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ON CUT COMPLETIONS OF ABELIAN LATTICE ORDERED GROUPS

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Abstract. We denote by F_a the class of all abelian lattice ordered groups H such that each disjoint subset of H is finite. In this paper we prove that if $G \in F_a$, then the cut completion of G coincides with the Dedekind completion of G.

Keywords: abelian lattice ordered group, disjoint subset, cut completion, Dedekind completion

MSC 2000: 06F20, 06F15

The notion of the cut completion of a lattice ordered group was introduced by Ball [1].

Let G be a lattice ordered group. We denote by G^c and $G^\wedge$ the cut completion and the Dedekind completion of G, respectively.

If G is a lexico extension of a lattice ordered group A, then we express this fact by writing $G = \langle A\rangle$.

Lattice ordered groups with a finite number of disjoint elements were investigated by Conrad [4].

Let F_a be the class of all abelian lattice ordered groups having only a finite number of disjoint elements.

In the present paper we prove the following result:

(A) Let G be an abelian lattice ordered group and let $A \neq \{0\}$ be an ℓ-subgroup of G such that $G = \langle A\rangle$. Then
  - (i) $G^c = \langle A^c\rangle$,
  - (ii) the linearly ordered groups G/A and G^c/A^c are isomorphic.

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By applying (A) we obtain

(B) Let $G \in F_a$. Then

(i) $G^c \in F_a$,

(ii) $G^c = G^\wedge$.

A result analogous to the relation given in (ii) of (B) concerning distinguished extensions of linearly ordered groups was proved by Ball [3].

The question whether (A) and (B) are valid also for the non-abelian case remains open.

1. Preliminaries

For lattice ordered groups we apply the notation as in Conrad [5]. In particular, the group operation in a lattice ordered group is written additively.

We recall some relevant definitions.

A lattice ordered group G is said to be a *lexico extension* of its ℓ-subgroup A if the following conditions are satisfied:

(i) A is a convex ℓ-subgroup of G;

(ii) if $0 < g \in G$ and $g \notin A$, then $g > a$ for each $a \in A$.

Under these conditions we write $G = \langle A \rangle$. It is well-known that then we have

(i_1) A is an ℓ-ideal of G;

(ii_1) the factor ℓ-group G/A is linearly ordered.

A subset X of a lattice ordered group G is called a (*Dedekind*) *cut* in G if X is an order closed lattice ideal (X is the set of all lower bounds of its upper bounds) such that $g + X \neq X \neq X + g$ for each $g \in G$ with $g > 0$.

G is said to be *cut complete* (*Dedekind complete*) if every (Dedekind) cut of G has a supremum in G. (Cf. [1], [3].)

An ℓ-subgroup G_1 of a lattice ordered group G_2 is said to be *order dense* in G_2 if for each $0 < g_2 \in G$ there exists $0 < g_1 \in G_1$ with $g_1 \leqslant g_2$.

For each lattice ordered group G there exist lattice ordered groups G^c and $G^\wedge$ such that

(i) G^c is cut complete and $G^\wedge$ is Dedekind complete;

(ii) both G^c and $G^\wedge$ contain G as an order dense ℓ-subgroup;

(iii) if $G \leqslant K < G^c$ ($G \leqslant K < G^\wedge$), then K fails to be cut complete (Dedekind complete).

G^c and $G^\wedge$ are called the *cut completion* or the *Dedekind completion* of G, respectively.

G^c and $G^\wedge$ are uniquely determined up to isomorphisms leaving all the elements of G fixed.

2. Lexico extensions

Let us suppose that G and B are abelian lattice ordered groups which satisfy the following conditions:

(i) $G = \langle A \rangle$;
(ii) A is a convex ℓ-subgroup of B;
(iii) $G \cap B = A$.

We denote by H_0 the set of all pairs (g, b) with $g \in G$ and $b \in B$. For $(g_i, b_i) \in H$ $(i = 1, 2)$ we put

$$(g_1, b_1) \equiv (g_2, b_2)$$

if both $g_1 - g_2$, $b_2 - b_1$ belong to A and if these elements are equal.

The relation $\equiv$ on H_0 is reflexive, symmetric and transitive. Denote

$$\overline{(g, b)} = \{(g_1, b_1) \in H_0 \colon (g, b) \equiv (g_1, b_1)\},$$
$$H = \{\overline{(g, b)} \colon (g, b) \in H_0\}.$$

For $\overline{(g_1, b_1)}, \overline{(g_2, b_2)} \in H$ put

$$\overline{(g_1, b_1)} + \overline{(g_2, b_2)} = \overline{(g_3, b_3)},$$

where $g_3 = g_1 + g_2$ and $b_3 = b_1 + b_2$. It is easy to verify that $+$ is a correctly defined binary operation on H which is associative and commutative. Further, $\overline{(0, 0)}$ is the neutral element of $(H, +)$. Moreover,

$$\overline{(g, b)} + \overline{(-g, -b)} = \overline{(0, 0)}.$$

Thus we have

2.1. Lemma. *$(H, +)$ is an abelian group.*

We define a binary relation $\leqslant$ on H as follows. Let $\overline{(g_1, b_1)}, \overline{(g_2, b_2)} \in \overline{H}$. We put

$$\overline{(g_1, b_1)} \leqslant \overline{(g_2, b_2)}$$

if either

$$(\alpha) \qquad g_1 < g_2 \quad \text{and} \quad g_1 - g_2 \notin A$$

or

$$(\beta) \qquad g_1 - g_2 \in A \quad \text{and the relation}$$
$$g_1 - g_2 \leqslant b_2 - b_1$$

is valid in B.

Then in view of the definition of $\equiv$, $\leqslant$ is a correctly defined binary relation on the set H.

2.2. Lemma. *$\leqslant$ is a partial order on H.*

Proof. a) Reflexivity: Let $\overline{(g_1, b_1)} = \overline{(g_2, b_2)}$. Then

$$g_1 - g_2 = b_2 - b_1.$$

Hence $g_1 - g_2 \in B \cap G$ and thus in view of (iii), $g_1 - g_2 \in A$. Further, according to (β), we obtain $\overline{(g_1, b_1)} \leqslant \overline{(g_2, b_2)}$.

b) Transitivity: Let $\overline{(g_1, b_1)} \leqslant \overline{(g_2, b_2)}$ and $\overline{(g_2, b_2)} \leqslant \overline{(g_3, b_3)}$. We distinguish the following cases:

(α_1) Suppose that

$$g_1 < g_2,\ g_1 - g_2 \notin A,\ g_2 < g_3,\ g_2 - g_3 \notin A.$$

Thus $g_1 < g_3$. If $g_1 - g_3 \in A$, then $g_1 + A = g_3 + A$. Since $g_1 + A$ is a convex subset of G we get $g_2 \in g_1 + A$, whence $g_1 - g_2 \in A$, which is a contradiction. Thus $g_1 - g_3 \notin A$ and then $\overline{(g_1, b_1)} \leqslant \overline{(g_3, b_3)}$.

(α_2) Suppose that

$$g_1 - g_2 \in A,\ g_1 - g_2 \leqslant b_2 - b_1;$$
$$g_2 - g_3 \in A,\ g_2 - g_3 \leqslant b_3 - b_2.$$

Then $g_1 - g_3 \in A$ and

$$g_1 - g_3 = (g_1 - g_2) + (g_2 - g_3) \leqslant (b_2 - b_1) + (b_3 - b_2) = b_3 - b_1,$$

whence $\overline{(g_1, b_1)} \leqslant \overline{(g_2, b_2)}$.

(α_3) Suppose that

$$g_1 < g_2,\ g_1 - g_2 \notin A,$$
$$g_2 - g_3 \in A,\ g_2 - g_3 \leqslant b_3 - b_2.$$

Then we have

$$g_1 < g_3, \; g_1 - g_3 \notin A,$$

thus $\overline{(g_1, b_1)} \leqslant \overline{(g_3, b_3)}$.

(α_4) If the relations

$$g_1 - g_2 \in A, \quad g_1 - g_2 \leqslant b_2 - b_1,$$
$$g_2 < g_3 \quad \text{and } g_2 - g_3 \notin A$$

are valid, then we can proceed analogously as in the case (α_3).

c) Antisymmetry: Let $\overline{(a_i, b_i)}$ $(i = 1, 2, 3)$ be as in b) and suppose that $\overline{(a_1, b_1)} = \overline{(a_3, b_3)}$. Without loss of generality we can assume that $g_1 = g_3$ and $b_1 = b_3$. Again, we can distinguish the cases (α_1)-(α_4).

The case (α_1) cannot hold, since we would have $g_1 < g_3$, which is a contradiction. Analogously, neither (α_3) nor (α_4) can be valid.

Suppose that (α_2) is satisfied. Hence $g_1 - g_2 \in A$. Thus we have also $g_2 - g_1 \in A$. Then the relations

$$\overline{(g_1, b_1)} \leqslant \overline{(g_2, b_2)}, \quad \overline{(g_2, b_2)} \leqslant \overline{(g_1, g_2)}$$

yield

$$g_1 - g_2 \leqslant b_2 - b_1,$$
$$g_2 - g_1 \leqslant b_1 - b_2,$$

whence $g_1 - g_2 = b_2 - b_1$. Therefore $\overline{(g_1, b_1)} = \overline{(g_2, b_2)}$. □

2.3. Lemma. *With respect to the operation + and to the relation $\leqslant$, H is a partially ordered group.*

P r o o f. Let $\overline{(g_i, b_i)} \in H$ $(i = 1, 2, 3)$,

$$\overline{(g_1, b_1)} \leqslant \overline{(g_2, b_2)}.$$

Denote

$$g_1' = g_1 + g_3, \quad b_1' = b_1 + b_3,$$
$$g_2' = g_2 + g_3, \quad b_2' = b_2 + b_3.$$

Suppose that (α) holds. Then

$$g_1' < g_2' \quad \text{and} \quad g_1' - g_2' \in A,$$

whence $\overline{(g_1', b_1')} \leqslant \overline{(g_2', b_2')}$.

Further suppose that (β) is valid. Thus

$$g_1' - g_2' \in A, \quad g_1' - g_2' \leqslant b_2' - b_1'.$$

Again, we obtain $\overline{(g_1', b_1')} \leqslant \overline{(g_2', b_2')}$. □

2.4. Lemma. *H is a lattice ordered group.*

Proof. In view of 2.3 it suffices to verify that for each $(g, b) \in H$ there exists

$$\sup\{\overline{(g, b)}, \overline{(0, 0)}\}$$

in H.

Let $\overline{(g, b)}$ be an arbitrary element of H. If $g \notin A$, then we have either $g > 0$ or $g < 0$. In the first case

$$\overline{(0, 0)} < \overline{(g, b)},$$

and in the other,

$$\overline{(0, 0)} > \overline{(g, b)}.$$

It remains to consider the situation when $g \in A$. Hence $g + b \in B$ and thus there exists $b_1 \in B$ such that the relation

$$b_1 = \sup\{0, g + b\}$$

is valid in B. Then we clearly have

$$\overline{(0, 0)} \leqslant \overline{(0, b_1)}, \quad \overline{(g, b)} \leqslant \overline{(0, b_1)}.$$

Let $\overline{(g', b')} \in H$, $\overline{(0, 0)} \leqslant \overline{(g', b')}$, $\overline{(g, b)} \leqslant \overline{(g', b')}$.

If $g' \notin A$, then $g' > 0$ and then $\overline{(g', b')} \geqslant \overline{(g, b)}$. Suppose that $g' \in A$. We have

$$\overline{(g', b')} = \overline{(0, g' + b')}, \quad \overline{(g, b)} = \overline{(0, g + b)},$$

hence

$$g' + b' \geqslant 0, \quad g' + b' \geqslant g + b.$$

This yields that $g' + b' \geqslant b_1$ and therefore

$$\overline{(g', b')} \geqslant \overline{(0, b_1)}.$$

Thus we obtain that the relation

$$\overline{(0, b_1)} = \sup\{\overline{(g, b)}, \overline{(0, 0)}\}$$

is valid in H. □

For each $g \in G$ we put

$$\varphi(g) = \overline{(g, 0)}.$$

Then φ is an isomorphism of the lattice ordered group G into the lattice ordered group H. Hence, if g and $\varphi(g)$ are identified, then we can view G as an ℓ-subgroup of H.

Further, for each $b \in B$ we set

$$\psi(b) = \overline{(0, b)}.$$

The mapping ψ is an isomorphism of the lattice ordered group B into H. If $b \in B \cap G$, then $\psi(b) = \varphi(b)$. We can identify b and $\psi(b)$ for each $b \in B$. Thus B turns out to be an ℓ-subgroup of H.

Under the above mentioned identification we have

2.5. Lemma. $H = \langle B \rangle$.

Proof. Let $\overline{(g, b)} \in H$ be such that $\overline{(g, b)} \geqslant \overline{(0, 0)}$ and $\overline{(g, b)} \notin B$. Then $g \notin A$ and thus $0 < g$. Further let $b_1 \in B$. Hence b_1 is identified with $\overline{(0, b_1)}$. We get $\overline{(0, b_1)} < \overline{(g, b)}$. Therefore $H = \langle B \rangle$. □

In view of (i), G/A is a linearly ordered group. Also, according to 2.5, H/B is a linearly ordered group. Let $g + A \in G/A$. If $g_1 \in G$ and $g_1 + A = g + A$, then $g - g_1 \in A$, whence $g - g_1 \in B$, thus $g + B = g_1 + B$. Hence the correspondence

$$\chi\colon\ G/A \to H/B$$

defined by

$$\chi(g + A) = g + B$$

is a correctly defined mapping of G/A into H/B.

2.6. Lemma. *χ is an isomorphism of G/A into H/B.*

Proof. Let $\overline{(g, b)} + B$ be an arbitrary element of H/B. Then $\overline{(g, 0)} \in \overline{(g, b)} + B$, whence $\overline{(g, b)} + B = g + B$ and thus χ is an epimorphism.

Next, since

$$(g_1 + A) + (g_2 + A) = (g_1 + g_2) + A,$$

the mapping χ is a homomorphism with respect to the group operation.

If $\chi(g + A) = B$, then $g \in B$, whence $g \in G \cap B = A$, yielding that $g + A = A$. Hence χ is an isomorphism with respect to the group operation.

We have already remarked that both B/A and G/B are linearly ordered. Let $g_1 + A$, $g_2 + A \in G/A$. Then the relation

$$g_1 + A \leqslant g_2 + A$$

is equivalent to

$$(g_1 \wedge g_2) + A = g_1 + A$$

and this is equivalent to

$$(g_1 \wedge g_2) + B = g_1 + B.$$

The last relation holds if and only if

$$g_1 + B \leqslant g_2 + B.$$

This completes the proof. □

Summarizing, we have

2.7. Proposition. *Let A, B and G be abelian lattice ordered groups which satisfy the conditions* (i), (ii) *and* (iii) *above. Then there exists a lattice ordered group H such that*

(a) $H = \langle B \rangle$;
(b) *G is an ℓ-subgroup of H;*
(c) *the mapping defined by*

$$g + A \to g + B$$

(where g runs over G) is an isomorphism of G/A onto H/B.

3. Proof of (A)

In order to prove (A) we apply the result of the previous section.

3.1. Lemma. *Let H be an abelian lattice ordered group, $H = \langle B \rangle$, $B \neq \{0\}$. Suppose that B is cut complete. Then H is cut complete.*

Proof. Let X be a cut in H. Hence $h + X \neq X$ for each $h \in H$ with $h > 0$. Denote

$$\overline{G_1} = \{g + B \in H/B \colon (g + B) \cap X \neq \emptyset\}.$$

Then the set $\overline{G_1}$ is nonempty and it is linearly ordered (by the linear order induced from that of H/B).

a) First suppose that if $g+B \in \overline{G_1}$, then $g+B \subseteq X$. Since $B \neq \{0\}$, there exists $0 < g_1 \in B$. Thus for each $g+B \in \overline{G_1}$ we have

$$g_1 + (g+B) = g + (g_1 + B) = B,$$

whence $g_1 + X = X$, which is a contradiction.

b) In view of a), there exists $g+B \in \overline{G_1}$ such that

$$(g+B) \cap X \neq g+B.$$

Then $g+B$ is the greatest element of the set $\overline{G_1}$.

There exists $g_1 \in (g+B) \cap X$. Denote

$$X - g_1 = Y, \quad Y \cap B = Z.$$

Then Y is an order closed lattice ideal in H and

$$h + Y \neq Y \quad \text{for each } 0 < h \in B. \tag{1}$$

Further we have

$$\emptyset \neq Z \neq B,$$

Z being an order closed lattice ideal in B; moreover, (1) yields that

$$b + Z \neq Z \quad \text{for each } 0 < b \in B.$$

Thus Z is a cut in B. Since B is cut complete, there exists $b_1 \in B$ such that the relation

$$b_1 = \sup Z$$

is valid in B. From this we conclude that

$$b_1 = \sup Y$$

is valid in H and therefore

$$b_1 + g_1 = \sup X$$

holds in H. Thus H is cut complete. □

3.2. Lemma. *Let A, B, G and H be as in 2.7. Suppose that $B = A^c$. Further suppose that H' is an ℓ-subgroup of H such that $G \subseteq H' \subset H$. Then H' is not cut complete.*

Proof. Since $H' \subset H$ we infer that $(H')^+ \subset H^+$. Hence there exists $\overline{(g,b)} \in H^+$ such that $\overline{(g,b)}$ does not belong to H'.

Under the embeddings considered in Section 2, the element $\overline{(g,b)}$ can be identified with $g + b$. Since $G \subseteq H'$ we obtain $g \in H'$, thus b cannot belong to H'.

Denote $B_1 = H' \cap B$. Then $A \subseteq B_1 \subset B$. Thus B_1 fails to be cut complete. Hence there exists a cut Z in B_1 such that Z has no supremum in B_1. We have

$$b_1 + Z \neq Z \tag{2}$$

for each $0 < b_1 \in B_1$.

Let Z_1 be the order closed lattice ideal in H' which is generated by the set Z. Then Z_1 is a cut in H'. Moreover, from (2) we obtain that

$$h' + Z' \neq Z'$$

for each $0 < h' \in H'$. The fact that Z has no supremum in B_1 implies that Z' has no supremum in H'. Therefore H' is not cut complete. □

Proof of (A). Suppose that the assumption of (A) is satisfied. Put $B = A^c$ and let H be as in 2.7. In view of 2.7 we have $H = \langle B \rangle$. Then A is order dense in B and B is order dense in H, whence A is order dense in H. This yields that G is order dense in H. From this and from 3.1 and 3.2 we conclude that $H = G^c$. □

3.3. Lemma. *Let A, B, G and H be as in 2.7. Suppose that $B = A^\wedge$. Let H' be an ℓ-subgroup of H such that $G \subseteq H' \subset H$. Then H' is not Dedekind complete.*

Proof. We apply the same method as in the proof of 3.2 with the distinction that instead of cuts we now deal with Dedekind cuts. □

3.4. Proposition. *Let G be an abelian lattice ordered group, $G = \langle A \rangle$. Suppose that $A^c = A^\wedge$. Then $G^c = G^\wedge$.*

Proof. In view of the proof of (A) we have $G^c = H$, where H is as in 2.7 and $B = A^c$. Each cut complete lattice ordered group is Dedekind complete, hence H is Dedekind complete. In view of 3.3 we then conclude that H is a Dedekind completion of G. □

For a lattice ordered group G we denote by G^{dist} the distinguished completion of G (cf. Ball [3]).

From the definitions of $G^c, G^\wedge$ and G^{dist} we obtain (cf. also Ball [2])

4.1. Lemma. *For each lattice ordered group G we have*

$$G \subseteq G^\wedge \subseteq G^c \subseteq G^{\mathrm{dist}}.$$

4.2. Lemma. *Let G be a linearly ordered group. Then*

(i) *G^{dist} is linearly ordered;*

(ii) *$G^{\mathrm{dist}} = G^c = G^\wedge$.*

Proof. According to 4.3 in [3], G^{dist} is linearly ordered and $G^{\mathrm{dist}} = G^\wedge$. Then in view of 4.1, $G^c = G^\wedge$. □

In the remaining part of this section we suppose that a lattice ordered group G is represented as a direct product

$$G = \prod_{i\in I} G_i.$$

For $g \in G$ we denote by g_i or by $g(G_i)$ the component of g in G_i. If $Y \subseteq G$, then we put

$$Y(G_i) = \{y(G_i) \colon\ y \in Y\}.$$

From the definition of the direct product we immediately obtain

4.3. Lemma. *Let $Y \subseteq G$. Then the following conditions are equivalent:*

(i) *Y is an order closed lattice ideal in G;*

(ii) *for each $i \in I$, $Y(G_i)$ is an order closed lattice ideal in G_i.*

4.4. Lemma. *Let Y be an order closed lattice ideal in G. Then the following conditions are equivalent:*

(i) *Y is a cut in G;*

(ii) *for each $i \in I$, $Y(G_i)$ is a cut in G_i.*

Proof. Let (i) be valid and let $i \in I$. Further let $g^i \in G_i$, $g^i > 0$. There exists $g \in G$ such that $g_i = g^i$ and $g_j = 0$ whenever $j \in I$ and $j \neq i$. Then $g > 0$. Put $g + Y = Z$. Thus

$$Z_i = g^i + Y_i,$$
$$Z_j = Y_j \quad \text{for } j \in I \setminus \{i\}$$

(where $Y_i = Y(G_i)$ and similarly for the other symbols applied above). Since $Z \neq Y$, we must have $Z_i \neq Y_i$, i.e., $g^i + Y_i \neq Y_i$. Analogously we obtain $Y_i + g^i \neq Y_i$. Hence (ii) holds.

Conversely, suppose that (ii) is valid. Let $0 < g \in G$. Then there is $i \in I$ such that $g_i > 0$. We have (under analogous notation as above)

$$(g + Y)(G_i) = g_i + Y_i \neq Y_i,$$

whence $g + Y \neq Y$. Similarly, $Y + g \neq Y$. Thus (i) holds. □

4.5. Lemma. *G is cut closed if and only if all G_i are cut closed.*

Proof. Assume that G is cut closed. Let $i \in I$ and let Y^i be a cut in G_i. Denote

$$\overline{Y}^i = \{g \in G\colon\ g_i \in Y^i \quad \text{and } g_j \leqslant 0 \text{ for each } j \in I \setminus \{i\}\}.$$

Then for each $j \in I$, $\overline{Y}^i(G_j)$ is a cut in G_j, whence in view of 4.4, $\overline{Y}^i$ is a cut in G. Thus there exists $g \in G$ such that

$$g = \sup \overline{Y}^i$$

is valid in G. Further we have $\overline{Y}^i(G_i) = Y^i$. Hence

$$g_i = \sup Y^i$$

holds in G_i. Therefore G_i is cut complete.

Conversely, assume that all G_i's are cut complete. Let Y be a cut in G. For each $i \in I$ we denote $Y_i = Y(G_i)$. In view of 4.4, Y_i is a cut in G_i, hence there is $z^i \in G_i$ such that the relation

$$z^i = \sup Y_i$$

is valid in G_i. There exists $g \in G$ such that

$$g_i = z^i \quad \text{for each } i \in I.$$

Then $g = \sup Y$ in G; therefore G is cut closed. □

4.6. Corollary. *The lattice ordered group $\prod\limits_{i\in I} G_i^c$ is cut closed.*

The following result was proved in [6] under the assumption that G is abelian, but the proof remains valid also without this assumption.

4.7. Proposition. (Cf. [6], Theorem 2.7.) *Let* $G = \prod_{i\in I} G_i$. *Then* $G^\wedge = \prod_{i\in I} G_i^\wedge$.

Denote $H = \prod_{i\in I} G_i^c$.

4.8. Lemma. *Let* K *be an* ℓ*-subgroup of* H *such that* $G \subseteq K \subset H$. *Assume that* $G_i^\wedge = G_i^c$ *for each* $i \in I$. *Then* K *is not cut closed.*

P r o o f. By way of contradiction, assume that K is cut closed. Then K is Dedekind complete. Thus $G \subseteq K$ yields $G^\wedge \subseteq K$. By applying 4.7 we obtain

$$K \supseteq \prod_{i\in I} G_i^\wedge = \prod_{i\in I} G_i^c = H,$$

which is a contradiction. $\square$

4.9. Proposition. *Let* G *be a lattice ordered group which can be represented as a direct product* $G = \prod_{i\in I} G_i$. *Suppose that* $G_i^c = G_i^\wedge$ *for each* $i \in I$. *Then* $G^c = \prod_{i\in I} G_i^c$.

P r o o f. It is easy to verify that G is order dense in $\prod_{i\in I} G_i^c$. Hence it suffices to apply 4.6 and 4.8. $\square$

5. Proof of (B)

We recall that a subset D of a lattice ordered group G is called disjoint if $d \geqslant 0$ for each $d \in D$, and $d_1 \wedge d_2 = 0$ whenever d_1 and d_2 are distinct elements of D.

Let F be the class of all lattice ordered groups such that each disjoint subset of G is finite.

According to [4] the structure of a lattice ordered group $G \neq \{0\}$ belonging to F can be described as follows.

There exist a positive integer n_0 and finite nonempty systems $S_1, S_2, \ldots, S_{n_0}$ of convex ℓ-subgroups of G such that the following conditions are satisfied:

1) All lattice ordered groups belonging to S_1 are nonzero and linearly ordered.

2) Let $1 < n \leqslant n_0$. Then there is a positive integer $k(n)$ such that

$$S_n = \{G_1^n, G_2^n, \ldots, G_{k(n)}^n\}$$

and for each $j \in \{1, 2, \ldots, k(n)\}$ there is a subset T_j^{n-1} of S_{n-1} with

$$G_j^n = \langle X_j^{n-1} \rangle,$$

where X_j^{n-1} is a direct product of lattice ordered groups belonging to T_j^{n-1}. Moreover,

$$S_{n-1} = \bigcup T_j^{n-1} \quad (j = 1, 2, \dots, k(n))$$

and

$$T_{j(1)}^{n-1} \cap T_{j(2)}^{n-1} = \emptyset$$

whenever $j(1), j(2)$ are distinct elements of the set $\{1, 2, \dots, k(n)\}$.

3) $S_{n_0} = \{G\}$.

Conversely, we obviously have

5.1. Lemma. *Let $S_1 = \{G_1^1, G_2^1, \dots, G_{k(1)}^1\}$ be a finite system of nonzero linearly ordered groups. Suppose that we consecutively construct systems $S_2, S_3, \dots, S_{n_0}$ such that the conditions* (2) *and* (3) *are satisfied. Then G belongs to F; namely, if D is a disjoint set of strictly positive elements of G, then* $\operatorname{card} D \subseteq k(1)$.

In the remaining part of this section we assume that G is a lattice ordered group belonging to F_a. The case $G = \{0\}$ being trivial we suppose that $G \neq \{0\}$. Hence there are systems $S_1, S_2, \dots, S_{n_0}$ of convex ℓ-subgroups of G satisfying the conditions 1), 2) and 3).

For each $n \in \{1, 2, \dots, n_0\}$ and each $j \in \{1, 2, \dots, k(n)\}$ we put

$$H_j^n = (G_j^n)^c.$$

Further, if $n \in \{2, 3, \dots, n_0\}$ and $j \in \{1, 2, \dots, k(n)\}$, then we set

$$Y_j^{n-1} = (X_j^{n-1})^c.$$

Also, for $n \in \{1, 2, \dots, n_0\}$ we denote

$$S_n' = \{H_j^n \colon j = 1, 2, \dots, k(n)\}.$$

5.2. Lemma. *Let $H_j^1 \in S_1'$. Then H_j^1 is linearly ordered.*

P r o o f. It suffices to apply 4.2. □

5.3. Lemma. *Let $G_j^1 \in S_1$. Then $(G_j^1)^c = (G_j^1)^\wedge$.*

P r o o f. In view of the assumption, G_j^1 is linearly ordered. Then the assertion is a consequence of 4.2. □

5.4. Lemma. *Let $j \in \{1, 2, \ldots, k(2)\}$. Then*

$$(X_j^2)^c = (X_j^2)^\wedge.$$

Proof. X_j^2 is the direct product of a finite number of elements of S_1. Thus the assertion follows from 5.3, 4.7 and 4.9. $\square$

5.5. Lemma. *Let $j \in \{1, 2, \ldots, k(2)\}$. Then*

$$(G_j^2)^c = (G_j^2)^\wedge.$$

Proof. It suffices to apply 3.4 and 5.4. $\square$

5.6. Lemma. *Let $j \in \{1, 2, \ldots, k(2)\}$. Then $H_j^2 \in F_a$.*

Proof. We have

$$H_j^2 = (G_j^2)^c = (\langle X_j^1 \rangle)^c.$$

Hence in view of (A),

$$H_j^2 = \langle (X_j^1)^c \rangle = \langle Y_j^1 \rangle.$$

In view of 4.9 and 5.3, Y_j^1 is the direct product of a finite number of elements of S_1'. Thus we have $H_j^2 \in F_a$ (cf. also 5.1). $\square$

Proof of (B). From 5.6 and 5.2, by applying the obvious induction we obtain that (B) holds. $\square$

We conclude by remarking that if $n \in \{2, 3, \ldots, n_0\}$ and $j \in \{1, 2, \ldots, k(n)\}$, then according to 2.7, the linearly ordered groups

$$G_j^n / X_n^{n-1} \quad \text{and} \quad H_j^n / Y_j^{n-1}$$

are isomorphic. This and 5.2 (together with the definition of H_j^1 for $j \in \{1, 2, \ldots, k(1)\}$) yield that the structure of G^c is very near to the structure of G; roughly speaking, constructing G^c we proceed in the same way as when constructing G with the distriction that for $j \in \{1, 2, \ldots, k(1)\}$ we replace G_j^1 by $(G_j^1)^c$.

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