Czechoslovak Mathematical Journal

Said R. Grace

Oscillation of certain difference equations

Czechoslovak Mathematical Journal, Vol. 50 (2000), No. 2, 347–358

Persistent URL: http://dml.cz/dmlcz/127574

Czechoslovak Mathematical Journal, 50 (125) (2000), 347–358

OSCILLATION OF CERTAIN DIFFERENCE EQUATIONS

S. R. Grace, Orman

(Received September 17, 1997)

Abstract. Some new criteria for the oscillation of difference equations of the form

$$\Delta^2 x_n - p_n \Delta x_{n-h} + q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n} = 0$$

and

$$\Delta^i x_n + p_n \Delta^{i-1} x_{n-h} + q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n} = 0, \ i = 2, 3,$$

are established.

1. Introduction

In this paper we will discuss the oscillatory property of certain difference equations of the form

$$\Delta^2 x_n - p_n \Delta x_{n-h} + q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n} = 0 \tag{(E_1)}$$

and

$$\Delta^i x_n + p_n \Delta^{i-1} x_{n-h} + q_n |x_{g_n}|^c \operatorname{sgn} x_{g_n} = 0, \ i = 2, 3, \tag{(E_i)}$$

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$, $\{p_n\}$ and $\{q_n\}$ are sequences of nonnegative real numbers, $\{g_n\}$ is a sequence of integers, h is an integer and c is any positive real number, and $g_n \to \infty$ as $n \to \infty$.

The oscillation, nonoscillation and asymptotic behavior of Eq. (E_1) when $p_n = 0$ have been considered by many authors, we refer to [4–7, 9, 10, 12] and the references cited therein.

A real solution $\{x_n\}$, $n \geqslant 0$ of Eq. (E_1) (or Eq. (E_i), $i = 2, 3$) is said to be nonoscillatory if there exists $N \geqslant 0$ such that $x_n x_{n+1} > 0$ for all $n \geqslant N$, and is oscillatory otherwise. Eq. (E_i), $i=1,2$ or 3 is said to be almost oscillatory if every solution $\{x_n\}$ of Eq. (E_i), $i=1,2$ or 3 is oscillatory or $\{\Delta x_n\}$ is oscillatory for Eq. (E_i), $i=1$ or 2, or $\{\Delta^2 x_n\}$ is oscillatory for Eq. (E_3).

Eq. (E_1) and Eq. (E_i), $i=2,3$ may be viewed as discrete analogues of the functional differential equations

$$x''(t)-p(t)x'(t-h)+q(t)|x(g(t))|^c \operatorname{sgn} x(g(t))=0 \tag{(F_1)}$$

and

$$x^{(i)}(t)+p(t)x^{(i-1)}(t-h)+q(t)|x(g(t))|^c \operatorname{sgn} x(g(t))=0,\ i=2,3 \tag{(F_i)}$$

respectively, where $g,p,q:[t_0,\infty)\to R$, $t_0\geqslant 0$ are continuous, $g(t)\to\infty$ as $t\to\infty$, $p(t)\geqslant 0$ and $q(t)\geqslant 0$ eventually, c and h are real numbers and $c>0$. In fact the results in this paper are motivated by similar results for Eq. (F_1) and Eq. (F_i), $i=2,3$, see [1–3].

The purpose of this paper is to establish some new criteria for the almost oscillation of Eq. (E_i), $i=1,2,3$. In Section 2 we establish two criteria for the almost oscillation of Eq. (E_1) when $c>0$ and $c>1$. In Section 3 we deal with the oscillatory and asymptotic behavior of Eq. (E_2) and obtain sufficient conditions for any solution $\{x_n\}$ of Eq. (E_2) either to be oscillatory or else approach zero monotonically as $n\to\infty$. Also, we give sufficient conditions for all solutions of Eq. (E_2) to be almost oscillatory when $c=1$. The final section presents two criteria for the almost oscillation of Eq. (E_3) when $c>0$ and $c=1$.

2. Almost oscillation of Eq. (E_1)

The following result is concerned with the oscillation of Eq. (E_1) for any $c>0$.

Theorem 1. *Let h be any nonnegative integer and $\Delta p_n\geqslant 0$ for $n\geqslant n_0\geqslant 0$. If*

$$\sum_{i=n_0}^{\infty} q_i=\infty \tag{1}$$

and

$$\sum_{j=n_0}^{\infty} a_{j+1}\sum_{i=n_0}^{j-1} q_i=\infty, \tag{2}$$

where

$$a_{j+1}=\prod_{i=n_0}^{j}(1+p_i)^{-1},\quad j\geqslant 1,$$

then Eq. (E_1) is almost oscillatory.

P r o o f. Assume for the sake of contradiction that Eq. (E_1) has a nonoscillatory solution $\{x_n\}$, which we may and will assume to be eventually positive. There exists a positive integer $n_1 \geqslant n_0$ such that $x_{g_n} > 0$ for $n \geqslant n_1$.

Next, we consider the following two cases:

(A) $\Delta x_n < 0$ eventually, (B) $\Delta x_n > 0$ eventually.

(A) Assume $\Delta x_n < 0$ eventually. From Eq. (E_1), we observe that $\Delta^2 x_n \leqslant 0$ eventually and hence one can easily see that $x_n \to -\infty$ as $n \to \infty$, a contradiction.

(B) Assume $\Delta x_n > 0$ eventually. There exist $N \geqslant n_2$ and a constant $c_1 > 0$ such that

$$x_{g_n} \geqslant c_1 \quad \text{for} \quad n \geqslant N. \tag{3}$$

Using (3) in Eq. (E_1) we have

$$\Delta^2 x_n - p_n \Delta x_{n-h} + b q_n \leqslant 0 \quad \text{for} \quad n \geqslant N, \tag{4}$$

where $b = c_1^c$. Summing both sides of (4) from N to $n-1 \geqslant N$, we get

$$\Delta x_n - \Delta x_N - \sum_{i=N}^{n-1} p_i \Delta x_{i-h} + b \sum_{i=N}^{n-1} q_i \leqslant 0,$$

or, using summation by part,

$$\Delta x_n - \Delta x_N - p_n x_{n-h} + p_N x_{N-h} + \sum_{i=N}^{n-1} x_{n-h+1} \Delta p_i + b \sum_{i=N}^{n-1} q_i \leqslant 0.$$

Using the fact that $\Delta p_n \geqslant 0$ and $x_n > 0$ for $n \geqslant n_2$, we have

$$\Delta x_n - \Delta x_N - p_n x_n + b \sum_{i=N}^{n-1} q_i \leqslant 0, \ \ n \geqslant N+1.$$

From (1), there exists $N_1 \geqslant N+1$ such that

$$\Delta x_N \leqslant \frac{1}{2} b \sum_{i=N}^{n-1} q_i \quad \text{for} \quad n \geqslant N_1 + 1.$$

Thus,

$$\Delta x_n - p_n x_n + \frac{1}{2} b \sum_{i=N}^{n-1} q_i \leqslant 0 \quad \text{for} \quad n \geqslant N_1 + 1. \tag{5}$$

Define a sequence $\{r_n\}$ by the recurrence relation

$$r_{n+1} = \frac{1}{1+p_n}, \quad n \geqslant n_0 \geqslant 0 \text{ and } r_{n_0} > 0.$$

Next, we multiply (5) by r_{n+1}, obtaining

$$\Delta(r_n x_n) + \frac{1}{2} b r_{n+1} \sum_{i=N}^{n-1} q_i \leqslant 0 \quad \text{for} \quad n \geqslant N_1 + 1. \tag{6}$$

Summing both sides of (6) from $N_1 + 1$ to $k \geqslant N_1 + 1$, we have

$$0 < r_{k+1} x_{k+1} \leqslant r_{N_1+1} x_{N_1+1} - \frac{1}{2} b \sum_{n=N_1+1}^{k} r_{n+1} \sum_{i=N}^{n-1} q_i \to -\infty \text{ as } k \to \infty,$$

a contradiction. This completes the proof. □

The following theorem deals with the almost oscillation of Eq. (E_1) when $g_n \geqslant n+2$, $n \geqslant n_0 \geqslant 0$ and $c > 1$.

Theorem 2. *Let h be a nonnegative integer, $c > 1$, $g_n \geqslant n+2$ for $n \geqslant n_0 \geqslant 0$, and assume that there exists a real sequence $\{z_n\}$, $n \geqslant n_0$ such that*

$$z_n > 0,\ \Delta z_n \geqslant 0,\ \Delta^2 z_n \leqslant 0 \text{ and } \Delta(z_n p_n) \leqslant 0 \text{ for } n \geqslant n_0. \tag{7}$$

If

$$\sum_{n=n_0}^{\infty} z_n q_n = \infty, \tag{8}$$

then Eq. (E_1) is almost oscillatory.

P r o o f. Let $\{x_n\}$ be an eventually positive solution of Eq. (E_1), say $x_n > 0$ and $x_{g_n} > 0$ for $n \geqslant n_1 \geqslant n_0 \geqslant 0$. As in the proof of Theorem 1, we consider the cases (A) and (B) and observe that case (A) is impossible. Next, we consider the case (B):

(B) Assume $\Delta x_n > 0$ for $n \geqslant N \geqslant n_1 + h$. Set

$$w_n = z_n \Delta x_n / x_{n+1}^c \quad \text{for} \quad n \geqslant N.$$

Then

$$\begin{aligned} \Delta w_n &= z_{n+1}(\Delta x_{n+1}/x_{n+2}^c) - z_n(\Delta x_n / x_{n+1}^c) \\ &= -z_n q_n (x_{g_n}/x_{n+2})^c + z_n p_n (\Delta x_{n-h}/x_{n+2}^c) + z_n \Delta x_{n+1}(x_{n+2}^{-c} - x_{n+1}^{-c}) \\ &\quad + \Delta z_n (\Delta x_{n+1}/x_{n+2}^c), \text{ and hence we see that} \\ \Delta w_n &\leqslant -z_n q_n + z_n p_n (\Delta x_{n-h}/x_{n+2}^c) + z_n(\Delta x_{n+1}/x_{n+2}^c),\ n \geqslant N. \end{aligned} \tag{9}$$

Summing both sides of (9) from N to $k-1 \geqslant N$, using (7) and the fact that $x_{n+2} \geqslant x_{n-h+1}$, $n \geqslant N$, we obtain

$$w_k - w_N \leqslant -\sum_{n=N}^{k-1} z_n q_n + z_N p_N \sum_{n=N}^{k-1} \Delta x_{n-h}/x^c_{n-h+1} + \Delta z_N \sum_{n=N}^{k-1} \Delta x_{n+1}/x^c_{n+2}.$$

As in the proof of Theorem 4.1 in [7], we have

$$\sum^{\infty} \Delta x_i / x^c_{i+1} < \infty,$$

and hence by (8), it follows that

$$0 < w_k \leqslant C - \sum_{n=N}^{k-1} z_n q_n \to -\infty \text{ as } k \to \infty,$$

where C is a constant, a contradiction. This completes the proof. □

Remark 1. One can easily observe that Theorems 1 and 2 are applicable to equations of type (E_1) when $h = 0$ or $p_n = 0$ for $n \geqslant 0$.

3. Oscillation and asymptotic behavior of Eq. (E_2)

Theorem 3. *Let h be any positive integer,*

(10) $$\liminf_{n\to\infty} \sum_{k=n-h}^{n-1} p_k > \left(\frac{h}{1+h}\right)^{1+h},$$

and assume that there exists a real sequence $\{z_n\}$ such that

(11) $$z_n > 0, \Delta z_n \geqslant 0 \text{ and } \Delta(z_n p_n) \leqslant 0 \text{ for } n \geqslant n_0 \geqslant 0.$$

If condition (8) holds and

(12) $$\sum_{i=n_0}^{\infty} 1/z_n = \infty,$$

then every solution $\{x_n\}$ of Eq. (E_2) is oscillatory or $\{\Delta x_n\}$ is oscillatory or else $x_n \to 0$ monotonically as $n \to \infty$.

P r o o f. Let $\{x_n\}$ be an eventually positive solution of Eq. (E_2). There exists $n_1 \geqslant n_0 \geqslant 0$ such that $x_{g_n} > 0$ for $n \geqslant n_1$. Next, we consider the following two cases:

(A*) $\Delta x_n > 0$ eventually, (B*) $\Delta x_n < 0$ eventually.

(A*) Suppose $\Delta x_n > 0$ eventually. From Eq. (E_2) we see that

$$\Delta^2 x_n + p_n \Delta x_{n-h} = -q_n x_{g_n}^c \leqslant 0 \quad \text{eventually.}$$

Set $y_n = \Delta x_n > 0$ eventually. Then

$$\Delta y_n + p_n y_{n-h} \leqslant 0 \text{ eventually.} \tag{13}$$

In view of Theorem 3 in [11] and condition (10), inequality (13) has no eventually positive solution, which is a contradiction.

(B*) Suppose $\Delta x_n < 0$ for $n \geqslant N \geqslant n_2$. So, we have

$$x_n \to c_1 \geqslant 0 \quad \text{as} \quad n \to \infty.$$

Suppose that $c_1 > 0$ and consider the sequence $\{w_n\}$ defined by

$$w_n = z_{n-1} \Delta x_n \quad \text{for } n \geqslant N.$$

Then

$$\begin{aligned}
\Delta w_n = \Delta(z_{n-1}\Delta x_n) &= z_n \Delta^2 x_n + \Delta z_{n-1} \Delta x_n \\
&\leqslant -b z_n q_n - z_n p_n \Delta x_{n-h} + \Delta z_{n-1} \Delta x_n \\
&\leqslant -b z_n q_n - z_n p_n \Delta x_{n-h}, \ n \geqslant N,
\end{aligned}$$

where $b = c_1^c$. Summing both sides of the above inequality from N to $k-1 \geqslant N$, we obtain

$$w_k - w_N \leqslant -b \sum_{n=N}^{k-1} z_n q_n - \sum_{n=N}^{k-1} z_n p_n \Delta x_{n-h}.$$

Using (11), we have

$$w_k \leqslant -b \sum_{n=N}^{k-1} z_n q_n + z_N p_N (x_{N-h} - x_{k-h}) \leqslant C - b \sum_{n=N}^{k-1} z_n q_n,$$

where $C = z_N p_N x_{N-h}$. By condition (8), there exist $N^* \geqslant N$ and a constant $c^* > 0$ such that

$$w_k = z_{k-1} \Delta x_k \leqslant -c^*$$

or

$$\Delta x_k \leqslant -c^*/z_{k-1} \quad \text{for} \quad n \geqslant N^*.$$

Summing both sides of the above inequality from N^* to $m \geqslant N^*+1$, letting $m \to \infty$ and using (12), we obtain a contradiction to the fact that $x_n > 0$ eventually. This complete the proof. □

Theorem 4. *Let h be any integer and $\Delta p_n \leqslant 0$, $n \geqslant n_0 \geqslant 0$. If condition* (1) *holds, then the conclusion of Theorem* 3 *holds.*

Proof. Let $\{x_n\}$ be an eventually positive solution of Eq. (E_2). As in the proof of Theorem 3, we see that $x_{g_n} > 0$ for $n \geqslant n_1$. Next, we consider the following two cases:

(A*) $\Delta x_n > 0$ eventually, (B*) $\Delta x_n < 0$ eventually.

(A*) Suppose $\Delta x_n > 0$ for $n \geqslant n_2 \geqslant n_1$. There exist constants $c_1 > 0$ and $N \geqslant n_2$ such that (3) holds for $n \geqslant N$. Now, from Eq. (E_2) we have

$$\Delta^2 x_n + b q_n \leqslant 0 \text{ for } n \geqslant N, \text{ where } b = c_1^c.$$

Summing both sides of the above inequality from N to $m \geqslant N+1$, letting $m \to \infty$ and using (1), we obtain a contradiction to the fact that $\Delta x_n > 0$ for $n \geqslant n_2$.

(B*) Suppose $\Delta x_n < 0$ eventually. The proof of this case is similar to the proof of Theorem 3 (B*) with $z_n = 1$, and hence is omitted.

The following result is concerned with the almost oscillation of Eq. (E_2). □

Theorem 5. *Let h be a nonpositive integer, $c = 1$ and $\Delta p_n \leqslant 0$ for $n \geqslant n_0 \geqslant 0$, and let condition* (1) *hold. Moreover, assume that there exists a sequence $\{k_n\}$ of positive integers such that $g_n \leqslant n - k_n$, $n \geqslant n_0$, $\{n - k_n\}$, $n \geqslant 0$ is increasing. If*

$$\liminf_{n\to\infty} \sum_{j=n-k_n}^{n-1} Q_j > \limsup_{n\to\infty} \Big(\frac{k_n}{(1+k_n)}\Big)^{1+k_n}, \tag{14}$$

where

$$Q_j = \sum_{i=j-k_j}^{j-1} q_i - p_{j-k_j} > 0, \ n-1 \geqslant j \geqslant n-k_n, \tag{15}$$

then Eq. (E_2) *is almost oscillatory.*

Proof. Let x_n be an eventually positive solution of Eq. (E_2). As in the proof of Theorem 3, we observe that $x_{g_n} > 0$ for $n \geqslant n_1$. Next, we consider the following two cases:

(A*) $\Delta x_n > 0$ eventually, (B*) $\Delta x_n < 0$ eventually.

(A*) Suppose $\Delta x_n > 0$ eventually. The proof of this case is similar to the proof of Theorem 4 (A*) and hence is omitted.

(B*) Suppose $\Delta x_n < 0$ for $n \geqslant N \geqslant n_2$. From Eq. (E_2) and the fact that $g_n \leqslant n - k_n$, $n \geqslant N$, we have

$$\Delta^2 x_n + p_n \Delta x_{n-h} + q_n x_{n-k_n} \leqslant 0 \quad \text{for} \quad n \geqslant N. \tag{16}$$

Summing both sides of (16) from $n-k_n$ to $n-1 \geqslant n-k_n$, $n \geqslant N$, we have

$$\Delta x_n - \Delta x_{n-k_n} + \sum_{j=n-k_n}^{n-1} p_j \Delta x_{j-h} + \sum_{j=n-k_n}^{n-1} q_j x_{j-k_j} \leqslant 0$$

or, using summation by parts,

$$\Delta x_n + \Big[p_n x_{n-h} - p_{n-k_n} x_{n-k_n} - \sum_{j=n-k_n}^{n-1} x_{n-h+1} \Delta p_j\Big] + \sum_{j=n-k_n}^{n-1} q_j x_{j-k_j} \leqslant 0, \ n \geqslant N.$$

Using the fact that $\Delta p_n \leqslant 0$ and $h < 0$, we obtain

$$\Delta x_n - p_{n-k_n} x_{n-k_n} + x_{n-k_n} \sum_{j=n-k_n}^{n-1} q_j \leqslant 0$$

or

$$\Delta x_n + Q_n x_{n-k_n} \leqslant 0 \quad \text{for} \quad n \geqslant N, \tag{17}$$

where Q_n is defined as in (15). But Theorem 3 in [11] and condition (14) imply that inequality (17) has no eventually solution, which is a contradiction. This completes the proof. □

Next, we consider the special case of Eq. (E_2), namely the equation

$$\Delta^2 x_n + p\Delta x_{n-h} + q x_{n-k} = 0, \tag{(L_2)}$$

where p and q are positive constants, h is a nonnegative integer and k is any positive integer.

The following corollary is a consequence of Theorem 5.

Corollary 1. *If*

$$kq - p > \frac{k^{k+1}}{(1+k)^{1+k}}, \tag{18}$$

then Eq. (L_2) *is almost oscillatory.*

Remark 2. (i) If we set $p_n = 0$, $n \geqslant 0$ in Theorems 3 and 4, we can easily check that Theorem 3 with $c > 1$ (or $0 < c < 1$) and Theorem 2.3 (or Theorem 2.4) in [4] are similar and Theorem 4 and Theorem 2.5 in [4] are the same and hence, we conclude that Eq. (E_2) with c as given above is oscillatory.

We note that the presence of the term—$p_n \Delta x_{n-h}$ makes the coexistence of oscillatory and monotonically decreasing positive (increasing negative) solutions for Eq. (E_2) possible.

(ii) We note that Theorem 5 is applicable to Eq. (E_2) when $p_n = 0$. Only condition (14) is disregarded.

4. Almost oscillation of Eq. (E_3)

Theorem 6. *Let h be a positive integer, $\Delta p_n \leqslant 0$ for $n \geqslant n_0 \geqslant 0$ and let conditions* (1) *and* (10) *hold. Then Eq.* (E_3) *is almost oscillatory.*

Proof. Let $\{x_n\}$ be an eventually positive solution of Eq. (E_3), say $x_n > 0$ and $x_{g_n} > 0$ for $n \geqslant n_1 \geqslant n_0 \geqslant 0$. Next, we consider the following two cases:
(A) $\Delta^2 x_n > 0$ eventually, (B) $\Delta^2 x_n < 0$ eventually.

(A) Suppose $\Delta^2 x_n > 0$ eventually. From Eq. (E_3) we have

$$\Delta^3 x_n + p_n \Delta^2 x_{n-h} = -q_n x_{g_n}^c \leqslant 0 \text{ eventually.}$$

Set $y_n = \Delta^2 x_n > 0$ eventually. Then

$$\Delta y_n + p_n y_{n-h} \leqslant 0 \text{ eventually.}$$

The rest of the proof is similar to that of Theorem 3 (A*) and hence is omitted.

(B) Suppose $\Delta^2 x_n < 0$ for $n \geqslant n_2 \geqslant n_1 + h$. It is easy to check that $\Delta x_n > 0$ for $n \geqslant n_1$ and there exist $N \geqslant n_2$ and a constant $c_1 > 0$ such that (3) holds for $n \geqslant N$. From Eq. (E_3) it follows that

$$\Delta^3 x_n + p_n \Delta^2 x_{n-h} + b q_n \leqslant 0, \tag{19}$$

where $b = c_1^c$. Summing both sides of (19) from N to $m-1 \geqslant N$, we have

$$\Delta^2 x_m - \Delta^2 x_N + \sum_{n=N}^{m-1} p_n \Delta^2 x_{n-h} + b \sum_{n=N}^{m-1} q_n \leqslant 0,$$

or

$$\Delta^2 x_m + \left[p_m \Delta x_{m-h} - p_N \Delta x_{N-h} - \sum_{n=N}^{m-1} \Delta p_n \Delta x_{n-h+1} \right] + b \sum_{n=N}^{m-1} q_n \leqslant 0.$$

Using $\Delta p_n \leqslant 0$ for $n \geqslant n_0$, we have

$$\Delta^2 x_m - p_N \Delta x_{N-n} + b \sum_{n=N}^{m-1} q_n \leqslant 0 \quad \text{for } m-1 \geqslant n \geqslant N.$$

From (1) it follows that there exist $N^* \geqslant N+1$ and $c^* > 0$ such that

$$\Delta^2 x_m \leqslant -c^* \quad \text{for } m \geqslant N^*,$$

and consequently

$$0 < \Delta x_j \to -\infty \quad \text{as} \quad j \to \infty,$$

a contradiction. This completes the proof. □

Theorem 7. *Let h be a nonpositive integer, $c = 1$, $\Delta p_n \leqslant 0$ and $g_n = n - k$, $n \geqslant n_0 \geqslant 0$ where k is a positive integer, and let condition* (1) *hold. If every bounded solution of*

$$\Delta^3 y_n + q_n y_{n-k} = 0 \tag{20}$$

is oscillatory and

$$\liminf_{n\to\infty} \Big(\frac{n-k}{2}\Big)\Big[\sum_{j=n-k}^{n-1} q_j - p_{n-k}\Big] > \Big(\frac{k}{1+k}\Big)^{1+k}, \tag{21}$$

then Eq. (E_3) *is almost oscillatory.*

P r o o f. Let $\{x_n\}$ be an eventually positive solution of Eq. (E_3), say $x_n > 0$ and $x_{n-k} > 0$ for $n \geqslant n_1 \geqslant n_0 \geqslant 0$. As in the proof of Theorem 6, we consider the following two cases:

(A) $\Delta^2 x_n > 0$ eventually, (B) $\Delta^2 x_n < 0$ eventually.

(A) Suppose $\Delta^2 x_n > 0$ eventually. Then there are two possibilities:

(A_1) $\Delta^2 x_n > 0$ and $\Delta x_n > 0$ eventually, (A_2) $\Delta^2 x_n > 0$ and $\Delta x_n < 0$ eventually.

(A_1) Suppose $\Delta^2 x_n > 0$ and $\Delta x_n > 0$ for $n \geqslant n_2 \geqslant n_1 + h$. There exist constants $c_1 > 0$ and $N \geqslant n_2$ such that (3) holds for $n \geqslant N$. From Eq. (E_3) and (3) we have

$$\Delta^3 x_n + c_1 q_n \leqslant 0 \quad \text{for } n \geqslant N. \tag{22}$$

Summing both sides of (22) from N to $m - 1 \geqslant N$, we have

$$0 < \Delta^2 x_m \leqslant \Delta^2 x_N - c_1 \sum_{n=N}^{m-1} q_n \to -\infty \quad \text{as} \quad m \to \infty,$$

a contradiction.

(A_2) Suppose $\Delta^2 x_n > 0$ and $\Delta x_n < 0$ eventually. From Eq. (E_3) we have

$$\Delta^3 x_m + q_n x_{n-k} \leqslant 0 \text{ eventually.} \tag{23}$$

But, by Theorem 1′ in [8], if (23) has an eventually positive solution, then (20) has an eventually positive solution as well, a contradiction.

(B) Suppose $\Delta^2 x_n < 0$ for $n \geqslant n_2 \geqslant n_1 + h$. Then $\Delta x_n > 0$ for $n \geqslant n_2$, and by Lemma 4.1 (d) in [7] there exists N sufficiently large, $N \geqslant 2n_2 + k$ such that

$$x_{n-k} \geqslant \Big(\frac{n-k}{2}\Big)\Delta x_{n-k} \quad \text{for} \quad n \geqslant N. \tag{24}$$

Using (24) in Eq. (E_3) and setting $y_n = \Delta x_n > 0$ for $n \geqslant N$, we have

$$\Delta^2 y_n + p_n \Delta y_{n-h} + \Big(\frac{n-k}{2}\Big) q_n y_{n-k} \leqslant 0 \quad \text{for} \quad n \geqslant N.$$

The rest of the proof is similar to the proof of Theorem 5 (B) and hence is omitted. This completes the proof. □

Next, we consider a special case of Eq. (E_3), namely the equations

$$\Delta^3 x_n + p\Delta^2 x_{n-h} q x_{n-k} = 0 \tag{(L_3)}$$

and

$$\Delta^3 x_n + q x_{n-k} = 0, \tag{(L_3^*)}$$

where p and q are positive constants, h and k are nonnegative integers.

From Corollary 2 in [8] we obtain the following results:

Corollary 2. *All bounded solutions of Eq.* (L_3^*) *are oscillatory if one of the following conditions holds:*

(i) $k = 0$ *and* $q \geqslant 1$;

(ii) $k \geqslant 1$ *and* $q > \frac{27k^k}{(3+k)^{3+k}}$.

Now, from Theorem 6 *and* 7 *and Corollary* 2, *we obtain the following result:*

Corollary 3. *Eq.* (L_3) *is almost oscillatory if one of the following conditions holds:*

(I) $h > 0$ *is odd and* $p > \frac{h^h}{(1+h)^{1+h}}$;

(II) $h \leqslant 0$ *is odd and* $q > \frac{27k^k}{(3+k)^{3+k}}$.

Remark 3. (i) If we set $p_n = 0$ in Theorem 6, we see that condition (1) is not sufficient to allow every solution of Eq. (E_3) with $p_n = 0$ to oscillate. This can be shown by consider the equation

$$\Delta^3 x_n + \Big(1 - \frac{1}{e}\Big)^3 x_n = 0,$$

which has a nonoscillatory solution $x_n = e^{-n}$.

Therefore, we conclude that the presence of p_n in Eq. (E_3) generates oscillations.

(ii) We note that Theorem 7 is applicable to Eq. (E_3) when $p_n = 0$. Only condition (21) is disregarded.

Acknowledgement. The author wishes to thank the referee for his very helpful comments and suggestions.

References

[1] *S. R. Grace*: On the oscillatory and asymptotic behavior of damped functional differential equations. Math. Japon. *36* (1991), 229–237.

[2] *S. R. Grace*: Oscillatory and asymptotic behavior of damped functional differential equations. Math. Nachr. *142* (1989), 279–305.

[3] *S. R. Grace*: Oscillation theorems for damped functional differential equations. Funkcial. Ekvac. *35* (1992), 261–278.

[4] *S. R. Grace, B. S. Lalli*: Oscillation theorems for second order delay and neutral difference equations. Utilitas Math. *45* (1994), 197–211.

[5] *S. R. Grace, B. S. Lalli*: Oscillation theorems for forced neutral difference equations. J. Math. Anal. Appl. *187* (1994), 91–106.

[6] *I. Györi, G. Ladas*: Oscillation Theory of Delay Differential Equations with Applications. Oxford University Press, Oxford, 1991.

[7] *J. W. Hooker, W. T. Patula*: A second order nonlinear difference equation: Oscillation and asymptotic behavior. J. Math. Anal. Appl. *91* (1983), 9–29.

[8] *G. Ladas, C. Qian*: Comparison results and linearized oscillations for higher order difference equations. Internat. J. Math. & Math. Sci. *15* (1992), 129–142.

[9] *B. S. Lalli, S. R. Grace*: Oscillation theorems for second order neutral difference equations. Appl. Math. Comput. *62* (1994), 47–60.

[10] *W. T. Patula*: Growth and oscillation properties of second order linear difference equations. SIAM J. Math. Anal. *10* (1979), 55–61.

[11] *Ch. G. Philos*: On oscillation of some difference equations. Funkcial. Ekvac. *34* (1991), 157–172.

[12] *F. Weil*: Existence theorem for the difference equation $y_{n+1} - 2y_n + y_{n-1} = h^2 f(y_n)$. Internat. J. Math. & Math. Sci. *3* (1990), 69–77.

Author's address: Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12211, A.R. of Egypt.