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RANDOM FIXED POINT THEOREMS FOR A CERTAIN CLASS OF MAPPINGS IN BANACH SPACES

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Abstract. Let (Ω, Σ) be a measurable space and C a nonempty bounded closed convex separable subset of p-uniformly convex Banach space E for some $p > 1$. We prove random fixed point theorems for a class of mappings $T\colon \Omega \times C \to C$ satisfying: for each $x, y \in C$, $\omega \in \Omega$ and integer $n \geqslant 1$,

$$\begin{aligned} &\|T^n(\omega, x) - T^n(\omega, y)\| \\ &\quad \leqslant a(\omega) \cdot \|x - y\| + b(\omega)\{\|x - T^n(\omega, x)\| + \|y - T^n(\omega, y)\|\} \\ &\qquad + c(\omega)\{\|x - T^n(\omega, y)\| + \|y - T^n(\omega, x)\|\}, \end{aligned}$$

where $a, b, c\colon \Omega \to [0, \infty)$ are functions satisfying certain conditions and $T^n(\omega, x)$ is the value at x of the n-th iterate of the mapping $T(\omega, \cdot)$. Further we establish for these mappings some random fixed point theorems in a Hilbert space, in L^p spaces, in Hardy spaces H^p and in Sobolev spaces $H^{k,p}$ for $1 < p < \infty$ and $k \geqslant 0$. As a consequence of our main result, we also extend the results of Xu [43] and randomize the corresponding deterministic ones of Casini and Maluta [5], Goebel and Kirk [13], Tan and Xu [37], and Xu [39, 41].

Keywords: p-uniformly convex Banach space, normal structure, asymptotic center, random fixed points, generalized random uniformly Lipschitzian mapping

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1. Introduction

In recent years randomizations of deterministic fixed point theorems of nonlinear mappings have received much attention in nonlinear functional analysis (see

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Bharucha-Reid [2, 3], Boscan [4], Castaing and Valadiez [7], Chang [8], Engl [12], Itoh [14, 15], Lin [20], Nowak [23], Papageorgiou [24, 25], Rybinski [30], Sehgal and Singh [32], Sehgal and Waters [31], Tan and Yuan [35, 36], and Xu [40, 42, 43]). In particular, Xu [43] obtained some random fixed point theorems for nonlinear uniformly Lipschitzian mappings in Banach spaces.

In this paper, we prove certain random fixed point theorems for a class of mappings, which we call generalized uniformly Lipschitzian mappings in the Banach space. Our results extend the result of Xu [43] and also randomize the corresponding deterministic ones of Casini and Maluta [5], Goebel and Kirk [13], Tan and Xu [37], and Xu [39,41].

2. Preliminaries

Let (Ω, Σ) be a measurable space with Σ a sigma algebra of subsets of Ω. Let (E, d) be a metric space. We denote by $CL(E)$ (resp. CB(E), K(E)) the family of all nonempty closed (resp. closed bounded, compact) subsets of E, and by H the Hausdorff metric on $CB(E)$ induced by d, i.e.,

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

for $A, B \in CB(E)$, where $d(x, B) = \inf\{d(x, y)\colon\ y \in B\}$ is the distance from x to $B \subset E$. A multifunction $f\colon\ \Omega \to E$ is called (Σ-) measurable if, for any open subset B of E, the set $f^{-1}(B) = \{\omega \in \Omega\colon\ f(\omega) \cap B \neq \emptyset\} \in \Sigma$. Note that in Himmelberg [16], this is called weakly measurable. Since in the present paper only this type of measurability is used, we omit the term 'weakly' for simplicity. Note also that if $f(\omega) \in K(E)$ for all $\omega \in \Omega$, then f is measurable if and only if $f^{-1}(F) \in \Sigma$ for all closed subsets F of E. A measurable operator $x\colon\ \Omega \to E$ is called a measurable selector for a measurable multifunction $f\colon\ \Omega \to E$ if $x(\omega) \in f(\omega)$. Let M be a nonempty closed subset of E. Then a mapping $f\colon\ \Omega \times M \to E$ is called a random operator if, for each $x \in M$, the mapping $f(\cdot, x)\colon\ \Omega \to E$ is measurable. An operator $x\colon\ \Omega \to E$ is said to be a random fixed point of f if x is measurable and $x(\omega) \in f(\omega, x(\omega))$ for all $\omega \in \Omega$.

Let C be a nonempty subset of a normed linear space E. Then a mapping $f\colon\ C \to C$ is said to be uniformly Lipschitzian if there exists a constant $k > 0$ such that

$$\|f^n x - f^n y\| \leqslant k\|x - y\|$$

for all $x, y \in C$ and integers $n \geqslant 1$. A uniformly Lipschitzian mapping f is said to be nonexpansive if $k = 1$. A mapping $f\colon\ C \to C$ is said to be generalized uniformly

Lipschitzian if there exist constants $a, b, c > 0$ with $3b + 3c < 1$ such that

$$\|f^n x - f^n y\| \leqslant a \cdot \|x - y\| + b\{\|x - f^n x\| + \|y - f^n y\|\} \\ + c\{\|x - f^n y\| + \|y - f^n x\|\}$$

for each $x, y \in C$ and integers $n \geqslant 1$. By taking $b = c = 0$, it will be seen that this class of mappings is more general than uniformly Lipschitzian mappings.

A random mapping $f\colon \Omega \times C \to C$ is said to be continuous (resp. uniformly Lipschitzian, etc.) if, for fixed $\omega \in \Omega$, the mapping $f(\omega, \cdot)\colon C \to C$ has the above particular property.

Here we list for convenience the following two theorems.

Theorem A [38]. *Let (Ω, Σ) be a measurable space, E a Polish space and $F\colon \Omega \to CL(E)$ a measurable mapping. Then F has a measurable selector.*

Theorem B [35]. *Let (Ω, Σ) be a measurable space, E a separable metric space and X a metric space. If $f\colon \Omega \times E \to X$ is measurable in $\omega \in \Omega$ and continuous in $x \in E$ and if $x\colon \Omega \to E$ is measurable, then $f(\cdot, x(\cdot))\colon \Omega \to X$ is measurable.*

We also need the following propositions.

Proposition 1 [3]. *Let C be a closed convex separable subset of a Banach space and (Ω, Σ) a measurable space. Suppose $f\colon \Omega \to C$ is a multifunction that is w-measurable, i.e. for each $x^* \in E^*$, the dual space of E, the numerically-valued multifunction $x^* f\colon \Omega \to (-\infty, \infty)$ is measurable. Then f is measurable.*

Proposition 2 [14]. *Suppose $\{T_n\}$ is a sequence of measurable set-valued operators from Ω to $CB(E)$ and $T\colon \Omega \to CB(E)$ is an operator. If, for each $\omega \in \Omega$, $H(T_n(\omega), T(\omega)) \to 0$, then T is measurable.*

The normal structure coefficient $N(E)$ of E is defined (cf. Bynum [5]) by

$$N(E) = \inf \left\{ \frac{\operatorname{diam} C}{\gamma_C(C)} \right\},$$

where the infimum is taken over all bounded convex subsets C of E consisting of more than one point, $\operatorname{diam} C = \sup\{\|x - y\|\colon x, y \in C\}$ is the diameter of C and $\gamma_C(C) = \inf_{x \in C} \bigl(\sup_{y \in C} \|x - y\|\bigr)$ is the Chebyshev radius of C relative to itself. A space E is said to have uniformly normal structure if $N(E) > 1$. It is known that every uniformly convex Banach space has uniformly normal structure (cf. Daneš [9]) and that $N(H) = \sqrt{2}$ for a Hilbert space H. Recently, Pichugov [26] (cf. Prus [28]) calculated that $N(L^p) = \min\{2^{\frac{1}{p}}, 2^{\frac{p-1}{p}}\}$, $1 < p < \infty$. Some estimates for the normal structure coefficient in other Banach spaces may be found in Prus [29].

Recall that the modulus of convexity of a Banach space E is the function $\delta(\cdot)$ define on [0,2] by

$$\delta(\varepsilon) = \inf\{1 - \tfrac{1}{2}\|x + y\| \colon \ \|x\| \leqslant 1, \ \|y\| \leqslant 1, \ \|x - y\| \geqslant \varepsilon\}.$$

E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for all $0 < \varepsilon \leqslant 2$.

Let $p > 1$ and denote by λ a number in $[0, 1]$ and by $W_p(\lambda)$ the function $\lambda \cdot (1 - \lambda)^p + \lambda^p \cdot (1 - \lambda)$.

The functional $\|\cdot\|^p$ is said to be uniformly convex (cf. Zalinescu [44]) on the Banach space E if there exists a positive constant c_p such that for all $\lambda \in [0, 1]$ and $x, y \in E$, the following inequality holds:

$$\|\lambda x + (1 - \lambda)y\|^p \leqslant \lambda\|x\|^p + (1 - \lambda)\|y\|^p - W_p(\lambda) \cdot c_p \cdot \|x - y\|^p. \tag{1}$$

Xu [41] proved that the functional $\|\cdot\|^p$ is uniformly convex on the whole Banach space E if and only if E is p-uniformly convex, i.e. there exists a constant $c > 0$ such that the moduli of convexity $\delta_E(\varepsilon) \geqslant c \cdot \varepsilon^p$ for all $0 \leqslant \varepsilon \leqslant 2$.

3. Main results

In this section we always assume that (Ω, Σ) is a measurable space, C a nonempty bounded closed convex subset of a Banach space E, and $T\colon \Omega \times C \to C$ is a generalized random uniformly Lipschitzian mapping, i.e., there exist functions $a, b, c\colon \Omega \to [0, \infty) =: \mathbb{R}^+$ with $3b(\omega) + 3c(\omega) < 1$ and

$$\begin{aligned} &\|T^n(\omega, x) - T^n(\omega, y)\| \\ &\quad \leqslant a(\omega) \cdot \|x - y\| + b(\omega)\{\|x - T^n(\omega, x)\| + \|y - T^n(\omega, y)\|\} \\ &\qquad + c(\omega)\{\|x - T^n(\omega, y)\| + \|y - T^n(\omega, x)\|\} \end{aligned} \tag{2}$$

for all $x, y \in C$, $\omega \in \Omega$ and integers $n \geqslant 1$. Here $T^n(\omega, x)$ is the value at x of the n-th iterate of the mapping $T(\omega, \cdot)$.

The following lemma was given in [43]:

Lemma 1 [43]. *Let M be a separable metric space and $f\colon \Omega \times M \to \mathbb{R} =: (-\infty, \infty)$ a Carathéodory mapping, i.e., for every $x \in M$, the mapping $f(\cdot, x)\colon \Omega \to \mathbb{R}$ is measurable and for every $\omega \in \Omega$, the mapping $f(\omega, \cdot)\colon M \to \mathbb{R}$ is continuous. Then for any $s \in \mathbb{R}$, the mapping $\widetilde{F}_s\colon \Omega \to M$ defined by*

$$\widetilde{F}_s(\omega) = \{x \in M \colon \ f(\omega, x) < s\}, \qquad \omega \in \Omega$$

is measurable. If, in addition, M is a closed convex separable subset of a normed linear space, $\widetilde{F}_s(\omega)$ is nonempty for all $\omega \in \Omega$, and f is convex in $x \in M$, then the mapping $F_s\colon \Omega \to M$ defined by

$$F_s(\omega) = \{x \in M\colon\ f(\omega, x) \leqslant s\}, \qquad \omega \in \Omega$$

is measurable.

Now, we are in position to give our main result:

Theorem 1. *Let (Ω, Σ) be a measurable space. Let E be a p-uniformly convex Banach space for some $p > 1$, C a nonempty bounded closed convex separable subset of E, and $T\colon \Omega \times C \to C$ a generalized random uniformly Lipschitzian mapping. If for each $\omega \in \Omega$*

$$\left[\frac{(\alpha(\omega)+\beta(\omega))^p \cdot \{(\alpha(\omega)+\beta(\omega))^p - 1\}}{c_p \cdot N^p}\right]^{\frac{1}{p}} < 1,$$

where

$$\alpha(\omega) = \frac{a(\omega)+b(\omega)+c(\omega)}{1-b(\omega)-c(\omega)}, \qquad \beta(\omega) = \frac{2b(\omega)+2c(\omega)}{1-b(\omega)-c(\omega)},$$

N is the normal structure coefficient of E and c_p is the constant given in inequality (1), then T has a random fixed point.

Proof. Fix a measurable function $x_0\colon \Omega \to C$ and define function $f\colon \Omega \times C \to \mathbb{R}^+$ by

$$f(\omega, x) = \limsup_{n\to\infty} \|T^n(\omega, x_0(\omega)) - x\|, \qquad x \in E.$$

By Theorem B, it is easily seen that f is measurable in $\omega \in \Omega$ and continuous in $x \in E$. Now, by following the argument of Xu in [43], we show that there exists a measurable function $x\colon \Omega \to C$ such that

$$f(\omega, x(\omega)) = \inf_{x\in C} f(\omega, x), \qquad \omega \in \Omega. \tag{3}$$

To this end we set

$$r(\omega) = \inf_{x\in C} f(\omega, x)$$

and

$$F(\omega) = \{x \in C\colon\ f(\omega, x) = r(\omega)\}.$$

Since E is reflexive and f is convex in x, it is easily seen that each $F(\omega)$ is nonempty closed convex. We first show that $r(\cdot)$ is measurable. Suppose $\{y_n\}$ is a countable dense subset of C. Then we have for each $\omega \in \Omega$

$$r(\omega) = \inf_{n\geqslant 1} f(\omega, y_n).$$

It thus follows that $r(\cdot)$ is measurable since each $f(\cdot, y_n)$ is measurable. Next, for each integer $k \geqslant 1$ we set

$$F_k(\omega) = \left\{ x \in C\colon\ f(\omega, x) \leqslant r(\omega) + \frac{1}{k} \right\}.$$

Then each $F_k(\cdot)\colon\ \Omega \to C$ is measurable by Lemma 1, and is closed convex valued. It is clear that

$$F(\omega) = \bigcap_{k=1}^{\infty} F_k(\omega). \tag{4}$$

We now claim that $F\colon\ \Omega \to C$ is measurable. By separability of C, we have a metric, denoted d_w, on C which induces the weak topology on C. Let H_w be the corresponding Hausdorff metric. We now show that

$$\lim_{k\to\infty} H_w(F_k(\omega), F(\omega)) = 0, \qquad \omega \in \Omega. \tag{5}$$

In fact, since $\{F_k(\omega)\}$ is a decreasing sequence, we have from (4) that the limit in (5), denoted $h(\omega)$, exists and it is not difficult to see that

$$h(\omega) = \lim_{k\to\infty} \sup_{y\in F_k(\omega)} d_w(y, F(\omega)).$$

If $h(\omega) > 0$, then for each $k \geqslant 1$ there exists a $y_k \in F_k(\omega)$ such that

$$d_w(y_k, F(\omega)) > \frac{1}{2} h(\omega). \tag{6}$$

Since $\{y_k\}$ is contained in C and C is weakly compact, there exists a subsequence $\{y_{k'}\}$ of $\{y_k\}$ which is weakly convergent to some $y \in C$, i.e., $d_w(y'_k, y) \to 0$ as $k' \to \infty$. Again, since $\{F_k(\omega)\}$ is a decreasing sequence of closed convex (and hence weakly closed) subsets, it follows that

$$y \in \bigcap_{k=1}^{\infty} F_k(\omega) = F(\omega). \tag{7}$$

On the other hand, by continuity of the distance d_w, we have by (6) that $d_w(y, F(\omega)) \geqslant \frac{1}{2} h(\omega) > 0$, which implies that y does not belong to $F(\omega)$. This contradicts (7) and (5) is proved. From (5) and Proposition 2, it follows that there exists a w-measurable selector x for F. This x clearly satisfies (3). (Note that by uniform convexity of E, there is exactly one $x(\omega) \in C$ that satisfies (3).) Now by induction we can define a sequence $\{x_n(\omega)\}$ of measurable functions $x_n\colon\ \Omega \to C$

with $x_0(\omega) \equiv x_0$ such that for each $m \geqslant 0$, $x_{m+1}(\omega)$ is the asymptotic center of the sequence $\{T^n(\omega, x_m(\omega))\}$ in C, i.e.

$$\limsup_{n\to\infty} \|T^n(\omega, x_m(\omega)) - x_{m+1}(\omega)\| = \inf_{y\in C} \limsup_{n\to\infty} \|T^n(\omega, x_m(\omega)) - y\|.$$

Let for each $\omega \in \Omega$ and integer $m \geqslant 0$

$$r_m(\omega) = \limsup_{n\to\infty} \|T^n(\omega, x_m(\omega)) - x_{m+1}(\omega)\|$$

and

$$D_m(\omega) = \sup_{n\geqslant 1} \|x_m(\omega) - T^n(\omega, x_m(\omega))\|.$$

By using (2) after a simple calculation, we have for each x, y in C and $\omega \in \Omega$,

$$\begin{aligned} \|T^i(\omega, x) - T^j(\omega, y)\| \leqslant \frac{a(\omega) + b(\omega) + c(\omega)}{1 - b(\omega) - c(\omega)} \cdot \|x - T^{j-i}(\omega, y)\| \\ + \frac{2b(\omega) + 2c(\omega)}{1 - b(\omega) - c(\omega)} \cdot \|T^j(\omega, y) - x\|, \end{aligned}$$

i.e.,

$$\text{(8)} \qquad \|T^i(\omega, x) - T^j(\omega, y)\| \leqslant \alpha(\omega) \cdot \|x - T^{j-i}(\omega, y)\| + \beta(\omega) \cdot \|T^j(\omega, y) - x\|.$$

By the result of Lim [18, Theorem 1] and by (8) we have

$$\begin{aligned} r_m(\omega) &= \limsup_{i\to\infty} \|T^i(\omega, x_m(\omega)) - x_{m+1}(\omega)\| \\ &\leqslant \frac{1}{N} \cdot \limsup_{n\to\infty} \{\|T^i(\omega, x_m(\omega)) - T^j(\omega, x_m(\omega))\| \colon\ i, j \geqslant n\} \\ &\leqslant \frac{1}{N} \cdot \limsup_{n\to\infty} \{\alpha(\omega) \cdot \|x_m(\omega) - T^{j-i}(\omega, x_m(\omega))\| \\ &\quad + \beta(\omega) \cdot \|x_m(\omega) - T^j(\omega, x_m(\omega))\| \colon\ i, j \geqslant n\} \end{aligned}$$

and so

$$\text{(9)} \qquad r_m(\omega) \leqslant \frac{(\alpha(\omega) + \beta(\omega))}{N} \cdot D_m(\omega),$$

where N is the normal structure coefficient of E. For each fixed $m \geqslant 1$ and all $n > k \geqslant 1$, we have from (1) and (8)

$$\begin{aligned}
&\|\lambda x_{m+1}(\omega) + (1-\lambda)T^k(\omega, x_{m+1}(\omega)) - T^n(\omega, x_m(\omega))\|^p \\
&\quad + c_p \cdot W_p(\lambda) \cdot \|x_{m+1}(\omega) - T^k(\omega, x_{m+1}(\omega))\|^p \\
&\leqslant \lambda \|x_{m+1}(\omega) - T^n(\omega, x_m(\omega))\|^p \\
&\quad + (1-\lambda) \cdot \|T^k(\omega, x_{m+1}(\omega)) - T^n(\omega, x_m(\omega))\|^p \\
&\leqslant \lambda \|x_{m+1}(\omega) - T^n(\omega, x_m(\omega))\|^p \\
&\quad + (1-\lambda) \cdot \alpha(\omega) \cdot \|x_{m+1}(\omega) - T^{n-k}(\omega, x_m(\omega))\| \\
&\quad + \beta(\omega) \cdot \|x_{m+1}(\omega) - T^n(\omega, x_m(\omega))\|)^p.
\end{aligned}$$

Taking the limit superior as $n \to \infty$ on each side, by definition of $x_m(\omega)$ we get

$$\begin{aligned}
&r_m^p(\omega) + c_p \cdot W_p(\lambda) \cdot \|x_{m+1}(\omega) - T^k(\omega, x_{m+1}(\omega))\|^p \\
&\qquad \leqslant \{\lambda + (1-\lambda) \cdot (\alpha(\omega) + \beta(\omega))^p\} r_m^p(\omega).
\end{aligned}$$

It then follows that

$$\begin{aligned}
D_{m+1}^p(\omega) &\leqslant \frac{(1-\lambda)\{(\alpha(\omega)+\beta(\omega))^p - 1\}}{c_p \cdot W_p(\lambda)} \cdot r_m^p(\omega) \\
&\leqslant \frac{(1-\lambda)\{(\alpha(\omega)+\beta(\omega))^p - 1\}}{c_p \cdot W_p(\lambda)} \cdot \frac{(\alpha(\omega)+\beta(\omega))^p}{N^p} \cdot D_m^p(\omega).
\end{aligned}$$

Letting $\lambda \to 1$, we conclude that

$$\begin{aligned}
&D_{m+1}(\omega) \\
&\quad \leqslant \left[\frac{(\alpha(\omega)+\beta(\omega))^p\{(\alpha(\omega)+\beta(\omega))^p - 1\}}{c_p \cdot N^p}\right]^{\frac{1}{p}} \cdot D_m(\omega) \qquad (10) \\
&\quad = A \cdot D_m(\omega), \quad m = 1, 2, \ldots
\end{aligned}$$

where $A = \left[\frac{(\alpha(\omega)+\beta(\omega))^p \cdot \{(\alpha(\omega)+\beta(\omega))^p - 1\}}{c_p \cdot N^p}\right]^{\frac{1}{p}} < 1$ by the assumption of the theorem. So, in general,

$$D_{m+1}(\omega) \leqslant A \cdot D_m(\omega) \leqslant \ldots \leqslant A^{m+1} D_0(\omega).$$

Since

$$\begin{aligned}
\|x_{m+1}(\omega) - x_m(\omega)\| &\leqslant \|x_{m+1}(\omega) - T^n(\omega, x_m(\omega))\| \\
&\qquad + \|T^n(\omega, x_m(\omega)) - x_m(\omega)\|,
\end{aligned}$$

taking the limit superior as $n \to \infty$ on each side, we have

$$\begin{aligned}
\|x_{m+1}(\omega) - x_m(\omega)\| &\leqslant r_m(\omega) + D_m(\omega) \\
&\leqslant 2 \cdot D_m(\omega) \leqslant \ldots \leqslant 2 \cdot A^{m+1} D_0(\omega), \\
&\to 0
\end{aligned}$$

as $m \to \infty$. It then follows that $\{x_m(\omega)\}$ is a Cauchy sequence. Let $x(\omega) = \lim_{m\to\infty} x_m(\omega)$ for each $\omega \in \Omega$. Then we have from the triangle inequality and by (8)

$$\begin{aligned}
&\|x(\omega) - T(\omega, x(\omega))\| \\
&\quad \leqslant \|x(\omega) - x_m(\omega)\| + \|x_m(\omega) - T(\omega, x_m(\omega))\| \\
&\qquad + \|T(\omega, x_m(\omega)) - T(\omega, x(\omega))\| \\
&\quad \leqslant \|x(\omega) - x_m(\omega)\| + \|x_m(\omega) - T(\omega, x_m(\omega))\| \\
&\qquad + \alpha(\omega) \cdot \|x_m(\omega) - x(\omega)\| + \beta(\omega) \cdot \|T(\omega, x(\omega)) - x_m(\omega)\|
\end{aligned}$$

and so

$$\begin{aligned}
\|x(\omega) - T(\omega, x(\omega))\| \leqslant {}& \frac{1 + \alpha(\omega) + \beta(\omega)}{1 - \beta(\omega)} \cdot \|x(\omega) - x_m(\omega)\| \\
&+ \frac{1}{1 - \beta(\omega)} \cdot \|x_m(\omega) - T(\omega, x_m(\omega))\| \to 0
\end{aligned}$$

as $m \to +\infty$. Hence $T(\omega, x(\omega)) = x(\omega)$ for each $\omega \in \Omega$. This $x(\omega)$ is obviously measurable and thus it is a random fixed point of T. This completes the proof. □

If we put $b(\omega) = c(\omega) = 0$ in Theorem 1, then we have the following result.

Corollary 1 [43, Theorem 1]. *Let (Ω, Σ) be a measurable space. Let E be a p-uniformly convex Banach space for some $p > 1$, C a nonempty bounded closed convex separable subset of E, and $T\colon \Omega \times C \to C$ a random uniformly Lipschitzian mapping. If for each $\omega \in \Omega$*

$$\alpha(\omega) < \left[\frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot c_p \cdot N^p}\right)\right]^{\frac{1}{p}},$$

where N is the normal structure coefficient of E and c_p is the constant given in the inequality (1)*, then T has a random fixed point.*

Now we give applications of the above established inequalities analogous to (1) in some Banach spaces. Let us begin with the following wellknown result.

Lemma 2. (i) *In a Hilbert space H, the following equality holds:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{11}$$

for all x, y in H and $\lambda \in [0,1]$.

(ii) *If $1 < p \leqslant 2$, then we have for all x, y in L^p and $\lambda \in [0,1]$*

$$\|\lambda x + (1-\lambda)y\|^2 \leqslant \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\cdot(p-1)\cdot\|x-y\|^2. \tag{12}$$

(The inequality (12) *is contained in Lim, Xu and Xu* [19] *and Smarzewski* [34].*)*

(iii) *Assume that* $2 < p < \infty$ *and* t_p *is the unique zero of the function* $g(x) = -x^{p-1} + (p-1)x + p - 2$ *in the interval* $(1, \infty)$. *Let*

$$c_p = (p-1) \cdot (1+t_p)^{2-p} = \frac{1 + t_p^{p-1}}{(1+t_p)^{p-1}}.$$

Then we have the inequality

$$\|\lambda x + (1-\lambda)y\|^p \leqslant \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda) \cdot c_p \cdot \|x-y\|^p \tag{13}$$

for all x, y *in* L^p *and* $\lambda \in [0,1]$. *(The inequality* (13) *is essentially due to Lim, Xu and Xu* [19] *and Xu* [41].*)*

By Theorem 1 and Lemma 2, we immediately obtain the following results:

Theorem 2. *Let* (Ω, Σ) *be a measurable space. Let* C *be a nonempty closed convex separable subset of a Hilbert space* H *and* $T\colon \Omega \times C \to C$ *a generalized random uniformly Lipschitzian mapping. If for each* $\omega \in \Omega$

$$\left[\frac{(\alpha(\omega)+\beta(\omega))^2\{(\alpha(\omega)+\beta(\omega))^2 - 1\}}{2}\right]^{\frac{1}{2}} < 1,$$

where $\alpha(\omega)$, $\beta(\omega)$ *are as in Theorem* 1, *then* T *has a random fixed point.*

Theorem 3. *Let* (Ω, Σ) *be a measurable space. Let* C *be a nonempty closed convex separable subset of* L^p, $1 < p < \infty$, *and* $T\colon \Omega \times C \to C$ *a generalized uniformly Lipschitzian mapping. If for each* $\omega \in \Omega$

$$\left[\frac{(\alpha(\omega)+\beta(\omega))^2\{(\alpha(\omega)+\beta(\omega))^2 - 1\}}{(p-1)\cdot 2^{\frac{p-1}{p}}}\right]^{\frac{1}{2}} < 1 \quad \text{for} \quad 1 < p \leqslant 2$$

and

$$\left[\frac{(\alpha(\omega)+\beta(\omega))^p \cdot \{(\alpha(\omega)+\beta(\omega))^p - 1\}}{c_p \cdot 2}\right]^{\frac{1}{p}} < 1 \quad \text{for} \quad 2 < p < \infty,$$

where $\alpha(\omega)$, $\beta(\omega)$ *are as in Theorem* 1, *then* T *has a random fixed point.*

If we put $b(\omega) = c(\omega) = 0$ in Theorem 2 and Theorem 3, then we obtain the following results.

Corollary 2 [43, Corollary 1]. *Let* (Ω, Σ) *be a measurable space. Let* C *be a nonempty closed convex separable subset of a Hilbert space* H *and* $T\colon \Omega \times C \to C$ *a* *random uniformly Lipschitzian mapping. If $\alpha(\omega) < \sqrt{2}$ for each $\omega \in C$, then T has a random fixed point.*

Corollary 3 [43, Corollary 2]. *Let (Ω, Σ) be a measurable space. Let C be a nonempty closed convex separable subset of L^p, $1 < p < \infty$, and $T\colon \Omega \times C \to C$ a uniformly Lipschitzian mapping. If for each $\omega \in \Omega$*

$$\alpha(\omega) < \left[\frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot (p-1) \cdot 2^{\frac{p-1}{p}}}\right)\right]^{\frac{1}{2}} \quad \text{if } 1 < p \leqslant 2$$

and

$$\alpha(\omega) < \left[\frac{1}{2}\left(1 + \sqrt{1 + 8 \cdot c_p}\right)\right]^{\frac{1}{p}} \qquad \text{if } 2 < p < \infty,$$

where c_p is as in (1), then T has a random fixed point.

Suppose now that E is a uniformly convex Banach space whose modulus of convexity is denoted by $\delta(\cdot)$. Let $\tau > 1$ be the unique solution of the equation $\tau \cdot (1 - \delta_E(\frac{1}{\tau})) = 1$. Goebel and Kirk [13] proved that if T is a uniformly α-Lipschitzian self-mapping of a nonempty bounded closed convex subset C of E and if $\alpha < \tau$, then T has a fixed point. For a Hilbert space H, $\tau = \frac{\sqrt{5}}{2}$ and for L^p, we have $\tau = (1 + \frac{p}{2})^{\frac{1}{p}}$. Lifshitz [22] and Lim [17] extended the Geobel and Kirk's result in the setting of Hilbert space and L^p spaces, respectively (see also [6, 19, 33 and 39]). In [43], Xu presented its stochastic version.

It is also wellknown that if E is a uniformly convex Banach space, then the equation

$$r^2 \delta_E^{-1}\left(1 - \frac{1}{r}\right)\frac{1}{N} = 1 \tag{14}$$

has a unique solution $r > 1$, where N is the normal structure coefficient of E.

Now we give more a general stochastic version of the result of Goebel and Kirk [13].

Theorem 4. *Let (Ω, Σ) be a measurable space. Let E be a uniformly convex Banach space, C a nonempty bounded closed convex separable subset of E, and $T\colon \Omega \times C \to C$ a generalized random uniformly Lipschitzian mapping. Let*

$$(\alpha(\omega) + \beta(\omega)) < r$$

for all $\omega \in \Omega$, where

$$\alpha(\omega) = \frac{a(\omega) + b(\omega) + c(\omega)}{1 - b(\omega) - c(\omega)}, \qquad \beta(\omega) = \frac{2b(\omega) + 2c(\omega)}{1 - b(\omega) - c(\omega)},$$

and $r > 1$ is the unique solution of (14). Then T has a random fixed point.

P r o o f. As in the proof of Theorem 1 above, taking $x_0(\omega) \equiv x_0 \in C$, we can inductively construct a sequence $\{x_m(\omega)\}$ of measurable mappings $x_m \colon \Omega \to C$ such that for each $m \geqslant 0$, $x_{m+1}(\omega)$ is the asymptotic center of the sequence $\{T^n(\omega, x_m(\omega))\}$ in C, i.e.

$$\limsup_{n\to\infty} \|T^n(\omega, x_m(\omega)) - x_{m+1}(\omega)\| = \inf_{y\in C} \limsup_{n\to\infty} \|T^n(\omega, x_m(\omega)) - y\|.$$

Let for each $\omega \in \Omega$ and integer $m \geqslant 0$

$$r_m(\omega) = \limsup_{n\to\infty} \|T^n(\omega, x_m(\omega)) - x_{m+1}(\omega)\|$$

and

$$D_m(\omega) = \sup_{n\geqslant 1} \|x_m(\omega) - T^n(\omega, x_m(\omega))\|.$$

Then by the proof of Theorem 1, we also have

$$r_m(\omega) \leqslant \frac{(\alpha(\omega) + \beta(\omega))}{N} \cdot D_m(\omega), \tag{15}$$

where N is the normal structure coefficient of E. We may assume $D_m(\omega) > 0$ for all $m \geqslant 0$. Let $m \geqslant 0$ be fixed and let $\varepsilon > 0$ be small enough. First choose $j \geqslant 1$ such that

$$\|T^j(\omega, x_{m+1}(\omega)) - x_{m+1}(\omega)\| > D_{m+1}(\omega) - \varepsilon$$

and then choose $n_0 \geqslant 1$ so large that

$$\|T^n(\omega, x_m(\omega)) - x_{m+1}(\omega)\| < r_m(\omega) + \varepsilon$$

and

$$\begin{aligned}\|T^n(\omega, x_m(\omega)) - T^j(\omega, x_{m+1}(\omega))\| &\leqslant \alpha(\omega) \cdot \|T^{n-j}(\omega, x_m(\omega)) - x_{m+1}(\omega)\| \\ &\quad + \beta(\omega) \cdot \|T^n(\omega, x_m(\omega)) - x_{m+1}(\omega)\| \\ &\leqslant \alpha(\omega)(r_m(\omega) + \varepsilon) + \beta(\omega)(r_m(\omega) + \varepsilon) \\ &= (\alpha(\omega) + \beta(\omega))(r_m(\omega) + \varepsilon)\end{aligned}$$

for all $n \geqslant n_0$. It then follows that

$$\begin{aligned}&\|T^n(\omega, x_m(\omega)) - \tfrac{1}{2}(x_{m+1}(\omega) + T^j(\omega, x_{m+1}(\omega)))\| \\ &\quad\leqslant (\alpha(\omega) + \beta(\omega))(r_m(\omega) + \varepsilon)\left(1 - \delta_E\left(\frac{D_{m+1}(\omega) - \varepsilon}{(\alpha(\omega) + \beta(\omega))(r_m(\omega) + \varepsilon)}\right)\right)\end{aligned}$$

for all $n \geqslant n_0$ and hence

$$\begin{aligned}
&r_m(\omega)\\
&\leqslant \limsup_{n\to\infty} \|T^n(\omega, x_m(\omega)) - \frac{1}{2}(x_{m+1}(\omega) + T^j(\omega, x_{m+1}(\omega)))\|\\
&\leqslant (\alpha(\omega)+\beta(\omega))(r_m(\omega)+\varepsilon)\left(1-\delta_E\left(\frac{D_{m+1}(\omega)-\varepsilon}{(\alpha(\omega)+\beta(\omega))(r_m(\omega)+\varepsilon)}\right)\right).
\end{aligned}$$

Taking the limit as $\varepsilon \to 0$, we obtain

$$r_m(\omega) \leqslant (\alpha(\omega)+\beta(\omega))r_m(\omega)\left(1-\delta_E\left(\frac{D_{m+1}(\omega)}{(\alpha(\omega)+\beta(\omega))r_m(\omega)}\right)\right),$$

which together with (15) leads to the inequality

$$D_{m+1}(\omega) \leqslant (\alpha(\omega)+\beta(\omega))^2\delta_E^{-1}\left(1-\frac{1}{(\alpha(\omega)+\beta(\omega))}\right)\frac{1}{N}D_m(\omega).$$

Hence we have

$$D_{m+1}(\omega) \leqslant AD_m(\omega) \leqslant A^{m+1}D_0(\omega), \tag{16}$$

where $A = (\alpha(\omega)+\beta(\omega))^2\delta_E^{-1}(1-\frac{1}{(\alpha(\omega)+\beta(\omega))})\frac{1}{N}D_m(\omega) < 1$ by assumption. Noticing

$$\begin{aligned}
\|x_{m+1}-x_m\| &\leqslant \limsup_{n\to\infty}\|T^n(\omega, x_m(\omega)) - x_{m+1}(\omega)\|\\
&\quad + \limsup_{n\to\infty}\|T^n(\omega, x_m(\omega)) - x_m(\omega)\|\\
&\leqslant r_m(\omega) + D_m(\omega) \leqslant 2\cdot D_m(\omega) \leqslant \ldots \leqslant 2\cdot A^m D_0(\omega),
\end{aligned}$$

we obtain from (16) that $\{x_m(\omega)\}$ is a Cauchy sequence. Let

$$x(\omega) = \lim_{m\to\infty} x_m(\omega)$$

for each $\omega \in \Omega$. Then by the proof of Theorem 1, we conclude that this $x(\omega)$ is a random fixed point of T. □

The following is also an improvement of Theorem 3 of Xu [43], which is the random version of Theorem 3.1 of Casini and Maluta [6].

Theorem 5. *Let (Ω, Σ) be a measurable space. Let E be a Banach space with uniformly normal structure, C a nonempty bounded closed convex separable subset of E, and $T\colon \Omega\times C\to C$ a generalized random uniformly Lipschitzian mapping. Let*

$$(\alpha(\omega)+\beta(\omega)) < N^{\frac{1}{2}}$$

for all $\omega \in \Omega$, *where*

$$\alpha(\omega) = \frac{a(\omega) + b(\omega) + c(\omega)}{1 - b(\omega) - c(\omega)}, \qquad \beta(\omega) = \frac{2b(\omega) + 2c(\omega)}{1 - b(\omega) - c(\omega)},$$

and N *is the normal structure coefficient of* E. *Then* T *has a random fixed point.*

Proof. Let x_0 be an arbitrary point of C and set $x_0(\omega) \equiv x_0$. Now by Lemma 2 of Xu [43] and Theorem B, we can inductively construct a sequence $\{x_m\}$ of measurable functions $x_m\colon \Omega \to C$ such that for each $\omega \in \Omega$ and integer $m \geqslant 0$,

(i) $\|x_{m+1}(\omega) - z\| \leqslant \limsup\limits_{n\to\infty} \|T^n(\omega, x_m(\omega)) - z\|$ for all $z \in E$, and

(ii) $\limsup\limits_{n\to\infty} \|T^n(\omega, x_m(\omega)) - x_{m+1}(\omega)\| \leqslant \frac{1}{N} A(\{T^n(\omega, x_m(\omega))\})$, where $A(\{z_n\}) = \limsup\limits_{n\to\infty}\{\|z_i - z_j\|\colon\ i, j\ \geqslant n\}$ is the asymptotic diameter of $\{z_n\}$.

Set for each $\omega \in \Omega$ and integer $m \geqslant 0$

$$D_m(\omega) = \sup_{k\geqslant 1} \|T^k(\omega, x_m(\omega)) - x_m(\omega)\|$$

and

$$r = \frac{[\alpha(\omega) + \beta(\omega)]^2}{N}.$$

Then $r < 1$. From (i), (ii) and (8), it follows that

$$\begin{aligned} D_m(\omega) &\leqslant \sup_{i\geqslant 1} \limsup_{n\to\infty} \|T^n(\omega, x_{m-1}(\omega)) - T^i(\omega, x_m(\omega)))\| \\ &\leqslant \alpha(\omega) \limsup_{n\to\infty} \|T^{n-i}(\omega, x_{m-1}(\omega)) - x_m(\omega)\| \\ &\quad + \beta(\omega) \limsup_{n\to\infty} \|T^n(\omega, x_{m-1}(\omega)) - x_m(\omega)\| \\ &\leqslant \frac{\alpha(\omega) + \beta(\omega)}{N} A(\{T^n(\omega, x_{m-1}(\omega))\}). \end{aligned}$$

However, by (8) we have for all $i > j$

$$\begin{aligned} \|T^i(\omega, x_{m-1}(\omega)) - T^j(\omega, x_{m-1}(\omega))\| &\leqslant \alpha(\omega)\|x_{m-1}(\omega) - T^{i-j}(\omega, x_{m-1}(\omega))\| \\ &\quad + \beta(\omega)\|x_{m-1}(\omega) - T^i(\omega, x_{m-1}(\omega))\| \\ &\leqslant (\alpha(\omega) + \beta(\omega)) D_{m-1}(\omega). \end{aligned}$$

Therefore we conclude that

$$\begin{aligned} D_m(\omega) &\leqslant \frac{[\alpha(\omega) + \beta(\omega)]^2}{N} D_{m-1}(\omega) \\ &= r D_{m-1}(\omega) \leqslant \ldots \leqslant r^m D_0(\omega), \end{aligned}$$

and

$$\begin{aligned}\|x_m(\omega)-x_{m+1}(\omega)\| &\leqslant \sup_{n\geqslant 1}\|x_m(\omega)-T^n(\omega,x_m(\omega))\| \\ &\quad+\limsup_{n\to\infty}\|T^n(\omega,x_m(\omega))-x_{m+1}(\omega)\| \\ &\leqslant D_m(\omega)+\frac{1}{N}A(\{T^n(\omega,x_m(\omega))\}) \\ &\leqslant \Big(1+\frac{\alpha(\omega)+\beta(\omega)}{N}\Big)D_m(\omega) \\ &\leqslant \Big(1+\frac{\alpha(\omega)+\beta(\omega)}{N}\Big)r^m D_0(\omega),\end{aligned}$$

which implies that $\{x_m(\omega)\}$ is a Cauchy sequence whose limit is denoted by $x_(\omega)$. From the proof of Theorem 1, it follows that this x is a random fixed point of T. This completes the proof. □

If we put $b(\omega)=c(\omega)=0$ in Theorem 4 and Theorem 5, then we also have the following results.

Corollary 4 [43, Theorem 2]. *Let (Ω,Σ) be a measurable space. Let E be a uniformly convex Banach space, C a nonempty bounded closed convex separable subset of E, and $T\colon \Omega\times C\to C$ a random uniformly Lipschitzian mapping such that $\alpha(\omega)<r$ for all $\omega\in\Omega$, where $r>1$ is the unique solution of* (14). *Then T has a random fixed point.*

Corollary 5 [43, Theorem 3]. *Let (Ω,Σ) be a measurable space. Let E be a Banach space with uniformly normal structure, C a nonempty bounded closed convex separable subset of E, and $T\colon \Omega\times C\to C$ a random uniformly Lipschitzian mapping such that $\alpha(\omega)<N^{\frac{1}{2}}$ for all $\omega\in\Omega$, where N is the normal structure coefficient of E. Then T has a random fixed point.*

4. Additional results

Using the results of Prus and Smarzewski [27], Smarzewski [33] and Xu [41], we can obtain from Theorem 1 fixed point theorems, for example, for Hardy and Sobolev spaces.

Let H^p, $1<p<\infty$, denote the Hardy space [11] of all functions x analytic in the unit disc $|x|<1$ of the complex plane and such that

$$\|x\|=\lim_{r\to 1^-}\left(\frac{1}{2\pi}\int_0^{2\pi}|x(re^{i\theta})|^p\,d\theta\right)^{\frac{1}{p}}<\infty.$$

Now, let Ω be an open subset of R^n. Denote by $H^{k,p}(\Omega)$, $k \geqslant 0$, $1 < p < \infty$ the Sobolev space [1, p. 149] of distributions x such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \ldots + \alpha_n \leqslant k$ equipped with the norm

$$\|x\| = \left(\sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha x(\omega)|^p \, \mathrm{d}\omega \right)^{\frac{1}{p}}.$$

Let $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, $\alpha \in \Lambda$, be a sequence of positive measure spaces, where the index set Λ is finite or countable. Given a sequence of linear subspaces X_α in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$, $1 < p < \infty$ and $q = \max\{2, p\}$ [21], the linear space of all sequences $x = \{x_\alpha \in X_\alpha \colon \alpha \in \Lambda\}$ equipped with the norm

$$\|x\| = \left(\sum_{\alpha \in \Lambda} (\|x_\alpha\|_{p,\alpha})^q \right)^{\frac{1}{q}},$$

where $\| \cdot \|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$.

Finally, let $L_p = L^p(S_1, \Sigma_1, \mu_1)$ and $L_q = L^q(S_2, \Sigma_2, \mu_2)$, where $1 < p < \infty$, $q = \max\{2, p\}$ and (S_i, Σ_i, μ_i) are positive measure spaces. Denote by $L_q(L_p)$ the Banach spaces [10, III. 2.10] of all measurable L_p-valued functions x on S_2 such that

$$\|x\| = \left(\int_{S_2} (\|x(s)\|_p)^q \mu_2(ds) \right)^{\frac{1}{q}}.$$

These spaces are q-uniformly convex with $q = \max\{2, p\}$ [27, 33] and the norm in these spaces satisfies

$$\|\lambda x + (1-\lambda)y\|^q \leqslant \lambda \|x\|^q + (1-\lambda)\|y\|^q - d \cdot W_q(\lambda) \cdot \|x - y\|^q$$

with a constant

$$d = d_p = \begin{cases} \dfrac{p-1}{8} & \text{if } \ 1 < p \leqslant 2, \\ \dfrac{1}{p \cdot 2^p} & \text{if } \ 2 < p < \infty. \end{cases}$$

Hence from Theorem 1 we have the following result.

Theorem 6. *Let (Ω, Σ) be a measurable space. Let C be a nonempty closed convex separable subset of the space E, where $E = H^p$, or $E = H^{k,p}(\Omega)$ or $E = L_{q,p}$ or $E = L_q(L_p)$, and $1 < p < \infty$, $q = \max\{2, p\}$, $k \geqslant 0$. Let $T \colon \Omega \times C \to C$ be a generalized random uniformly Lipschitzian mapping. If for each $\omega \in \Omega$*

$$\left[\frac{(\alpha(\omega) + \beta(\omega))^q \{(\alpha(\omega) + \beta(\omega))^q - 1\}}{d \cdot N^2} \right]^{\frac{1}{q}} < 1,$$

where $\alpha(\omega)$, $\beta(\omega)$ are as in Theorem 1, then T has a random fixed point.

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