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PTÁK HOMOMORPHISM THEOREM REVISITED

J. R. Ferrer, M. López Pellicer and L. M. Sánchez Ruiz, Valencia

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Abstract. Rodrigues' extension (1989) of the classical Pták's homomorphism theorem to a non-necessarily locally convex setting stated that a nearly semi-open mapping between a semi-B-complete space and an arbitrary topological vector space is semi-open. In this paper we study this extension and, as a consequence of the results obtained, provide an improvement of Pták's homomorphism theorem.

Keywords: homomorphism theorem, B-complete space

MSC 2000: 46A30

1. Introduction

Pták's theory characterizes B-complete spaces as those locally convex spaces (l.c.s.) E for which every continuous nearly open linear mapping of E into any l.c.s. F is open (cf. [2, 11]). These results were carried over to a non-locally convex setting by Adasch (cf. [1]) and Sánchez Ruiz [9] who mixed locally convex and non-locally convex spaces, both of them getting a generalization of Pták spaces in the absence of local convexity conditions and preserving completeness. This did not happen in Rodrigues' extension to the non-locally convex setting [4, 5] which gives Pták's theorem when l.c.s. are considered in the range. And even when a homomorphism theorem in his general setting is not achieved since only semi-openness is ensured in the conclusion, there are interesting ideas in his development that we aim to focus on and clarify in this paper from a different viewpoint. As a consequence of the results obtained here we are able to furnish a new homomorphism theorem with weaker conditions than in Pták's.

Throughout this paper the word 'space' will stand for 'Hausdorff topological vector space over the field of real numbers' and our notation is standard. Let us just recall

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that if $\mathscr{U}$ and $\mathscr{V}$ are the sets of all neighbourhoods of the spaces E and F, then a linear mapping f of E into F is nearly open if $\overline{f(U)} \in \mathscr{V}$ for each $U \in \mathscr{U}$. If E is locally convex, then E is B-complete (or a Pták space) if each subspace L of its dual E', such that $L \cap U^{\circ}$ is $\sigma(E', E)$-closed for all $U \in \mathscr{U}$, is $\sigma(E', E)$-closed. And following [1], given a space $E(\tau)$, $\tau^{\circ\circ}$ will denote its associated locally convex topology which is generated by all absolutely convex neighbourhoods of the origin.

2. New reading and consequences of Rodrigues' ideas

One of the concepts used by Rodrigues [4] is that of an adequate map between spaces. To this aim he first considers what he calls *small adjoint* of a linear map T between two spaces E and F, i.e. the mapping T^t of the set $\Delta = \{d \in \overline{T(E)}'\colon\ d \circ T \in E'\}$, with $\overline{T(E)}$ endowed with the topology induced by F, into E' defined by $T^t(d) = d \circ T$.

Definition 2.1. A linear map T between spaces E and F is said to be adequate if given $x \in E$, $a \in E'$, then $\langle x, T^t(\Delta)\rangle = \{0\}$ and $\langle \operatorname{Ker} T, a\rangle = \{0\}$ implies $\langle x, a\rangle = 0$.

Remark 2.1. It is easy to check the statement on [4, p. 324] that if F is locally convex and T is continuous, then T is adequate. In fact, under these circumstances continuity of T implies that $\Delta = \overline{T(E)}'$ and since $\langle \overline{T(E)}, \overline{T(E)}'\rangle$ is a dual pairing, condition $\langle x, T^t(\Delta)\rangle = \{0\}$ implies that $\langle Tx, d\rangle = 0$ for all $d \in \overline{T(E)}'$ and consequently $Tx = 0$, i.e. $x \in \operatorname{Ker} T$. Then, condition $\langle \operatorname{Ker} T, a\rangle = \{0\}$ gives that $\langle x, a\rangle = 0$ and T is adequate. The existence of non-continuous adequate maps between locally convex spaces will be easily concluded from our Proposition 2.2 which gives a characterization of adequate maps via duality arguments.

We will identify Δ with $\Lambda = \{d \in (T(E))'\colon\ d \circ T \in E'\}$, with $T(E)$ endowed with the topology induced by F in a natural way since each continuous linear form on $T(E)$ can be easily extended to a continuous linear form on its closure.

Lemma 2.1. *Let τ_T be the final topology on $T(E)$ induced by T, and let τ_0 be the infimum of the topologies τ_T and the one induced by F on $T(E)$. Then $\Lambda = \big(T(E)(\tau_0)\big)'$.*

P r o o f. Assume $d \in \Lambda$, and let U be an open zero neighbourhood in $\mathbb{K}$. Then $T^{-1}(d^{-1}(U))$ is an open neighbourhood of the origin in E, and therefore $d^{-1}(U)$ is an open neighbourhood of the origin in $T(E)(\tau_T)$, thus $d \in \big(T(E)(\tau_T)\big)'$. On the other hand, $d^{-1}(U)$ is also an open neighbourhood of the origin in $T(E)$ with the topology induced by F since d is continuous on $T(E)$. Hence $d^{-1}(U)$ is an open neighbourhood of the origin in $T(E)(\tau_0)$, thus $d \in \big(T(E)(\tau_0)\big)'$.

Conversely, given $h \in \big(T(E)(\tau_0)\big)'$, $h \in \big(T(E)(\tau_T)\big)'$ and since the linear map of E onto $T(E)(\tau_T)$ is continuous, $h \circ T \in E'$. Also, h is continuous on $T(E)$ endowed with the topology induced by F, which implies that $h \in \Lambda$. $\square$

With the notation of the above lemma, we have the following

Proposition 2.2. *Let T be a linear map of E into F. The following assertions are equivalent:*

(i) *T is adequate.*

(ii) *$\big(T(E)(\tau_0)\big)'$ is $\sigma\big(\big(T(E)(\tau_T)\big)', T(E)(\tau_T)\big)$-dense in $\big(T(E)(\tau_T)\big)'$.*

Proof. [(i) $\Rightarrow$ (ii)]. Let A be the $\sigma\big(\big(T(E)(\tau_T)\big)', T(E)(\tau_T)\big)$-closure of $\big(T(E)(\tau_0)\big)'$ in $\big(T(E)(\tau_T)\big)'$ and suppose that $A \subsetneq \big(T(E)(\tau_T)\big)'$. Then the theorem of Hahn-Banach applied to each $\psi \in \big(T(E)(\tau_T)\big)' \setminus A$ gives a $Tx \in T(E)$ such that $\psi(Tx) = 1$ and $\langle Tx, A\rangle = \{0\}$. In particular this yields that $\langle Tx, \Delta\rangle = \{0\}$, or equivalently $\langle x, T^t(\Delta)\rangle = \{0\}$.

On the other hand if we consider $a \in E'$ defined by $\langle x, a\rangle = \langle Tx, \psi\rangle$, then it satisfies $\langle \operatorname{Ker} T, a\rangle = \langle T(\operatorname{Ker} T), \psi\rangle = \{0\}$. However, $\langle x, a\rangle = \psi(Tx) = 1$ and therefore T is not adequate.

[(ii) $\Rightarrow$ (i)]. Let $x \in E$, $a \in E'$ be such that $\langle x, T^t(\Delta)\rangle = \{0\}$ and $\langle \operatorname{Ker} T, a\rangle = \{0\}$. By the above lemma the former of these conditions is equivalent to $\varphi(Tx) = 0$ for all $\varphi \in \big(T(E)(\tau_0)\big)'$. Then (ii) implies that $\big\langle Tx, \big(T(E)(\tau_T)\big)'\big\rangle = \{0\}$.

Now $\langle \operatorname{Ker} T, a\rangle = \{0\}$ enables us to consider $\widehat{a} \in \big(T(E)(\tau_T)\big)'$ defined by $\widehat{a}(Tx) = ax$. Therefore $\widehat{a}(Tx) = 0$, i.e. $\langle x, a\rangle = 0$ and T is adequate. $\square$

Remark 2.2. Proposition 2.2 enables us to find non-continuous adequate maps between locally convex spaces, just by considering the identity map $i\colon E(\tau_1) \to E(\tau_2)$, with τ_1 strictly coarser than τ_2, since then $\tau_0 = \inf\{\tau_1, \tau_2\} = \tau_1$ and $(E(\tau_0))'$ is $\sigma(E^*, E)$-dense in E^*.

Given a linear map T between the spaces E and F and a seminorm P on E, a new seminorm P/T is considered on $T(E)$ by means of $P/T(y) = \inf P(T^{-1}y)$ for each $y \in T(E)$. Then T is semi-open [4] if P_T is continuous on $T(E)$ whenever P is continuous on E. This definition is tightly related to openness with the associated locally convex topologies and thus Theorem 10 of [4] provides a homomorphism theorem for these topologies. This follows from the fact that P is continuous if and only if $\{x \in E\colon P(x) < 1\}$ is open in E, that P/T is continuous if and only if $\{y \in T(E)\colon P/T(y) < 1\}$ is open in $T(E)$, and that $T(\{x \in E\colon P(x) < 1\}) = \{y \in T(E)\colon P/T(y) < 1\}$. And therefore T is semi-open if and only if for each seminorm P on E such that $\{x \in E\colon P(x) < 1\}$ is open, then $T(\{x \in E\colon P(x) < 1\})$ is open. Bearing this in mind, we have

Proposition 2.3. *Let T be a linear map between the spaces $E(\tau_1)$ and $F(\tau_2)$. The following assertions are equivalent:*

(i) *T is semi-open.*

(ii) *$T\colon E(\tau_1^{\circ\circ}) \to F(\tau_2^{\circ\circ})$ is open.*

A space E is semi-B-complete [4] if each subspace L of E' such that $L \cap E_P^*$ is $\sigma(E', E)$-compact in E' for all continuous seminorms P on E, is $\sigma(E', E)$-closed. In this definition $E_P^* = \{a \in E^*\colon a \leqslant P \text{ on } E\}$. A standard argument gives the following characterization of this space as well as of $E'_P = \{a \in E'\colon a \leqslant P \text{ on } E\}$ through duality.

Lemma 2.4. *$E'_P = \{x \in E\colon P(x) < 1\}^\circ$ with the polar taken in $\langle E, E'\rangle$ and $E_P^* = \{x \in E\colon P(x) < 1\}^\circ$ with the polar taken in $\langle E, E^*\rangle$.*

Therefore E is semi-B-complete if and only if each subspace L of E' such that $L \cap E'_P = L \cap \{x \in E\colon P(x) < 1\}^\circ$ is $\sigma(E', E)$-compact in E' for all continuous seminorms P on E, is $\sigma(E', E)$-closed. Hence it follows that if a space $E(\tau)$ is semi-B-complete, then $E(\tau^{\circ\circ})$ is B-complete. This property reminds the one of subbarrelled spaces [8] (introduced for the first time in [6] and called semi-barrelled in [4])—those in which every barrel is a neighbourhood of the origin (cf. [10, Remark 2.1])—for which the implication $E(\tau)$ subbarrelled $\Rightarrow E(\tau^{\circ\circ})$ barrelled holds, the question whether the converse is true remaining open. Subbarrelled spaces are precisely the class of spaces characterized by satisfying the closed graph theorem when Fréchet or Banach spaces are considered in the range, [8], whilst the spaces $E(\tau)$ for which $E(\tau^{\circ\circ})$ is barrelled are the class of spaces characterized by satisfying the Banach-Steinhaus theorem when Fréchet or Banach spaces are considered in the range, [7].

Rodrigues [4, Theorem 10] states that a nearly semi-open map between a semi-B-complete space and an arbitrary topological vector space is semi-open. Here a nearly semi-open map stands for a map T between two spaces E and F such that for every continuous seminorm P on E, there exists a continuous seminorm Q on $T(E)$ such that, if $b \in T(E)'$ and $b \circ T \in E'_P$, then $b \in T(E)'_Q$. Next we point out that, in the same way as with semi-open and open maps, there is a tight relation between this property and nearly openness with the associated locally convex topologies.

Proposition 2.5. *Let T be a linear map between the spaces $E(\tau_1)$ and $F(\tau_2)$. The following assertions are equivalent:*

(i) *T is nearly semi-open.*

(ii) *$T\colon E(\tau_1^{\circ\circ}) \to F(\tau_2^{\circ\circ})$ is nearly open.*

P r o o f. Note that $b \circ T \in E'_P$ is equivalent to saying that $|b(Tx)| \leqslant 1$ whenever $P(x) < 1$ since $E'_P = \{x \in E\colon\ P(x) < 1\}^\circ$. Thus (i) means that $\{b \in T(E)'\colon |b(Tx)| \leqslant 1 \text{ whenever } P(x) < 1\} \subseteq T(E)'_Q$, which is equivalent to $T(\{x \in E\colon\ P(x) < 1\})^\circ \subseteq \{y \in T(E)\colon\ Q(y) < 1\}^\circ$, where the polars are taken in $\langle T(E), T(E)'\rangle$. Taking polars again, the above is equivalent to the inclusion $\{y \in T(E)\colon\ Q(y) \leqslant 1\} \subseteq \overline{T(\{x \in E\colon\ P(x) < 1\})}$ with the closure taken in $\langle T(E), T(E)'\rangle$, i.e. that the closure of the image by means of T of a $\tau_1^{\circ\circ}$-neighbourhood of the origin is a neighbourhood of the origin in $T(E)$ with the topology induced by $\tau_2^{\circ\circ}$, which is precisely (ii). □

This proposition clearly generalizes [4, Lemma 7] making its proof immediate.

3. The homomorphism theorem

Theorem 3.1. *Let E be a locally convex space. The following assertions are equivalent:*

(i) *Each adequate nearly open linear mapping T of E into any l.c.s. F is a strict morphism.*

(ii) *E is a Pták space.*

P r o o f. [(i) $\Rightarrow$ (ii)]. This implication follows from Pták's theory and Remark 2.1.

[(ii) $\Rightarrow$ (i)] We will follow the notation of Lemma 2.1 and assume that T is onto, which means no loss of generality. Then Proposition 2.2 implies that $\Delta = \{d \in F'\colon d \circ T \in E'\} = (F(\tau_0))'$ is $\sigma\big((F(\tau_T))', F(\tau_T)\big)$-dense in $(F(\tau_T))'$. On the other hand, for each neighbourhood of the origin U in E, $U^\circ \cap T^t(\Delta) = T^t(T(U)^\circ) = T^t(\overline{T(U)}^\circ)$ is $\sigma(E', E)$-compact and therefore $\sigma(E', E)$-closed. Under (ii) this gives that $T^t(\Delta)$ is $\sigma(E', E)$-closed and therefore Δ is $\sigma\big((F(\tau_T))', F(\tau_T)\big)$-closed. Hence $\Delta = (F(\tau_T))'$ and both τ_0 and τ_T generate the same dual.

In order to see that T is open it suffices to check that the topology on F is finer than τ_T. Let V be a τ_T-barrel τ_T -neighbourhood of the origin in F. Then there is a neighbourhood of the origin in E such that $T(U) \subseteq V$, and therefore $\overline{T(U)}^{\tau_T} \subseteq V$. The above coincidence of duals yields $\overline{T(U)}^{\tau_T} = \overline{T(U)}^{\tau_0}$, and since the topology on F is finer than τ_0, it follows that $\overline{T(U)} \subseteq \overline{T(U)}^{\tau_0} \subseteq V$. Finally, nearly openness of T implies that $\overline{T(U)}$ is a neighbourhood of the origin in F, therefore V is also a neighbourhood of the origin in F, and T is a strict morphism. □

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Authors' address: Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, E-46071 Valencia, Spain, e-mail: `lmsr@mat.upv.es`.