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LIGHT PATHS WITH AN ODD NUMBER OF VERTICES IN POLYHEDRAL MAPS

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Abstract. Let P_k be a path on k vertices. In an earlier paper we have proved that each polyhedral map G on any compact 2-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \leqslant 0$ contains a path P_k such that each vertex of this path has, in G, degree $\leqslant k \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor$. Moreover, this bound is attained for $k = 1$ or $k \geqslant 2$, k even. In this paper we prove that for each odd $k \geqslant \frac{4}{3} \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor + 1$, this bound is the best possible on infinitely many compact 2-manifolds, but on infinitely many other compact 2-manifolds the upper bound can be lowered to $\left\lfloor (k-\frac{1}{3}) \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor$.

Keywords: graphs, path, polyhedral map, embeddings

MSC 2000: 05C10, 05C38, 52B10

1. Introduction

In this paper all manifolds are compact 2-dimensional manifolds. If a graph G is embedded in a manifold $\mathbb{M}$ then the closures of the connected components of $\mathbb{M} - G$ are called the *faces* of G. If each face is a closed 2-cell and each vertex has valence at least three then G is called a *map* in $\mathbb{M}$. If, in addition, no two faces have a multiply connected union then G is called a *polyhedral map* in $\mathbb{M}$. This condition on the union of two faces is equivalent to saying that any two faces that meet, meet on a single vertex or a single edge. When two faces in a map meet in one of these two ways we say that they *meet properly*.

In the sequel let $\mathbb{S}_g$ ($\mathbb{N}_q$) be an orientable (a non-orientable) surface of genus g (q, respectively). We say that H is a *subgraph* of a polyhedral map G if H is a subgraph of the underlying graph of the map G.

The *degree* of a face α of a polyhedral map is the number of edges incident to α. Vertices and faces of degree i are called *i-vertices* and *i-faces*, respectively. Let $v_i(G)$ and $p_j(G)$ denote the number of i-valent vertices and j-valent faces, respectively. For a polyhedral map G let $V(G)$, $E(G)$ and $F(G)$ be the vertex set, the edge set and the face set of G, respectively. The degree of a vertex A in G is denoted by $\deg_G(A)$ or $\deg(A)$ if G is known from the context. A path and a cycle on k vertices is defined to be the *k-path* and the *k-cycle*, respectively. A k-path passing through vertices $A_1, A_2, \dots, A_k$ is denoted by $[A_1, A_2, \dots, A_k]$ provided $A_iA_{i+1} \in E(G)$ for any $i = 1, 2, \dots, k-1$.

It is an old classical consequence of the famous Euler's formula that each planar graph contains a vertex of degree at most 5. A beautiful theorem of Kotzig [Ko1, Ko2] states that every 3-connected planar graph contains an edge with degree-sum of its endvertices at most 13. This result was further developed in various directions and served as a starting point for discovering many structural properties of embeddings of graphs. For example, Ivančo [Ivan] has proved that every polyhedral map on $\mathbb{S}_g$ contains an edge with degree sum of their end vertices at most $2g + 13$ if $0 \leqslant g \leqslant 3$ and at most $4g + 7$, if $g \geqslant 4$. For other results in this topic see e.g. [FaJe, GrSh, Jen, JeVo1, Zaks].

Fabrici and Jendroľ [FaJe] have proved that every 3-connected planar graph G of maximum degree at least k contains a path P_k on k-vertices such that each vertex of this path has, in G, degree $\leqslant 5k$, the bound $5k$ being the best possible. In the same paper [FaJe] they have asked if an analogous result can be established for closed 2-manifolds other than the sphere. More precisely, they have asked the following

Problem 1. For a given connected graph H let $\mathcal{G}(H, \mathbb{M})$ be the family of all polyhedral maps on a closed 2-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ having a subgraph isomorphic to H. What is the minimum integer $\varphi(H, \mathbb{M})$ such that every polyhedral map $G \in \mathcal{G}(H, \mathbb{M})$ contains a subgraph K isomorphic to H for which

$$\deg_G(A) \leqslant \varphi(H, \mathbb{M}) \text{ for every vertex } A \in V(K)?$$

(If such minimum does not exist we write $\varphi(H, \mathbb{M}) = \infty$. If $\varphi(H, \mathbb{M}) < +\infty$ then the graph H is called *light* in $\mathcal{G}(H, \mathbb{M})$.)

The answer to this question for $\mathbb{S}_0$ is contained in

Theorem 1 ([FaJe]). *Let k be an integer, $k \geqslant 1$. Then*

$$\varphi(P_k, \mathbb{S}_0) = 5k \quad \textit{for any } k \geqslant 1$$

and

$$\varphi(H, \mathbb{S}_0) = \infty \quad \textit{for any } H \neq P_k.$$

A slight modification of the method used in [FaJe] yields

Theorem 2. *For any integer $k \geqslant 1$ we have*

$$\varphi(P_k, \mathbb{N}_1) = 5k.$$

For compact 2-manifolds of higher genera we obtained

Theorem 3 ([JeVo1]). *Let k be an integer, $k \geqslant 1$, and let $\mathbb{M}$ be a closed 2-manifold with Euler characteristic $\chi(\mathbb{M}) \notin \{1, 2\}$. Then*

(i) $\varphi(P_1, \mathbb{M}) = \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor$,

(ii) $2\lfloor \frac{k}{2} \rfloor \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor \leqslant \varphi(P_k, \mathbb{M}) \leqslant k \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor$, *$k \geqslant 2$, and*

(iii) $\varphi(H, \mathbb{M}) = \infty$ *for any $H \neq P_k$.*

In Theorem 3 the upper bound is sharp for even $k \geqslant 2$ and $k = 1$. In this paper we investigate for which odd $k \geqslant 3$ the upper bound is attained.

The precise bounds for the torus $\mathbb{S}_1$ and the Klein bottle $\mathbb{N}_2$ have already been determined.

Theorem 4 ([JeVo2]). *Let k be an integer, $k \geqslant 1$. Then*

$$\varphi(P_k, \mathbb{S}_1) = \varphi(P_k, \mathbb{N}_2) = \begin{cases} 6k & \text{if } k = 1 \text{ or } k \text{ is even} \\ 6k - 2 & \text{if } k \text{ is odd}, k \geqslant 3. \end{cases}$$

Let K_n and K_n^- denote the complete graph on n vertices with no or one edge missing, respectively. For each large odd k we can show:

(i) the upper bound in Theorem 3 is attained at an infinite sequence of orientable 2-manifolds and at an infinite sequence of nonorientable 2-manifolds, these sequences being characterized by the fact that each member of them is a triangular embedding of a K_n^- (Theorems 5 and 6);

(ii) the upper bound in Theorem 3 is not attained at an infinite sequence of orientable 2-manifolds and at an infinite sequence of nonorientable 2-manifolds, these sequences being characterized by the fact that each member of them is a triangular embedding of a K_n (Theorems 7 and 8).

If $n \equiv 2$, or $5 (\mathrm{mod}\, 12)$ then K_n^- has a triangular embedding into an orientable 2-manifold $\mathbb{S}_g$ of minimal genus g, where $n = 12t + \frac{7}{2} \pm \frac{3}{2}$ and $g = 12t^2 \pm 3t$, see [Rin], [Jun].

Theorem 5. *Let k be an odd integer and $\mathbb{S}_g$ an orientable compact 2-manifold of genus $g = 12t^2 \pm 3t$, $t = 1, 2, \ldots$.*
If $k \geqslant \lfloor \frac{1}{2}(5 + \sqrt{1 + 48g}) \rfloor + 1$, then

$$\varphi(P_k, \mathbb{S}_g) = k \left\lfloor \frac{5 + \sqrt{1 + 48g}}{2} \right\rfloor .$$

If $n \equiv 2, 5$, or $11 \pmod{12}$ then K_n^- has a triangular embedding into a nonorientable 2-manifold $\mathbb{N}_q$ of minimal genus q, where $n = 12t + \frac{7}{2} \pm \frac{3}{2}$ and $q = 24t^2 \pm 6t$, $t = 1, 2, \ldots$, or $n = 12t + 11$ and $q = 24t^2 + 30t + 9$, $t = 1, 2, \ldots$, see [Rin].

Theorem 6. *Let k be an odd integer and $\mathbb{N}_q$ a nonorientable compact 2-manifold of genus $q = 24t^2 \pm 6t$, $t = 1, 2, \ldots$, or $q = 24t^2 + 30t + 9$, $t = 1, 2, \ldots$.*
If $k \geqslant \lfloor \frac{1}{2}(5 + \sqrt{1 + 24q}) \rfloor + 1$, then

$$\varphi(P_k, \mathbb{N}_q) = k \left\lfloor \frac{5 + \sqrt{1 + 24q}}{2} \right\rfloor .$$

If $n \equiv 0, 3, 4$, or $7 \pmod{12}$ then K_n has a triangular embedding into an orientable 2-manifold $\mathbb{S}_g$ of minimal genus g, where $n = 12t + \frac{7}{2} \pm \frac{7}{2}$ and $g = 12t^2 \pm 7t + 1$, or $n = 12t + \frac{7}{2} \pm \frac{1}{2}$ and $g = 12t^2 \pm t$, $t = 1, 2, \ldots$, see [Rin].

Theorem 7. *Let k be an odd integer and $\mathbb{S}_g$ an orientable compact 2-manifold of genus $g = 12t^2 \pm 7t + 1$, $t = 1, 2, \ldots$, or $g = 12t^2 \pm t$, $t = 1, 2, \ldots$.*
If $k > \frac{4}{3}\, \frac{5 + \sqrt{1 + 48g}}{2} - \frac{4}{3}$, then

$$\varphi(P_k, \mathbb{S}_g) \leqslant \left\lfloor \left(k - \frac{1}{3}\right) \frac{5 + \sqrt{1 + 48g}}{2} \right\rfloor =: m_k(\mathbb{S}_g).$$

Since K_7 has a triangular embedding into the torus $\mathbb{S}_1$, Theorem 7 is also true for the torus. It gives the bound $6k - 2$ already known by Theorem 4. Since K_7 has no embedding into the Klein bottle $\mathbb{N}_2$ the result of Theorem 4 for $\mathbb{N}_2$ cannot be deduced from Theorem 7.

If $n \equiv 0$, or $1 \pmod 3$, $6 \leqslant n \neq 7$ then K_n has a triangular embedding into a nonorientable 2-manifold $\mathbb{N}_q$ of minimal genus q, where $n = 3t$ and $q = \frac{1}{2}(3t^2 - 7t + 4)$, or $n = 3t + 1$ and $q = \frac{1}{2}(3t^2 - 5t + 2)$, $t = 2, 3, \ldots$, where $3t + 1 \neq 7$ [Rin].

Theorem 8. *Let k be an odd integer and $\mathbb{N}_q$ a nonorientable compact 2-manifold of genus $q = \frac{1}{2}(3t^2 - 7t + 4)$, $t = 2, 3, \ldots$, or $q = \frac{1}{2}(3t^2 - 5t + 2)$, $t = 3, 4, \ldots$.*

If $k > \frac{4}{3}\frac{5+\sqrt{1+24q}}{2} - \frac{4}{3}$, *then*

$$\varphi(P_k, \mathbb{N}_q) \leqslant \left\lfloor \left(k - \frac{1}{3}\right) \frac{5+\sqrt{1+24q}}{2} \right\rfloor =: m_k(\mathbb{N}_q).$$

2. Minimum degrees of graphs on $\mathbb{M}$

In this paper $\chi(\mathbb{M}) \leqslant 0$.

Let G be a graph embedded in a compact 2-dimensional manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$. If G is a map, i.e. each face is a 2-cell, then G fulfils Euler's formula

$$n - e + f = \chi(\mathbb{M}),$$

where

$$\chi(\mathbb{M}) = \begin{cases} 2(1-g) & \text{if } \mathbb{M} = \mathbb{S}_g, \\ 2 - q & \text{if } \mathbb{M} = \mathbb{N}_q. \end{cases}$$

If G contains a face F which is not a 2-cell then add an edge to its interior so that F is not subdivided. Add edges in this way until a 2-cell embedding is obtained. Let e^* denote the number of these edges, then Euler's formula is fulfilled with

$$n - (e + e^*) + f = \chi(\mathbb{M}),$$

where n, e and f denote the number of vertices, edges and faces of G, respectively. We summarize this in

Lemma 1. *Let G be the embedding of a graph in a compact 2-dimensional manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$. Let e^* denote the number of edges which can be added to G without changing the number of its faces. Then the Euler sum is*

$$n - e + f = \chi(\mathbb{M}) + e^*,$$

where n, e and f denote the number of vertices, edges and faces of G, respectively.

Lemma 2. *Let G be the embedding of a* simple *graph with minimum degree $\geqslant 2$ in a compact 2-dimensional manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$. Let e^* denote the maximum number of edges which can be added to G without changing the number of its faces. It is allowed that the new edges destroy the simplicity of G, i.e., the new edges can create loops or multiple edges. Then $p_0 = p_1 = p_2 = 0$, and the number of edges of G is*

$$e \leqslant 3(n + |\chi(\mathbb{M})| - e^*).$$

Equality holds if and only if all faces of the embedding of G are bounded by three edges.

P r o o f. By Lemma 1 we have

$$n - e + f = \chi(\mathbb{M}) + e^*. \tag{1}$$

On the boundary of each face F a vertex, say V, lies. Since $\delta(G) \geqslant 2$ and the graph G is simple, at least two edges incident with V belong to F. For the endvertices of these edges different from V the same is true. Hence F is bounded by at least three edges, $p_0 = p_1 = p_2 = 0$, and

$$3f \leqslant 2e, \tag{2}$$

where the equality holds if all faces of the embedding of G are bounded by three edges. The formulas (1) and (2) imply

$$3(\chi(\mathbb{M}) + e^*) = 3n - 3e + 3f \leqslant 3n - 3e + 2e$$

and

$$e \leqslant 3(n + |\chi(\mathbb{M})| - e^*).$$

□

Lemma 3. *Let G be the embedding of* a simple *graph with minimum degree $\geqslant 2$ in an orientable compact 2-dimensional manifold $\mathbb{S}_g$ of genus $g = 12t^2 \pm 7t + 1$, $t = 1, 2, \ldots$, or $g = 12t^2 \pm t$, $t = 1, 2 \ldots$. Then the minimum degree of G is $\delta(G) < \frac{5}{2} + \frac{1}{2}\sqrt{1 + 48g}$ or G is a triangulation of $\mathbb{S}_g$ which is a triangular embedding of K_n into $\mathbb{S}_g$ with $n = \frac{7}{2} + \frac{1}{2}\sqrt{1 + 48g}$.*

P r o o f. Let e^* denote the maximum number of edges which can be added to G without changing the number of its faces. Lemma 2 implies $p_0 = p_1 = p_2 = 0$, and the number e of edges of G is

$$e \leqslant 3(n + |\chi(\mathbb{M})| - e^*),$$

where the equality holds if and only if all faces are bounded by precisely 3 edges. From $2e \geqslant n \cdot \delta$ it follows that

$$n(\delta - 6) \leqslant 6|\chi(\mathbb{M})| - 6e^*$$

where the equality holds if and only if G is δ-regular and all faces are bounded by precisely 3 edges. If $\delta \leqslant 6$, then Lemma 3 is true.

Next, let $\delta > 6$. Then by $n \leqslant \delta + 1$ we have

$$(\delta + 1)(\delta - 6) \leqslant 6|\chi(\mathbb{M})| - 6e^*,$$

where the equality holds if and only if $n = \delta + 1$, G is δ-regular and all faces are bounded by precisely 3 edges. Hence

$$\delta \leqslant \frac{5 + \sqrt{49 - 24\chi(\mathbb{M}) - 24e^*}}{2}.$$

Consequently, $\delta \leqslant \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2}$, and the equality only holds if $e^* = 0$, i.e. all faces are 2-cells, and G is a triangular embedding of K_n in $\mathbb{S}_g$, $n = \frac{7+\sqrt{49-24\chi(\mathbb{M})}}{2}$.

By Ringel [Rin] a triangular embedding of K_n in $\mathbb{S}_g$ exists if $g = 12t^2 \pm 7t + 1$, $t = 1, 2, \ldots$, or $g = 12t^2 \pm t$, $t = 1, 2, \ldots$. From $\chi(\mathbb{M}) = 2 - 2g$ the validity of Lemma 3 follows. $\square$

Similarly the following Lemma 4 can be proved.

Lemma 4. *Let G be the embedding of a* simple *graph with minimum degree $\geqslant 2$ in a nonorientable compact 2-dimensional manifold $\mathbb{N}_q$ of genus $q = 24t^2 \pm 6t$, $t = 1, 2, \ldots$, or $q = 24t^2 + 30t + 9$, $t = 1, 2, \ldots$. Then the minimum degree of G is $\delta(G) < \frac{5}{2} + \frac{1}{2}\sqrt{1 + 24q}$ or G is a triangulation of $\mathbb{N}_q$ which is a triangular embedding of K_n into S_g with $n = 12t + \frac{7}{2} \pm \frac{3}{2}$, $t = 1, 2, \ldots$, or $n = 12t + 11$, $t = 1, 2, \ldots$.*

3. Proof of Theorems 7 and 8—upper bounds

The proof follows the ideas of [FaJe]. First the orientable case is proved. Suppose that there is a counterexample to our Theorem 7 having n vertices. Let G be a counterexample with the maximum number of edges among all counterexamples having n vertices. A vertex A of the graph G is major (minor) if $\deg_A(A) > m_k(\mathbb{S}_g)$ ($\leqslant m_k(\mathbb{S}_g)$, respectively), where $m_k(\mathbb{S}_g) := \lfloor (k - \frac{1}{3})\frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \rfloor = \lfloor (k - \frac{1}{3})\frac{5+\sqrt{1+48g}}{2} \rfloor$.

Lemma 5. *Every r-face α, $r \geqslant 4$, of G is incident only with minor vertices.*

P r o o f. Suppose there is a major vertex B incident with an r-face α, $r \geqslant 4$. Let C be a diagonal vertex on α with respect to B. Because G is a polyhedral map we can insert an edge BC into the r-face α. The resulting embedding is again a counterexample but with one edge more, a contradiction. $\square$

Let $H(G) = H$ be the subgraph of G induced by the set of major vertices of G. By Lemma 3 there is in H either a vertex A such that

$$\deg_H(A) \leqslant \frac{3}{2} + \frac{\sqrt{49 - 24\chi(\mathbb{M})}}{2} = \frac{5 + \sqrt{1 + 48g}}{2} - 1,$$

or H is a triangulation of $\mathbb{S}_g$ on $n = \frac{7+\sqrt{1+48g}}{2}$ vertices, where H is isomorphic to K_n.

Case 1. Assume that H contains a vertex A of degree $\deg_H(A) \leqslant \frac{3+\sqrt{1+48g}}{2}$. On the other hand, A is a major vertex in G, so the degree of A in G is $\deg_G(A) \geqslant (k - \frac{1}{3})\frac{5+\sqrt{1+48g}}{2} - \frac{2}{3}$. Because of Lemma 5 the subgraph of G induced on the set of vertices consisting of A and its neighbours is a wheel of length $\deg_G(A)$. The major vertices of the cycle of the wheel partition the minor vertices of this cycle into $\deg_H(A) \leqslant \frac{3+\sqrt{1+48g}}{2}$ paths, and one of these paths has a length

$$\begin{aligned}
&\geqslant \frac{\deg_G(A) - \deg_H(A)}{\deg_H(A)} \geqslant \frac{(k - \frac{1}{3})\frac{5+\sqrt{1+48g}}{2} - \frac{2}{3} - (\frac{5+\sqrt{1+48g}}{2} - 1)}{\frac{5+\sqrt{1+48g}}{2} - 1} \\
&\geqslant k - \frac{1}{3} + \frac{k - \frac{1}{3} - \frac{2}{3}}{\frac{5+\sqrt{1+48g}}{2} - 1} - 1 > k - \frac{1}{3} + \frac{4}{3} - 1 = k,
\end{aligned}$$

a contradiction! (Note that $k > \frac{4}{3}\frac{5+\sqrt{1+48g}}{2} - \frac{4}{3}$.) This contradiction completes the proof in Case 1.

Case 2. Assume that H is a triangulation of $\mathbb{S}_g$ on n vertices, where H is isomorphic to K_n. In Lemma 10 of [JeVo2] we studied precisely the properties of the components of the subgraph H' of G induced on the minor vertices of G.

Lemma 6 ([JeVo2]). *In each triangle D of H there exists a vertex A which is adjacent with only $\leqslant k - 2$ minor vertices of D.*

H has altogether n vertices and $\frac{n(n-1)}{3}$ faces (note that $3f = 2e = n(n-1)$). Therefore, one vertex B of H is incident with $\geqslant \lceil \frac{1}{n}(\frac{n(n-1)}{3}) \rceil = \lceil \frac{n-1}{3} \rceil$ faces F_i so that B has $\leqslant k - 2$ neighbours in the interior of F_i, $i = 1, 2, \ldots, \frac{n-1}{3}$. The number of neighbours of B in the interior of the other faces is $\leqslant k - 1$, $\deg_H(B) = n - 1$. Consequently, $\deg_G(B) \leqslant (n-1) + \lceil \frac{n-1}{3} \rceil (k-2) + (n - 1 - \lceil \frac{n-1}{3} \rceil)(k-1)$, and the major vertex B has a degree

$$\deg_G(B) \leqslant \Bigl(k - \frac{1}{3}\Bigr)(n-1).$$

This contradiction proves the theorem in Case 2.

The proof of Theorem 8 can be accomplished in a similar way. □

4. Proof of Theorems 5 and 6—lower bounds

The validity of the upper bounds follows from Theorem 3.

Here the lower bounds are shown by appropriate constructions.

Ringel [Rin] and Jungerman [Jun] presented a triangular embedding T_n of K_n^- in an orientable compact 2-manifold of genus g for $n \equiv 5 (\mathrm{mod}\, 12)$ or $n \equiv 2 (\mathrm{mod}\, 12)$, respectively. With the help of T_n they constructed a triangular embedding of K_n into $\mathbb{S}_{g+1}$, where $g+1$ is the smallest genus α such that K_n can be embedded into $\mathbb{S}_\alpha$. A consequence of Euler's formula reads

$$\sum_{j \geqslant 6}(6-j)v_j + 2\sum_{j \geqslant 3}(3-j)p_j = 6\chi(\mathbb{M}).$$

Since T_n is a triangulation of $\mathbb{S}_g$ and except two vertices of valency $n-2$, all vertices have valency $n-1$, this formula implies

$$(6-(n-1))(n-2) + (6-(n-2))2 = 6\chi(\mathbb{M}),$$

and

$$n = \frac{1}{2}\left(7 + \sqrt{57 - 24\chi(\mathbb{M})}\right) = \frac{1}{2}\left(7 + \sqrt{9 + 48g}\right). \tag{3}$$

For $n = 12t + \frac{7}{2} \pm \frac{3}{2}$ the genus of $\mathbb{S}_g$ is $g = 12t^2 \pm 3t$. Since T_n is a triangulation there is no embedding of K_n into $\mathbb{S}_g$. If a vertex of T_n of degree $n-2$ is deleted then an embedding of K_{n-1} into $\mathbb{S}_g$ is obtained.

By Ringel [Rin] we know: If K_s can be embedded into $\mathbb{S}_g$ but K_{s+1} has no embedding into $\mathbb{S}_g$ then $s = \lfloor \frac{1}{2}(7 + \sqrt{1+48g}) \rfloor$. Applied to K_{n-1}, this gives $n - 1 = \lfloor \frac{1}{2}(5 + \sqrt{1+48g}) \rfloor + 1$, and $n = \lfloor \frac{5+\sqrt{1+48g}}{2} \rfloor + 2 =: m(\mathbb{S}_g) + 2$. Hence T_n contains two nonadjacent vertices of degree $m(\mathbb{S}_g)$ and $n-2$ vertices of degree $m(\mathbb{S}_g)+1$.

Our construction ends in the following way: Into every triangle $[A_1A_2A_3]$ of T_n we insert a generalized 3-star consisting of a central vertex Z and three paths starting in Z, one of length $\frac{k+1}{2}$ and the other of length $\frac{k-1}{2}$. (By the length of a path we mean the number of vertices on it.) Let the paths p_1, p_2, p_3 of this star be ordered in the same way as the vertices of the face $[A_1A_2A_3]$ are ordered. The construction continues by joining the vertex A_i to all vertices of the paths p_i and p_{i+1}, $i = 1, 2, 3$, indices being taken modulo 3. If $[A_1A_2A_3]$ contains a vertex of degree $m(\mathbb{S}_g)$ then let A_1 be this vertex and p_1 the path of length $\frac{k+1}{2}$. Let $D_1, D_2, \dots, D_s$ denote the triangles of T_n incident with a vertex A. If A is a vertex of degree $\deg_{T_n}(A) = s = m(\mathbb{S}_g)$ then A is adjacent to $k-1$ vertices of the 3-star of D_i, $i = 1, 2, \dots, s = m(\mathbb{S}_g)$. Hence for $k - 1 \geqslant m(\mathbb{S}_g)$ we have

$$\deg_G(A) = (k-1)m(\mathbb{S}_g) + \deg_{T_n}(A) = (k-1)m(\mathbb{S}_g) + m(\mathbb{S}_g) = km(\mathbb{S}_g).$$

If A is a vertex of degree $\deg_{T_n}(A) = s = m(\mathbb{S}_g) + 1$ then A is adjacent to $\geqslant k-2$ vertices of the 3-star of D_i, $i = 1, 2, \ldots, s = m(\mathbb{S}_g) + 1$. Hence for $k - 1 \geqslant m(\mathbb{S}_g)$ we have

$$\begin{aligned}\deg_G(A) &\geqslant (k-2)(m(\mathbb{S}_g)+1) + \deg_{T_n}(A) = (k-2)(m(\mathbb{S}_g)+1) + (m(\mathbb{S}_g)+1)\\ &= (k-1)(m(\mathbb{S}_g)+1) = km(\mathbb{S}_g) - m(\mathbb{S}_g) + k - 1 \geqslant km(\mathbb{S}_g).\end{aligned}$$

This completes the proof in the orientable case.

The proof of the nonorientable case runs in a similar way. Ringel [Rin] presented a triangular embedding T_n of K_n^- in a nonorientable compact 2-manifold $\mathbb{N}_q$ of genus q if $n \equiv 2$, 5 or $11 \pmod{12}$. By formula (3) this implies

$$n = \frac{1}{2}(7 + \sqrt{57 - 24\chi(\mathbb{M})}) = \frac{1}{2}(7 + \sqrt{9 + 24q}),$$

and $n = 12t + \frac{7}{2} \pm \frac{3}{2}$ and $q = 24t^2 \pm 6t$, or $n = 12t + 11$ and $q = 24t^2 + 30t + 9$. The rest of the construction can be accomplished as in the orientable case. $\square$

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