma_i(A_i) \subseteq D$, $\gamma_i(x) = x$ for each $x \in A - A_i$. If $y \in C \cup D$, then we denote by $\overline{y}$ the element of B such that $\overline{y}(i) = y$ for each $i \in I$. We set

$$T_0 = \{\overline{y}\colon\ y \in C \cup D\}.$$

If $i \in I$, then put

$$T_i = \{b \in B\colon\ b(i) \in A_i, b(k) = \gamma_i(b(i)) \text{ for each } k \in I - \{i\}\}.$$

Let

$$T = \bigcup_{i \in I \cup \{0\}} T_i.$$

We define a mapping $\nu\colon\ T \to A$ as follows. If $p \in T_0$, $p = \overline{y}$, where $y \in C \cup D$, then we put $\nu(p) = y$. If $p \in T_i$, $i \in I$, then we put $\nu(p) = p(i)$. It can be easily shown that ν is an isomorphism, thus

(3) $A \cong T$.

Let us show that T is a retract of B. Let $b \in f^{-1}(T)$. Then $f(b) = t$, where either

(a) there is $y \in C \cup D$ with $t(i) = y$ for each $i \in I$,

or

(b) there is $i \in I$, $y \in A_i$ with

$$t(k) = \begin{cases} y & \text{if } k = i, \\ \gamma_i(y) & \text{if } k \in I - \{i\}. \end{cases}$$

First suppose that (a) is valid. Since $f(b(i)) = t(i) = y$ for $i \in I$, we have $f^{-1}(y) \neq \emptyset$, thus there is $y_1 \in f^{-1}(y) \cap D$. Denote $z = \overline{y_1}$. If $i \in I$, then

$$f(b(i)) = t(i) = y = f(y_1) = f(z(i)),$$

i.e., $f(b) = f(z)$. Further, if $i \in I$, then

$$s_f(b(i)) \leqslant s_f(z(i)),$$

hence $s_f(b) \leqslant s_f(z)$. Therefore

(4) $z \in T$, $f(z) = f(b)$, $s_f(b) \leqslant s_f(z)$.

Now let (b) hold. Take $z \in B$ such that

$$z(k) = \begin{cases} b(i) & \text{if } k = i, \\ \gamma_i(b(i)) & \text{if } k \in I - \{i\}. \end{cases}$$

Then $z \in T_i \subseteq T$. We have

$$f(z(i)) = f(b(i))$$

and, if $k \in I - \{i\}$, then

$$f(z(k)) = f(\gamma_i(b(i)) = \gamma_i(f(b(i)) = \gamma_i(t(i)) = \\ = \gamma_i(y) = t(k) = f(b(k)).$$

Hence $f(z) = f(b)$. Further, since γ_i is a homomorphism, we get

$$s_f(b) \leqslant s_f(b(i)) \leqslant \min\{s_f(b(i)), s_f(\gamma_i(b(i)))\} = s_f(z).$$

Thus if (b) is valid, then (1) is valid as well. According to (Thm), T is a retract of B, therefore A is DR-reducible. □

In the following notation assume that distinct symbols denote distinct elements.

2.4. Notation. Let $\delta \in \mathbb{N}$, $m \in \mathbb{N}$, $\tau_1, \tau_2, \ldots, \tau_m \in \mathbb{N} \cup \{\aleph_0\}$, $k_1, \ldots, k_{m-1} \in \mathbb{N}$, $k_l \leqslant \tau_l$ for each $l \in \{1, 2, \ldots, m-1\}$. Let

$$D_0 = \{d_{01}, d_{02}, \ldots, d_{0,\delta}\}.$$

If $l \in \{1, \ldots, m\}$ and $\tau_l \in \mathbb{N}$, then denote $I_l = \{1, 2, \ldots, \tau_l\}$, and if $l \in \{1, \ldots, m\}$ and $\tau_l = \aleph_0$, then $I_l = \mathbb{N}$. Further put

$$D_l = \{d_{lj} \colon j \in I_l\}$$

and let

$$D = D_0 \cup D_1 \cup \ldots D_m.$$

Now let us define a unary operation f on D as follows:

$$f(d_{lj}) = \begin{cases} d_{l,j-1} & \text{if } l \in \{1, \ldots, m\}, j \in I_l - \{1\}, \\ d_{l-1,k_{l-1}} & \text{if } l \in \{2, \ldots, m\}, j = 1, \\ d_{01} & \text{if } (l, j) \in \{(1,1), (0,\delta)\}, \\ d_{0,j+1} & \text{if } l = 0, j \in \{1, \ldots, \delta - 1\}. \end{cases}$$

The monounary algebra (D, f) defined above will be denoted by the symbol

$$\mathscr{D}(\delta; m; \tau_1, k_1; \tau_2, k_2; \ldots; \tau_m).$$

(For the case $D = \mathscr{D}(2; 4; 3, 2; 5, 1; 3, 2; 1)$ cf. Fig. 1.)

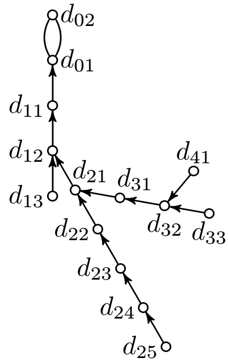

Fig. 1

2.5. Lemma. *Suppose that $A \neq C$ and that A is* DR*-irreducible. Then*

(i) *there are $\delta, m, \tau_1, \ldots, \tau_m, k_1, \ldots, k_{m-1}$ such that*

$$A \cong \mathscr{D}(\delta; m; \tau_1, k_1; \ldots; \tau_{m-1}, k_{m-1}; \tau_m),$$

(ii) *$\tau_{l-1} \geqslant \tau_l + k_{l-1}$ for each $l \in \{2, \ldots, m\}$.*

Proof. Let $A \neq C$, A be DR-irreducible. We denote elements of A by the symbols d_{lj}. The algebra A contains the cycle C with $\operatorname{card} C = n$; put $\delta = n$, $D_0 = C$. By 2.2 there is exactly one element c_0 of C with $Ch(c_0) \neq \emptyset$; denote it by d_{01} and let $D_1 \in Ch(c_0)$, $\tau_1 = \operatorname{card} D_1$. Further denote $d_{02} = f(d_{01}), \ldots, d_{0\delta} = f(d_{0,\delta-1})$. Under an appropriate notation we have

$$D_1 = \{d_{1j} \colon j \in I_1\}, \operatorname{card} I_1 = \tau_1,$$
$$f(d_{1j}) = \begin{cases} d_{1,j-1} & \text{for } j \in I_1 - \{1\}, \\ d_{01} & \text{if } j = 1. \end{cases}$$

By 2.3, $\operatorname{card}(f^{-1}(D_1) - D_1) \leqslant 1$.

Let us construct the sets D_m by induction. Let $m \in \mathbb{N}$, $m > 1$ and suppose that for each $m_1 \in \mathbb{N}$, $m_1 < m$

(2) $D_{m_1} = \{d_{m_1 j} \colon j \in I_{m_1}\}$ is defined, $\operatorname{card} D_{m_1} = \tau_{m_1}$,

(3) $f(d_{m_1 j}) = d_{m_1, j-1}$ for each $j \in I_{m_1} - \{1\}$,
$f(d_{m_1,1}) = d_{m_1-1, k_{m_1-1}}$ for some $k_{m_1-1} \in I_{m_1-1}$,

(4) $\operatorname{card}(f^{-1}(D_{m_1}) - D_{m_1}) \leqslant 1$.

If $f^{-1}(D_{m-1}) - D_{m-1} = \emptyset$, then

$$A = \mathscr{D}(\delta; m-1; \tau_1, k_1; \ldots; \tau_{m-2}, k_{m-2}; \tau_{m-1}).$$

Thus suppose that

$$\operatorname{card}(f^{-1}(D_{m-1}) - D_{m-1}) = 1;$$

denote $\{d_{m1}\} = f^{-1}(D_{m-1}) - D_{m-1}$. Then there is $k_{m-1} \in I_{m-1}$ with $f(d_{m1}) = d_{m-1,k_{m-1}}$. If $f^{-1}(d_{m1}) = \emptyset$, then put $I_m = \{1\}$ and then

$$A = \mathscr{D}(\delta; m; \tau_1, k_1; \ldots; \tau_{m-1}, k_{m-1}; 1).$$

If $f^{-1}(d_{m1}) \neq \emptyset$, then there exists a d_{m1}-chain, we can denote it by

$$D_m = \{d_{mj} \colon j \in I_m\}, \quad \operatorname{card} I_m = \tau_m,$$

thus (2) and (3) are valid for m. By way of contradiction, suppose

(5) $\operatorname{card}(f^{-1}(D_m) - D_m) \geqslant 2.$

Let

$$f^{-1}(D_m) - D_m = \{a_l \colon l \in L\}, \quad \operatorname{card} L \geqslant 2$$

and denote

$$E = \bigcup_{j=0}^{m} D_m.$$

For $l \in L$ let

$$A_l = \bigcup_{j \in \mathbb{N} \cup \{0\}} f^{-j}(a_l),$$

$$B_l = E \cup A_l.$$

Then B_l is a retract of A for each $l \in L$, and $A \notin R(B_l)$ for each $l \in L$. It can be proved analogously as in 2.3 that

$$A \in R\Bigl(\prod_{l \in L} B_l\Bigr)$$

and that A is DR-reducible, which is a contradiction, thus (5) fails to hold.

Let $l \in \mathbb{N}$, $l > 1$. If $\operatorname{card} I_{l-1} = \aleph_0$, then obviously

$$\operatorname{card} I_{l-1} \geqslant \operatorname{card} I_l + k_{l-1}.$$

Suppose that $\operatorname{card} I_{l-1} = \alpha < \aleph_0$. Then

$$D_{l-1} = \{d_{l-1,1}, d_{l-1,2}, \ldots, d_{l-1,\alpha}\},$$
$$f^{-1}(d_{l-1,\alpha}) = \emptyset.$$

Since D_{l-1} is a $d_{l-1,1}$-chain, we obtain

(6) $s_f(d_{l-1,1}) = \alpha - 1$,

(7) $s_f(d_{l-1,k_{l-1}-1}) = \alpha - (k_{l-1} + 1)$.

Further, we have

$$f(d_{l1}) = d_{l-1,k_{l-1}} = f(d_{l-1,k_{l-1}+1}), s_f(d_{l1}) \leqslant s_f(d_{l-1,k_{l-1}+1}),$$

hence

(8) $s_f(d_{l1}) \leqslant \alpha - (k_{l-1} + 1)$.

This relation yields that the set I_l is finite and that

(9) $s_f(d_{l1}) = \operatorname{card} I_l - 1$.

By (8) and (9) we get

$$\operatorname{card} I_l + k_{l-1} = s_f(d_{l1}) + 1 + k_{l-1} \leqslant \alpha = \operatorname{card} I_{l-1}.$$

Thus we have proved that the relation (ii) is valid.

According to (ii) we obtain

(10) $\tau_1 > \tau_2 > \tau_3 > \ldots$,

therefore the chain (10) is finite. Thus there is $m \in \mathbb{N}$ such that

$$A = \mathscr{D}(\delta; m; \tau_1, k_1; \ldots; \tau_{m-1}, k_{m-1}; \tau_m).$$

□

2.6. Lemma. *Let*

$$A = \mathscr{D}(\delta; m; \tau_1, k_1; \ldots; \tau_m)$$

and let (ii) *of* 2.5 *hold. Suppose that* $m \geqslant 3$ *and that there is* $l \in \{2, \ldots, m-1\}$ *with* $\tau_{l-1} = \tau_l + k_{l-1}$. *Then* A *is* DR*-reducible.*

Proof. Let the assumption hold. Denote

$$B_1 = A - D_{l-1},$$
$$B_2 = A - (D_{l+1} \cup \ldots \cup D_m).$$

The assumption yields

(1) $s_f(d_{l1}) = \tau_l - 1 = \tau_{l-1} + k_{l-1} - 1 = s_f(d_{l-1,k_{l-1}+1})$,

hence (Thm) implies

(2) $B_1 \in R(A)$.

Obviously,

(3) $B_2 \in R(A)$.

Since A is not isomorphic to any subalgebra of B_1 or B_2, we get

(4) $A \notin R(B_1)$, $A \notin R(B_2)$.

The proof that

$$A \in R(B_1 \times B_2)$$

is analogous to that of 2.3. Therefore A is DR-reducible. □

2.7. Lemma. *Let*

$$A = \mathscr{D}(\delta; m; \tau_1, k_1; \ldots; \tau_m)$$

and let (ii) *of* 2.5 *hold. Suppose that* $m \geqslant 2$ *and*

(1) $\tau_{m-1} = \tau_m + k_{m-1}$.

Then A is DR*-reducible.*

Proof. Let the assumption hold. By 2.5 and 2.6 it suffices to assume

(2) $\tau_{l-1} > \tau_l + k_{l-1}$ for each $l \in \{2, \ldots, m-1\}$.

Denote

$$B_1 = D_0 \cup D_1,$$
$$B_2 = A - D_m,$$
$$B = B_1 \times B_2.$$

Then

(3) $B_1 \in R(A)$, $B_2 \in R(A)$,
(4) $A \notin R(B_1)$, $A \notin R(B_2)$.

There is an endomorphism ψ of A such that $\psi(A) \subseteq D_0$ and $\psi(d_{m-1,k_{m-1}}) = d_{01}$. Define a mapping $\nu\colon A \to B$ as follows. If $x \in A - D_n$, then put $\nu(x) = (\psi(x), x)$. If $x = d_{mj} \in D_m$, $j \in \{1, \ldots, \tau_m\}$, then put

$$\nu(x) = (d_{1j}, d_{m-1,k_{m-1}+j}).$$

We obtain $\nu(d_{m1}) = (d_{11}, d_{m-1,k_{m-1}+1}), \ldots, \nu(d_{m\tau_m}) = (d_{1\tau_m}, d_{m-1,k_{m-1}+\tau_m}) =$
$= (d_{1\tau_m}, d_{m-1,\tau_{m-1}})$ by (1), thus ν is correctly defined. Obviously, ν is injective. Put $T = \nu(A)$. Then ν is an isomorphism, since

(a) if $x \in A - D_n$, then $f(x) \in A - D_n$ and $\nu(f(x)) = (\psi(f(x)), f(x)) =$
$= (f(\psi(x)), f(x)) = f(\nu(x))$,

(b) $\nu(f(d_{m1})) = \nu(d_{m-1,k_{m-1}}) = (\psi(d_{m-1,k_{m-1}}), d_{m-1,k_{m-1}}) =$
$= (d_{01}, d_{m-1,k_{m-1}}) = f((d_{11}, d_{m-1,k_{m-1}+1})) = f(\nu(d_{m1}))$,

(c) if $j \in \{2, \dots, \tau_m\}$, then $\nu(f(d_{mj})) = \nu(d_{m,j-1}) =$
$= (d_{1,j-1}, d_{m-1,k_{m-1}+j-1}) = f((d_{1j}, d_{m-1,k_{m-1}+j})) = f(\nu(d_{mj}))$.

Let us show that T is a retract of B. Let $b \in f^{-1}(T)$. Denote $f(b) = t$. Then either

(5.1) $t = (\psi(x), x)$ for some $x \in A - D_n$,

or

(5.2) $t = (d_{ij}, d_{m-1,k_{m-1}+j})$ for some $j \in \{1, \dots, \tau_m\}$.

Let (5.1) hold. Put $z = (\psi(b(2)), b(2))$. Then $z \in T$,

$$s_f(z) = \min\{s_f(z(1)), s_f(z(2))\} = s_f(b(2)) \geqslant s_f(b).$$

Further,

$$f(z) = (f(\psi(b(2))), f(b(2))) = (\psi(f(b(2)), f(b(2))) = (\psi(x), x) = t = f(b).$$

Suppose that (5.2) is valid. Then $b(1) \in f^{-1}\,(d_{1j}) \neq \emptyset$, i.e., $j < \tau_1$, $d_{1,j+1} \in f^{-1}(d_{1j})$. Similarly, $b(2) \in f^{-1}(d_{m-1,k_{m-1}+j}) = \{d_{m-1,k_{m-1}+j+1}\}$. Denote

$$z = (d_{1,j+1}, d_{m-1,k_{m-1}+j+1}).$$

Then $z \in T$,

$$f(z) = (f(d_{1,j+1}), f(d_{m-1,k_{m-1}+j+1})) = (d_{1j}, d_{m-1,k_{m-1}+j}) = t = f(b),$$
$$s_f(z) = \min\{s_f(z(1)), s_f(z(2))\} = s_f(z(2)) = s_f(b(2)) \geqslant s_f(b).$$

Therefore T is a retract of B and A is DR-reducible. □

2.8. Corollary. *Suppose that $A \neq C$ and that A is* DR*-irreducible. Then there are $\delta, m, \tau_1, \dots, \tau_m, k_1, \dots, k_{m-1}$ such that the following conditions are valid:*

(a) $A \cong \mathscr{D}\ (\delta; m; \tau_1, k_1; \dots; \tau_{m-1}, k_{m-1}; \tau_m)$;

(b) *either (i) $m = 1$, or (ii) $m > 1$ and*
(1) $\tau_{l-1} > \tau_l + k_{l-1}$ *for each $l \in \{2, \dots, m\}$.*

Remark. Notice that if $m > 1$, then $\tau_1 > \tau_2$, thus $\tau_2 \neq \aleph_0$. Further, (1) implies $\tau_l > k_l$ for each $l \in \{1, \dots, m-1\}$.

2.9. Theorem. *Let A be a connected monounary algebra possessing a cycle C. The following conditions are equivalent:*

(i) *A is* DR-*irreducible;*

(ii) *either $A = C$ or there are $\delta \in \mathbb{N}$, $m \in \mathbb{N}$, $\tau_1 \in \mathbb{N} \cup \{\aleph_0\}$, $\tau_2, \dots, \tau_m$, $k_1, \dots, k_{m-1} \in \mathbb{N}$ such that $A \cong \mathscr{D}(\delta; m; \tau_1, k_1; \dots; \tau_{m-1}, k_{m-1}; \tau_m)$,*

where either

(1) $m = 1$

or

(2) $m > 1$ *and* $\tau_{l-1} > \tau_l + k_{l-1}$ *for each* $l \in \{2, \dots, m\}$.

Proof. Let (i) hold. By 1.8, $\operatorname{card} f^{-1}(x) \leqslant 2$ for each $x \in A$. Then 2.8 implies that (ii) is valid.

Suppose that (ii) holds. If $A = C$, then obviously A is DR-irreducible. Let $A \neq C$, $m = 1$. If M is a retract of A, $M \neq A$, then $M = C$. By multiplying of cycles we cannot get an algebra with a subalgebra isomorphic to A, hence A is DR-irreducible.

Let (2) hold. By way of contradiction, assume that A is retract reducible. There are monounary algebras $B_\lambda, \lambda \in L$ such that

(3) $A \in R\Big(\prod\limits_{\lambda \in L} B_\lambda\Big)$,

(4) $B_\lambda \in R(A)$ for each $\lambda \in L$,

(5) $A \notin R(B_\lambda)$ for each $\lambda \in L$.

Without loss of generality we can suppose that B_λ is a retract of A for each $\lambda \in L$ and that $\cong$ in (ii) is equality. By (3) there is an isomorphism ν of A onto some retract M of $\prod\limits_{\lambda \in L} B_\lambda$. Denote $b \in \nu(d_{m\tau_m})$. Since $f^{-1}(d_{m\tau_m}) = \emptyset$, there is $\lambda_1 \in L$ such that $f^{-1}(b(\lambda_1)) = \emptyset$. Hence

(6) $b(\lambda_1) \in \{d_{1\tau_1}, \dots, d_{m\tau_m}\}$.

Let $\beta = \tau_m + k_{m-1} + \dots + k_1$. Then

$$f^\beta(d_{m\tau_m}) \in C,$$

thus $f^\beta(b)$ belongs to a cycle of M, i.e., $f^\beta(b(\lambda))$ belongs to a cycle of B_λ for each $\lambda \in L$. We have according to (2) that

(7) $f^\beta(d_{j\tau_j}) \notin C$ for each $j \in \{1, \dots, m-1\}$,

therefore

(8) $b(\lambda_1) = d_{m\tau_m}$.

Since each retract of A which contains $d_{m\tau_m}$ coincides with A, we obtain that

$$B_{\lambda_1} = A,$$

a contradiction to (5). □

2.10. Example. The algebra

$$A = \mathscr{D}(2; 4; 10, 1; 8, 2; 5, 2; 2)$$

is retract irreducible, because we have $m = 4 > 1$, $10 > 8 + 1$, $8 > 5 + 2$, $5 > 2 + 2$.

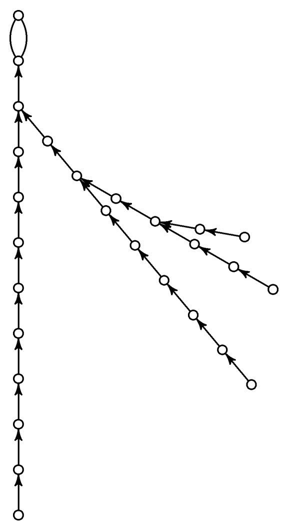

Fig. 2

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