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TRANSFORMATIONS $z(t) = L(t)y(\varphi(t))$ OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The paper describes the general form of an ordinary differential equation of an order $n+1$ ($n \geqslant 1$) which allows a nontrivial global transformation consisting of the change of the independent variable and of a nonvanishing factor. A result given by J. Aczél is generalized. A functional equation of the form

$$f\biggl(s, w_{00}v_0, \dots, \sum_{j=0}^{n} w_{nj}v_j\biggr) = \sum_{j=0}^{n} w_{n+1j}v_j + w_{n+1n+1}f(x, v, v_1, \dots, v_n),$$

where $w_{n+10} = h(s, x, x_1, u, u_1, \dots, u_n)$, $w_{n+11} = g(s, x, x_1, \dots, x_n, u, u_1, \dots, u_n)$ and $w_{ij} = a_{ij}(x_1, \dots, x_{i-j+1}, u, u_1, \dots, u_{i-j})$ for the given functions a_{ij} is solved on $\mathbb{R}$, $u \neq 0$.

Keywords: ordinary differential equations, linear differential equations, transformations, functional equations

MSC 2000: 34A30, 34A34, 39B40

1. Introduction

The theory of global pointwise transformations $z(t) = L(t)y(\varphi(t))$ of homogeneous linear differential equations was developed in the monograph [5] by F. Neuman (see historical remarks, definitions, results and applications). Transformations $z(t) = y(\varphi(t))$ were studied in [6] as a "motion" for n-th order linear differential equations. A general form

$$y''(x) = b(y(x))y'(x)^2 + p(x)y'(x)$$

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where φ satisfies a differential equation $\varphi''(x) = p(x)\varphi'(x) - p(\varphi(x))\varphi'(x)^2$ and b, p are arbitrary functions, was derived by J. Aczél [2] for the second order differential equations (eliminating regularity conditions from [4]). This general form allows a transformation $z(t) = y(\varphi(t))$ and transforms the equation into itself on the whole interval of definition. Aczél's result is generalized in [7] to ordinary differential equations of the order $n+1$ ($n \geqslant 1$).

The transformation $z(t) = L(t)y(\varphi(t))$ is the most general form of a pointwise transformation of homogeneous linear differential equations of an order greater than 2. Consider a differential equation

$$y''(x) = k\frac{y'(x)^2}{y(x)} + p(x)y'(x) + q(x)y(x), \quad x \in I \subseteq \mathbb{R}$$

and the conditions $\varphi(I) = I$,

$$\begin{aligned}\varphi''(t) &= 2(k-1)\frac{L'(t)}{L(t)}\varphi'(t) + p(t)\varphi'(t) - p(\varphi(t))\varphi'(t)^2,\\ \left(\frac{L'(t)}{L(t)}\right)' &= (k-1)\left(\frac{L'(t)}{L(t)}\right) + p(t)\frac{L'(t)}{L(t)} + q(t) - q(\varphi(t))\varphi'(t)^2\end{aligned}$$

on I. This second order differential equation is of a general form which allows the transformation $z(t) = L(t)y(\varphi(t))$ that transforms the equation into itself on I. The equation is a linear differential equation if solutions can vanish at some points in I (then $k\frac{y'(x)^2}{y(x)}$ exists only if $k = 0$). This result is not a special case of Aczél's result (see [8]).

In this paper we derive, similarly to [2, 4, 8], a general form of ordinary differential equations of the order $n+1$ ($n \geqslant 1$) which allows transformations $z(t) = L(t)y(\varphi(t))$ that transform the equation into itself on the whole interval of definition. Further on we assume that the solutions vanish at some points in I. We prove that the most general differential equation of the order $n+1$ ($n \geqslant 1$) of the above property, defined for $y \in \mathbb{R}$, is the linear differential equation.

2. Notation, preliminary results

Denote by (f) and (f^*) respectively the ordinary differential equations

$$\begin{aligned}y^{(n+1)}(x) &= f(x, y(x), \ldots, y^{(n)}(x)), \quad x \in I \subseteq \mathbb{R},\\ z^{(n+1)}(t) &= f^*(t, z(t), \ldots, z^{(n)}(t)), \quad t \in J \subseteq \mathbb{R},\end{aligned}$$

of the order $n+1$, $n \geqslant 1$.

Definition (see [5], pp. 25–26). We say that (f) is globally transformable into (f^*) with respect to the transformation $z(t) = L(t)y(\varphi(t))$ if there exist two functions L, φ such that

– the function L is of the class $C^{n+1}(J)$ and is nonvanishing on J,

– the function φ is a C^{n+1} diffeomorphism of the interval J onto the interval I

and the function

$$z(t) = L(t)y(\varphi(t)), \quad t \in J, \tag{1}$$

is a solution of the equation (f^*) whenever y is a solution of the equation (f).

If (f) is globally transformable into (f^*), then we say that (f), (f^*) are *equivalent equations*. We say that (1) is *a stationary transformation* if it globally transforms an equation (f) into itself on I, i.e. if L, φ satisfy the assumptions of Definition and the function $z(t) = L(t)y(\varphi(t))$ is a solution of $z^{(n+1)}(t) = f(t, z(t), \dots, z^{(n)}(t))$, $t \in I = \varphi(I)$, whenever $y(x)$ is a solution of $y^{(n+1)}(x) = f(x, y(x), \dots, y^{(n)}(x))$, $x \in I$.

We denote $y^{(i)}(\varphi(t)) = \mathrm{d}^i y(\varphi(t))/\mathrm{d}\varphi(t)^i$, $(y(\varphi(t)))^{(i)} = \mathrm{d}^i y(\varphi(t))/\mathrm{d}t^i$, $i \geqslant 0$.

Proposition 1 (Lemma 1, [9]). *Let $n \in \mathbb{N}$ and let the relation*

$$z(t) = L(t)y(\varphi(t))$$

be satisfied where the real functions $y\colon I \to \mathbb{R}$, $z\colon J \to \mathbb{R}$ belong to the classes $C^{n+1}(I), C^{n+1}(J)$ respectively, and $L\colon J \to \mathbb{R}$, $L \in C^r(J), L(t) \neq 0$ on J, and φ is a C^r diffeomorphism of J onto I for some integer $r \geqslant n+1$. Then

$$z^{(i)}(t) = \sum_{j=0}^{i} a_{ij}(t)y^{(j)}(\varphi(t)) = a_{i0}(t)y(\varphi(t)) + a_{i1}(t)y'(\varphi(t)) + \ldots + a_{ii}(t)y^{(i)}(\varphi(t)),$$
$$i \in \{0, 1, \dots, n+1\},$$

where

$$\begin{aligned} &a_{00}(t) = L(t), \dots, a_{i0}(t) = a'_{i-10}(t), \qquad i \geqslant 1;\\ &a_{ij}(t) = a'_{i-1j}(t) + a_{i-1j-1}(t)\varphi'(t), \qquad i > j > 1;\\ &a_{ii}(t) = a_{i-1i-1}(t)\varphi'(t); \qquad i \in \{0, 1, \dots, n+1\} \end{aligned}$$

are real functions, $a_{ij}(t) \in C^{r-(i-j)-1}(J)$ *for* $j > 0$*, and* $a_{i0}(t) \in C^{r-i}(J)$*. Moreover,*

$$
\begin{aligned}
a_{i0}(t) &= L^{(i)}(t), \quad i \geqslant 0;\\
a_{i1}(t) &= (L(t)\varphi(t))^{(i)} - L^{(i)}(t)\varphi(t) = \sum_{j=0}^{i-1} \binom{i}{j} L^{(j)}(t)\varphi^{(i-j)}(t), \quad i \geqslant 1;\\
&\dots\\
a_{ij}(t) &= \binom{i}{j} L^{(i-j)}(t)\varphi'(t)^j + \binom{i}{j-1} L(t)\varphi'(t)^{j-1}\varphi^{(i-j+1)}(t)\\
&\quad + r_{ij}(L, \dots, L^{(i-j-1)}, \varphi', \dots, \varphi^{(i-j)})(t), \quad i > j > 1;\\
&\dots\\
a_{ii-2}(t) &= \binom{i}{2} L''(t)\varphi'(t)^{i-2} + \binom{i}{3}(L(t)\varphi'''(t) + 3L'(t)\varphi''(t))\varphi'(t)^{i-3}\\
&\quad + 3\binom{i}{4} L(t)\varphi'(t)^{i-4}\varphi''(t)^2, \quad i \geqslant 2;\\
a_{ii-1}(t) &= \binom{i}{1} L'(t)\varphi'(t)^{i-1} + \binom{i}{2} L(t)\varphi'(t)^{i-2}\varphi''(t), \quad i \geqslant 2;\\
a_{ii}(t) &= L(t)\varphi'(t)^i, \quad i \geqslant 0
\end{aligned}
$$

and

$$
\begin{aligned}
&a_{i0}(t) = a_{i0}(L^{(i)})(t), && i \geqslant 0;\\
&a_{ij}(t) = a_{ij}(L, \dots, L^{(i-j)}, \varphi', \dots, \varphi^{(i-j+1)})(t), && i \geqslant j > 0;\ i \in \{0, 1, \dots, n+1\}.
\end{aligned}
$$

Let $\mathbf{V}_{n+1}$ denote an $(n+1)$-dimensional vector space, $\vec{c} = [c_0, \dots, c_n]^T = [c_i]_{i=0}^n \in \mathbf{V}_{n+1}$ being a vector of the space written in the column form; ^{T} means the transposition. Denote by $\vec{o} = [0, \dots, 0]^T$ the origin of $\mathbf{V}_{n+1}$ and by $\vec{e}_0, \dots, \vec{e}_n$ an orthonormal basis in $\mathbf{V}_{n+1}$. Let $\mathbf{V}_{n+1}$ be equipped with the scalar product $(\vec{p}, \vec{q}) = \sum\limits_{i=0}^{n} p_i q_i$ for any pair $\vec{p}, \vec{q}$ of its vectors. Let $\vec{p}_1, \dots, \vec{p}_m$ be m vectors from $\mathbf{V}_{n+1}$. Notation $P = [\vec{p}_1, \dots, \vec{p}_m] = [p_{ij}]_{j=1,\dots,m}^{i=0,\dots,n}$ denotes a matrix and $(P, Q) = \sum\limits_{j}^{i} p_{ij} q_{ij}$ the scalar product of two matrices of the same type. Similarly $P_{(j,\dots,k)} = [\vec{p}_j, \dots, \vec{p}_k]$ means a submatrix, $PQ = P_{(0,\dots,n)}Q_{(0,\dots,n)}$ is the matrix multiplication. For $y \in C^{n+1}(I)$ we denote $y_i(x) = y^{(i)}(x)$, $x \in I$, $i \in \{0, \dots, n+1\}$. Then

$$
y(x) = [y_0(x), \dots, y_n(x)]^T = [y(x), y'(x), \dots, y^{(n)}(x)]^T \in \mathbf{V}_{n+1}
$$

for each $x \in I$.

Remark 1. Let the assumptions of Proposition 1 be satisfied. Then

$$\vec{z}(t) = A(t)\vec{y}(\varphi(t))$$

is true on J for $A(t) = [a_{ij}(t)]_{j=1,\dots,m}^{i=0,\dots,n}$, where $a_{ij}(t) = 0$ for $j > i$. Moreover, $z_{n+1}(t) = (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}y_{n+1}(\varphi(t))$, where $\vec{a}_{n+1}(t) = [a_{n+10}(t), \dots, a_{n+1n}(t)]^T$, $t \in J$.

Observation 1 (see Corollary 1, [9])**.** *Every homogeneous linear differential equation of an order $n+1$ ($n \geqslant 1$) is a particular case of the equation (f), and for two equivalent linear equations*

$$\begin{aligned}
y_{n+1}(x) &= (\vec{p}(x), \vec{y}(x)) = p_0(x)y_0(x) + p_1(x)y_1(x) + \ldots + p_n(x)y_n(x),\\
y_i(x) &= y^{(i)}(x), \quad x \in I;\\
z_{n+1}(t) &= (\vec{q}(t), \vec{z}(t)) = q_0(t)z_0(t) + q_1(t)z_1(t) + \ldots + q_n(t)z_n(t),\\
z_i(t) &= z^{(i)}(t), \quad t \in J;
\end{aligned}$$

there always exist relations

$$\begin{aligned}
L^{(n+1)}(t) &= h(t, \varphi(t), \varphi'(t), L(t), \dots, L^{(n)}(t))\\
&= q_0(t)L(t) + \ldots + q_n(t)L^{(n)}(t) - L(t)\varphi'(t)^{n+1}p_0(\varphi(t));\\
\varphi^{(n+1)}(t) &= g(t, \varphi(t), \dots, \varphi^{(n)}(t), L(t), \dots, L^{(n)}(t))\\
&= \frac{1}{L(t)}\sum_{k=1}^{n}(a_{k1}(t)q_k(t) - \binom{k}{n+1}L^{(k)}(t)\varphi^{(n+1-k)}(t))\\
&\quad - p_1(\varphi(t))\varphi'(t)^{n+1}; \quad t \in J
\end{aligned}$$

between the functions L, φ and the coefficients of linear differential equations.

Here $a_{k1}(t) = a_{k1}(\varphi', \dots, \varphi^{(k)}, L, L', \dots, L^{(k-1)}(t))$ are defined by Proposition 1.

Assumption. For transformations $z(t) = L(t)y(\varphi(t))$ of ordinary differential equations of an order $n+1$ ($n \geqslant 1$) we assume that there exist differential equations

$$\begin{aligned}
L^{(n+1)}(t) &= h(t, \varphi(t), \varphi'(t), L(t), \dots, L^{(n)}(t)),\\
\varphi^{(n+1)}(t) &= g(t, \varphi(t), \dots, \varphi^{(n)}(t), L(t), \dots, L^{(n)}(t)), \quad t \in J.
\end{aligned}$$

3. Results

Lemma 1. *Let $n, r \in \mathbb{N}$ and $r \geqslant n+1$. Let φ satisfy the assumptions of Proposition 1. Then (1) is a stationary transformation of the equation (f) if and only if $\varphi(I) = I$ and the real function f satisfies the functional equation*

$$f(s, W\vec{v}) = (\vec{w}_{n+1}, \vec{v}) + w_{n+1n+1} f(x, \vec{v}), \tag{2}$$

where $W = [w_{ij}]_{j=0,\dots,n}^{i=0,\dots,n}$, $\vec{w}_{n+1} = [w_{n+10}, w_{n+11}, \dots, w_{n+1n}]^T$, $\vec{v} = [v_0, v_1, \dots, v_n]^T$ and $w_{i0} = a_{i0}(u_i)$, $w_{ij} = a_{ij}(x_1, x_2, \dots, x_{i-j+1}, u, u_1, \dots, u_{i-j})$ for $j > 0$ are defined by

$$\begin{aligned}
w_{i0} &= u_i, \qquad 1 \leqslant i \leqslant n;\\
w_{n+10} &= h(s, x, x_1, u, u_1, \dots, u_n);\\
w_{i1} &= \binom{i}{0} u x_i + \binom{i}{1} u_1 x_{i-1} + \dots + \binom{i}{i-1} u_{i-1} x_1, \qquad 1 \leqslant i \leqslant n;\\
w_{n+11} &= (n+1) u g(s, x, x_1, \dots, x_n, u, u_1, \dots, u_n) + \sum_{j=1}^{n} \binom{n}{j} u_j x_{n-j};\\
&\dots\\
w_{ij} &= \binom{i}{j} u_{i-j} x_1^j + \binom{j-1}{i} u x_1^{j-1} x_{i-j+1}\\
&\quad + r_{ij}(x_1, \dots, x_{i-j}, u_1, \dots, u_{i-j-1}), \qquad 1 < j < i;\\
&\dots\\
w_{ii-2} &= \binom{i}{2} u_2 x_1^{i-2} + \binom{i}{3} (u x_3 + 3 u_1 x_2) x_1^{i-3} + 3 \binom{i}{4} u x_1^{i-4} x_2^2, \qquad i \geqslant 2;\\
w_{ii-1} &= \binom{i}{1} u_1 x_1^{i-1} + \binom{i}{2} u x_1^{i-2} x_2, \qquad i \geqslant 2;\\
w_{ii} &= u x_1^i, \qquad i \geqslant 0;
\end{aligned} \tag{3}$$

where $s, x = x_0, x_i, v = v_0, v_i, u = u_0, \dots, u_i \in \mathbb{R}$, $u \neq 0$; a_{ij}, r_{ij} are real functions, $n \in \mathbb{N}$.

P r o o f. The transformation (1) is a global transformation of the equation (f) if and only if $\varphi(I) = I$ and at the same time the functions $y(x) = y(\varphi(t))$, $z(t) = L(t) y(\varphi(t))$ satisfy

$$\begin{aligned}
y^{(n+1)}(x) &= y^{(n+1)}(\varphi(t)) = f(\varphi(t), y(\varphi(t)), \dots, y^{(n)}(\varphi(t)))\\
&= f(\varphi(t), \vec{y}(\varphi(t))),\\
y^{(n+1)}(t) &= f(t, y(t), \dots, y^{(n)}(t)) = f(t, \vec{z}(t)), \quad t \in I = \varphi(I).
\end{aligned} \tag{4}$$

From (4), Proposition 1 and Remark 1 we get

$$\begin{aligned} z_{n+1}(t) &= (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t) y_{n+1}(\varphi(t)) \\ &= (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t) f(\varphi(t), \vec{y}(\varphi(t))) \\ &= f(t, \vec{z}(t)) = f(t, A(t)\vec{y}(\varphi(t))), \end{aligned}$$

i.e.

$$f(t, A(t)\vec{y}(\varphi(t))) = (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t) f(\varphi(t), \vec{y}(\varphi(t))),$$

where $z_{n+1}(t) = z^{(n+1)}(t)$ and the functions $a_{ij}(t)$ are defined by Proposition 1, $t \in J$. We denote $s = t, x_0 = x = \varphi(t), x_i = \varphi^{(i)}(t), u = u_0 = L(t),\ u_i = L^{(i)}(t), v_0 = v = y(\varphi(t)), v_i = y^{(i)}(\varphi(t)),\ w_{i0} = u_i, w_{ij} = a_{ij}(x_1, x_2, \ldots, x_{i-j+1}, u, u_1, \ldots, u_{i-j})$ for $i \geqslant j \geqslant 1$. Using the definitions of a_{ij} we obtain the assertion of Lemma 1. Here

$$\begin{aligned} &L^{(n+1)}(t) = h(t, \varphi(t), \varphi'(t), L(t), \ldots, L^{(n)}(t)), \\ &\varphi^{(n+1)}(t) = g(t, \varphi(t), \ldots, \varphi^{(n)}(t), L(t), \ldots, L^{(n)}(t)), \quad t \in J, \end{aligned}$$

i.e. $u_{n+1} = h(s, x, x_1, u, u_1, \ldots, u_n)$ and $x_{n+1} = g(s, x, x_1, \ldots, x_n, u, u_1, \ldots, u_n)$ in accordance with Assumption. □

Theorem 1. *The continuous general solution of the functional equation* (2) *is given by*

$$f(x, \vec{v}) = \sum_{j=0}^{n} p_j(x) v_j = (\vec{p}(x), \vec{v}),$$

$$w_{n+1j} = \sum_{k=j}^{n} p_k(s) w_{kj} - w_{n+1n+1} p_j(x), \quad j \in \{0, \ldots, n\}$$

where $p_0, p_1, \ldots, p_n$ *are arbitrary functions and* $w_{i0} = u_i,\ w_{ij} = a_{ij}(x_1, x_2, \ldots, x_{i-j+1}, u, u_1, \ldots, u_{i-j})$ *for* $j > 0$ *are defined by* (3), $i \geqslant j \geqslant 0$, $i \in \{0, 1, \ldots, n+1\}$, $n \in \mathbb{N}$. *Moreover,*

$$\begin{aligned} u_{n+1} &= h(s, x, x_1, u, u_1, \ldots, u_n) = \sum_{j=0}^{n} p_j(s) u_j - u x_1^{n+1} p_0(x), \\ x_{n+1} &= g(s, x, x_1, \ldots, x_n, u, u_1, \ldots, u_n) \\ &= \frac{1}{(n+1)u} \left(\sum_{j=1}^{n} \left(p_j(s) w_{j1} - \binom{n}{j} u_j x_{n-j} \right) - u x_1^{n+1} p_1(x) \right). \end{aligned}$$

P r o o f. Consider the functional equation (2),

$$f(s, W\vec{v}) = \sum_{j=0}^{n} w_{n+1j} v_j + w_{n+1n+1} f(x, \vec{v}) \tag{5}$$

and define functions $p_i(x) = f(x, \vec{e}_i)$, $i \in \{0, 1, \dots, n\}$. Substituting $\vec{v} = \vec{e}_i$ into (2) we obtain

$$w_{n+1i} = f(s, W\vec{e}_i) - w_{n+1n+1} p_i(x), \quad i \in \{0, 1, \dots, n\}. \tag{6}$$

The functional equation (5) becomes

$$f(s, W\vec{v}) = w_{n+1n+1}(f(x, \vec{v}) - (\vec{p}(x), \vec{v})) + \sum_{i=0}^{n} f(s, W\vec{e}_i) v_i. \tag{7}$$

We can put $f(x, \vec{v}) - (\vec{p}(x), \vec{v}) = \delta(\vec{v})$ because $w_{i0} = u_i, w_{ij} = a_{ij}(x_1, \dots, x_{i-j+1}, u, u_1, \dots, u_{i-j})$ are independent of x for $j > 0$. Then $\delta(\vec{e}_0) = f(x, \vec{e}_0) - p_0(x) = p_0(x) - p_0(x) = 0$ and similarly $\delta(\vec{e}_i) = 0$, $i \in \{0, 1, \dots, n\}$. Hence

$$f(x, \vec{v}) = (\vec{p}(x), \vec{v}) + \delta(\vec{v}); \quad \delta(\vec{e}_i) = 0, \quad i \in \{0, 1, \dots, n\} \tag{8}$$

for $x, v, v_1, \dots, v_n \in \mathbb{R}$.

Substituting (8) into (7) we obtain

$$\delta(W\vec{v}) = \sum_{i=0}^{n} \delta(W\vec{e}_i) v_i + w_{n+1n+1} \delta(\vec{v}). \tag{9}$$

Using $v_1 = \dots = v_n = 0$ and (3) we get

$$\delta(uv, u_1 v, \dots, u_n v) = \delta(u, u_1, \dots, u_n) v + u x_1^{n+1} \delta(v, 0, \dots, 0) \tag{10}$$

and for $x_1 = 1$ we have

$$\delta(uv, u_1 v, \dots, u_n v) = \delta(u, u_1, \dots, u_n) v + u \delta(v, 0, \dots, 0). \tag{11}$$

Comparison of (10), (11) gives $u(x_1^{n+1} - 1)\delta(v, 0, \dots, 0) = 0$ for $u, x_1, v \in \mathbb{R}$, $u \neq 0$. Hence $\delta(v, 0, \dots, 0) = 0$ for all $v \in \mathbb{R}$ and

$$\delta(\vec{u} v) = \delta(\vec{u}) v, \quad u, u_1, \dots, u_n, v \in \mathbb{R}. \tag{12}$$

Similarly, (9) together with $v_2 = \dots = v_n = 0$ gives

$$\delta(W(\vec{e}_0 v + \vec{e}_1 v_1)) = \delta(W\vec{e}_0) v + \delta(W\vec{e}_1) v_1 + w_{n+1n+1} \delta(\vec{e}_0 v + \vec{e}_1 v_1),$$

i.e.

$$\begin{aligned}(13)\qquad &\delta(w_{00}v, w_{10}v + w_{11}v_1, \dots, w_{n0}v + w_{n1}v_1)\\ &= \delta(w_{00}, w_{10}, \dots, w_{n0})v + \delta(0, w_{11}, \dots, w_{n1})v_1 + w_{n+1n+1}\delta(v, v_1, 0, \dots, 0).\end{aligned}$$

For $u_1 = \ldots = u_n = 0$ we have $w_{00} = u$, $w_{i0} = u_i = 0$ $(0 < i \leqslant n)$, $w_{i1} = ux_i$ $(1 \leqslant i \leqslant n)$, and (13) becomes

$$\begin{aligned}\delta(uv, ux_1v_1, \dots, ux_nv_1) = \delta(u, 0, \dots, 0)v &+ \delta(0, ux_1, \dots, ux_n)v_1\\ &+ ux_1^{n+1}\delta(v, v_1, 0, \dots, 0).\end{aligned}$$

Thus, using (12) and $\delta(\vec{e}_0) = 0$,

$$(14)\qquad \delta(v, x_1v_1, \dots, x_nv_1) = \delta(0, x_1, \dots, x_n)v_1 + x_1^{n+1}\delta(v, v_1, 0, \dots, 0).$$

For $v_1 = 1$ we obtain

$$(15)\qquad \delta(v, x_1, \dots, x_n) = \beta(x_1, \dots, x_n) + b(v)x_1^{n+1},$$

where $\beta(x_1, \dots, x_n) = \delta(0, x_1, \dots, x_n)$ and $b(v) = \delta(v, 1, 0, \dots, 0)$. Here

$$(16)\qquad \beta(cx_1, \dots, cx_n) = \beta(x_1, \dots, x_n)c,$$

$$(17)\qquad b(cv)c^{n+1} = b(v)c, \quad v \in \mathbb{R},$$

according to (12); $x_1, \dots, x_n, v \in \mathbb{R}$.

Choosing $v = 1$ in (17) we obtain $b(c) = \frac{k}{c^n}$ and the function b is continuous on $\mathbb{R}$ if and only if $b(c) = 0$ on $\mathbb{R}$. Hence,

$$(18)\qquad \delta(v, x_1, \dots, x_n) = \beta(x_1, \dots, x_n), \quad \beta(cx_1, \dots, cx_n) = \beta(x_1, \dots, x_n)c$$

on $\mathbb{R}$.

Now $\delta(v, v_1, 0, \dots, 0) = \beta(v_1, 0, \dots, 0) = v_1\beta(1, 0, \dots, 0) = v_1\delta(0, 1, 0, \dots, 0) = v_1\delta(\vec{e}_1) = 0$ and from (13) we get

$$\begin{aligned}\beta(w_{10}v + w_{11}v_1, \dots, w_{n0}v + w_{n1}v_1) &= \beta(w_{10}, \dots, w_{n0})v + \beta(w_{11}, \dots, w_{n1})v_1\\ &= \beta(vw_{10}, \dots, vw_{n0}) + \beta(v_1w_{11}, \dots, v_1w_{n1})\end{aligned}$$

and the function β satisfies Cauchy's functional equation in several variables

$$\beta(u_1 + v_1, \dots, u_n + v_n) = \beta(u_1, \dots, u_n) + \beta(v_1, \dots, v_n)$$

with the general continuous solution (see Aczél [1])

$$\beta(u_1, \dots, u_n) = \sum_{j=1}^{n} c_j u_j, \quad c_j \in \mathbb{R}. \tag{19}$$

In accordance with (8), (18), (19), the function f is of the form

$$f(x, v, v_1, \dots, v_n) = \sum_{j=0}^{n} p_j(x) v_j + \sum_{j=1}^{n} c_j v_j = \sum_{j=0}^{n} \tilde{p}_j v_j,$$

i.e.

$$f(x, v, v_1, \dots, v_n) = \sum_{j=0}^{n} p_j(x) v_j = (\vec{p}(x), \vec{v}) = f(x, \vec{v}), \tag{20}$$

where $p_0, p_1, \dots, p_n$ are arbitrary functions.

If we combine (20) with (6) we conclude

$$\begin{aligned} w_{n+1i} &= f(s, W\vec{e}_i) - w_{n+1n+1} p_i(x) \\ &= \sum_{k=i}^{n} p_k(s) w_{ki} - w_{n+1n+1} p_i(x), \quad i \in \{0, 1, \dots, n\}, \end{aligned}$$

where w_{kj} are defined by (3). Moreover, using (3) we have

$$\begin{aligned} h(s, x, x_1, u, u_1, \dots, u_n) = u_{n+1} = w_{n+10} &= \sum_{k=0}^{n} p_k(s) w_{k0} - w_{n+1n+1} p_0(x) \\ &= \sum_{k=0}^{n} p_k(s) u_k - u x_1^{n+1} p_0(x); \end{aligned}$$

$$\begin{aligned} g(s, x, x_1, \dots, x_n, u, u_1, \dots, u_n) &= \frac{1}{(n+1)u} \left(w_{n+11} - \sum_{j=1}^{n} \binom{n}{j} u_j x_{n-j} \right) \\ &= \frac{1}{(n+1)u} \left(\sum_{k=1}^{n} p_k(s) w_{k1} - w_{n+1n+1} p_1(x) - \sum_{j=1}^{n} \binom{n}{j} u_j x_{n-j} \right) \\ &= \frac{1}{(n+1)u} \left(\sum_{k=1}^{n} (p_k(s) w_{k1} - \binom{n}{k} u_k x_{n-k}) - w_{n+1n+1} p_1(x) \right). \end{aligned}$$

The assertion of the theorem is proved. □

Remark 2. By virtue of Theorem 1 and Proposition 1, if (1) is a stationary transformation of the equation (f) and the solutions of the equation (f) vanish at some points on I, then (f) is a linear differential equation. The criterion of global equivalence of the second order linear differential equations was published by O. Borůvka [3], of the third and higher order equations by F. Neuman [5]. In the monograph [5] there is a complete list of stationary groups for homogeneous linear differential equations of the n-th order. Some criteria for stationary transformations of linear differential and linear functional-differential equations are given in [9].

References

[1] *J. Aczél*: Lectures on Functional Equations and Their Applications. Academic Press, New York, 1966.

[2] *J. Aczél*: Über Zusammenhänge zwischen Differential- und Funktionalgleichungen. Jahresber. Deutsch. Math.-Verein. *71* (1969), 55–57.

[3] *O. Borůvka*: Linear Differential Transformations of the Second Order. The English Univ. Press, London, 1971.

[4] *A. Moór, L. Pintér*: Untersuchungen Über den Zusammenhang von Differential- und Funktionalgleichungen. Publ. Math. Debrecen *13* (1966), 207–223.

[5] *F. Neuman*: Global Properties of Linear Ordinary Differential Equations. Mathematics and Its Applications (East European Series) 52, Kluwer Acad. Publ., Dordrecht-Boston-London, 1991.

[6] *J. Posluszny, L. A. Rubel*: The motion of an ordinary differential equation. J. Differential Equations *34* (1979), 291–302.

[7] *V. Tryhuk*: On transformations $z(t) = y(\varphi(t))$ of ordinary differential equations. Czechoslovak Math. J. *50 (125)* (2000), 509–518.

[8] *V. Tryhuk*: On global transformations of ordinary differential equations of the second order. Czechoslovak Math. J. *50 (125)* (2000), 499–508.

[9] *V. Tryhuk*: Remark to transformations of linear differential and functional-differential equations. Czechoslovak Math. J. *50 (125)* (2000), 265–278.

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