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INEQUALITIES INVOLVING INDEPENDENCE DOMINATION, f-DOMINATION, CONNECTED AND TOTAL f-DOMINATION NUMBERS

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Abstract. Let f be an integer-valued function defined on the vertex set $V(G)$ of a graph G. A subset D of $V(G)$ is an f-dominating set if each vertex x outside D is adjacent to at least $f(x)$ vertices in D. The minimum number of vertices in an f-dominating set is defined to be the f-domination number, denoted by $\gamma_f(G)$. In a similar way one can define the connected and total f-domination numbers $\gamma_{c,f}(G)$ and $\gamma_{t,f}(G)$. If $f(x) = 1$ for all vertices x, then these are the ordinary domination number, connected domination number and total domination number of G, respectively. In this paper we prove some inequalities involving $\gamma_f(G), \gamma_{c,f}(G), \gamma_{t,f}(G)$ and the independence domination number $i(G)$. In particular, several known results are generalized.

Keywords: domination number, independence domination number, f-domination number, connected f-domination number, total f-domination number

MSC 2000: 05C90, 05C99

1. Introduction

A *dominating set* of a graph $G = (V(G), E(G))$ is a subset D of $V(G)$ such that each vertex outside D is adjacent to at least one vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum number of vertices in a dominating set of G. For a given positive integer n, a subset D of $V(G)$ is an *n-dominating set* if each vertex outside D is adjacent to at least n vertices in D [4, 5]. The smallest cardinality of an n-dominating set is the *n-domination number* [4, 5], denoted by $\gamma_n(G)$. Clearly, the 1-domination number is just the ordinary domination number. In [8], a more general domination concept was introduced. For a given integer-valued function f

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defined on the vertices of G, a subset D of $V(G)$ is an *f-dominating set* if each vertex $x \in V(G) - D$ is adjacent to at least $f(x)$ vertices in D. The *f-domination number* $\gamma_f(G)$ of G was defined in [10] to be the minimum cardinality of an f-dominating set of G. The authors of [8] discussed the $f_{j,k}$-domination number and thus gave some estimations for n-domination number, where j, k are given integers with $0 \leqslant j \leqslant k$, $f_{j,k}(x) = \min\{j, j-k+d(x)\}$ for $x \in V(G)$, and $d(x)$ is the degree of x in G. A study on general f-domination number was initiated in [10]. An f-dominating set D of G is said to be a *connected f-dominating set* of G [11] if the subgraph $G[D]$ induced by D is connected. Note that if G is connected, then connected f-dominating sets of G exist since $V(G)$ is such a set. In such a case the *connected f-domination number* $\gamma_{c,f}(G)$ was defined in [11] to be the minimum cardinality of a connected f-dominating set of G. A subset D is a *total f-dominating set* of G [11] if each vertex x of G is adjacent to at least $f(x)$ vertices in D. Obviously, G contains total f-dominating sets if and only if $f(x) \leqslant d(x)$ for all vertices $x \in V(G)$. If this is the case we define [11] the *total f-domination number* of G, denoted by $\gamma_{t,f}(G)$, to be the minimum cardinality of a total f-dominating set of G. Results for $\gamma_{c,f}(G)$ and $\gamma_{t,f}(G)$ were obtained in [11], and several Gallai-type equalities for $\gamma_f(G)$, $\gamma_{c,f}(G)$ and some other invariants concerning f were given in [12]. In particular, it was shown that $\gamma_f(G) + \beta_{f^*}(G) = |V(G)|$ for any f, where f^* is defined by $f^*(x) = d(x) - f(x) + 1$ for $x \in V(G)$ and $\beta_f(G)$ is the maximum cardinality of an *f-independent set* of G, that is a subset X of $V(G)$ such that each vertex $x \in X$ has degree less than $f(x)$ in $G[X]$. This tightens the inequality $\gamma_f(G) + \beta_{f^*}(G) \leqslant |V(G)|$ observed in [11] earlier. We note that Theorems 2, 3 and 5 of [4] can be generalized to f-domination number immediately. In fact, one can check that $\gamma_f(G) \geqslant \frac{f(V(G))}{\Delta(G)+M(f)}$, and that $\gamma_f(G) \geqslant \gamma(G) + \max\{0, m(f)-2\}$ if $m(f) \geqslant 1$, where we denote $m(f) = \min\limits_{x \in V(G)} f(x)$, $M(f) = \max\limits_{x \in V(G)} f(x)$. Furthermore, we have $\gamma_f(G) = \min \gamma_f(H)$, where the minimum is taken over all spanning bipartite subgraphs H of G.

Until recently we noticed that the concept of f-domination appeared in [7] in a slightly different way. Let the vertices of G be $x_1, x_2, \ldots, x_p$ and the degrees of these vertices be $d_1, d_2, \ldots, d_p$, respectively. Suppose that an integer b_i is associated with each vertex x_i, where $0 \leqslant b_i \leqslant d_i$, and denote $\mathbf{b} = (b_1, b_2, \ldots, b_p)$. A set D of vertices of G is a **b**-*dominating set* [7] if each $x_i \in V(G) - D$ is adjacent to at least b_i vertices in D. The minimum number of vertices in a **b**-dominating set was defined in [7] to be the **b**-*domination number* of G. Clearly, if f is the function defined by $f(x_i) = b_i, 1 \leqslant i \leqslant p$, then the **b**-domination number is just the f-domination number.

The concept of f-domination number has the following practical interpretation. Suppose we are given, say, a communication network, and we are asked to construct information centers at some of the existing nodes of the network in such a way that each node is either a center or can communicate directly with at least the given number of centers. At least how many centers should we construct? If the given number for node x is $f(x)$, then the minimum number of centers required is exactly the f-domination number of the network.

As a continuation of [10, 11], we will in this paper prove some inequalities involving $\gamma_f(G), \gamma_{c,f}(G), \gamma_{t,f}(G)$ and $i(G)$, where $i(G)$ is the *independence domination number*, that is, the minimum cardinality of a maximal independent set of G. In the paper we always suppose G is a simple graph with p vertices and f is a function from $V(G)$ to the set of nonnegative integers. We say f is *proper* if $1 \leqslant f(x) \leqslant d(x)$ for each vertex x. Note that G admits a proper f only if it contains no isolated vertices. An f-dominating set with the minimum cardinality is called a *minimum f-dominating set*. The similar terminology will be used for connected and total f-dominating sets. For $X \subseteq V(G)$, let $\overline{X} = V(G) - X$ and $G[X]$ be the subgraph of G induced by X. Denote $f(X) = \sum\limits_{x\in X} f(x)$ and $N(X) = \{y \in \overline{X}\colon$ there exists a vertex in X which is adjacent to $y\}$. In particular, $N(x)$ is the set of neighbours of x. In the case where a possible ambiguity exists we write $d_G(x)$, $N_G(X)$, $N_G(x)$ instead of $d(x)$, $N(X)$, $N(x)$ to emphasis that the underlying graph is G. The maximum and minimum degrees of the vertices of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Let $K_{1,k+1}$ denote the star on $k+2$ vertices (i.e., the tree on $k+2$ vertices with maximum degree $k+1$). The graph G is said to be $K_{1,k+1}$-*free* if it has no induced subgraph isomorphic to $K_{1,k+1}$. For a real number a, $\lceil a \rceil$ denotes the smallest integer no less than a.

2. Relationships between $\gamma_f(G)$ and $i(G)$

It was shown in [10] that there exists a subset of $V(G)$ which is both f-dominating and f-independent. Evidently such a subset must be a maximal f-independent set, but not conversely even if f is proper. For example, if G is the windmill graph with vertices $x_0, x_1, \ldots, x_6$ and edges x_1x_2, x_3x_4, x_5x_6 and $x_0x_i, 1 \leqslant i \leqslant 6$, then for the proper function f defined by $f(x_0) = 6$, $f(x_i) = 1, 1 \leqslant i \leqslant 6$, $\{x_1, x_3, x_5\}$ is a maximal f-independent set but not an f-dominating set of G. This is quite different from the situation of the ordinary case where a set $D \subseteq V(G)$ is a maximal independent set if and only if it is both dominating and independent. Thus in that case we have $\gamma(G) \leqslant i(G)$. Allan and Laskar [1] proved that if G is $K_{1,3}$-free, then $i(G) \leqslant \gamma(G)$ and hence $\gamma(G) = i(G)$. This was generalized by Bollobás and Cockayne [2] who proved that if G is $K_{1,k+1}$-free $(k \geqslant 2)$, then $i(G) \leqslant (k-1)\gamma(G) - (k-2)$. Based on a similar idea, we now give a further generalization of this latter result.

Theorem 1. *If G is $K_{1,k+1}$-free and $k \geqslant \max\{2, m(f)+1\}$, then*

$$m(f)i(G) \leqslant (k-1)\gamma_f(G) - (k - m(f) - 1). \tag{1}$$

Proof. Let D be a minimum f-dominating set of G and D_1 a maximal independent set of $G[D]$. Let W be the subset of $\overline{D}$ consisting of such vertices that are not adjacent to any vertex in D_1. We divide the proof into two cases.

Case 1. $W = \emptyset$.

Then each vertex in $\overline{D}$ is adjacent to at least one vertex in D_1. Thus, D_1 is a maximal independent set of G and hence $i(G) \leqslant |D_1| \leqslant |D| = \gamma_f(G)$. Since $m(f) \leqslant k-1$, we have $\gamma_f(G) \geqslant i(G) \geqslant \frac{m(f)(i(G)-1)}{k-1} + 1$, which implies the required inequality.

Case 2. $W \neq \emptyset$.

Let W_1 be a maximal independent set of $G[W]$. By the definition of W, $W_1 \cup D_1$ is a maximal independent set of G and hence $i(G) \leqslant |W_1| + |D_1|$. Let $\varepsilon(D - D_1, W_1)$ be the number of edges joining the vertices of $D - D_1$ and W_1. Since each vertex in W_1 is adjacent to at least $f(x)$ vertices in $D - D_1$, we have $f(W_1) \leqslant \varepsilon(D - D_1, W_1)$. On the other hand, since G is $K_{1,k+1}$-free and each vertex in $D - D_1$ must be adjacent to at least one vertex in D_1, we know that each vertex in $D - D_1$ is adjacent to at most $k-1$ vertices in W_1. Thus, $\varepsilon(D - D_1, W_1) \leqslant (k-1)|D - D_1| = (k-1)(\gamma_f(G) - |D_1|)$. So $m(f)|W_1| \leqslant f(W_1) \leqslant \varepsilon(D - D_1, W_1) \leqslant (k-1)(\gamma_f(G) - |D_1|)$. Therefore we have $m(f)i(G) \leqslant m(f)|W_1| + m(f)|D_1| \leqslant (k-1)(\gamma_f(G) - |D_1|) + m(f)|D_1| = (k-1)\gamma_f(G) - (k - m(f) - 1)|D_1|$, which implies (1) since $|D_1| \geqslant 1$. □

In general the extremal graphs for (1) are not unique but the structure of them is clear. Suppose G is such an extremal graph, that is $m(f)i(G) = (k-1)\gamma_f(G) - (k - m(f) - 1)$. If, using the notations in the proof above, $W = \emptyset$, then $m(f) = k-1$ and (1) becomes $i(G) = \gamma_f(G)$. From the proof we know $i(G) = |D_1| = |D| = \gamma_f(G)$ and hence $D_1 = D$. That is, D is a maximal independent set of G with the minimum cardinality $i(G)$. Furthermore, each vertex in $\overline{D}$ is adjacent to at most k vertices in D since G is $K_{1,k+1}$-free. Now we suppose G is an extremal graph with $W \neq \emptyset$. From the proof of (1) we have

(a) $|D_1| = 1$ for any choice of D_1, and hence $G[D]$ is a complete graph;

(b) $i(G) = |W_1| + |D_1| = |W_1| + 1 = \frac{(k-1)(\gamma_f(G)-1)}{m(f)} + 1$;

(c) $\forall x \in W_1$, $f(x) = m(f)$ and x is adjacent to exactly $m(f)$ vertices in $D - D_1$ and

(d) $\forall y \in D - D_1$, y is adjacent to exactly $k-1$ vertices in W_1.

Since W_1 is any maximal independent set of $G[W]$, we have from (c) that

(c1) $\forall x \in W$, $f(x) = m(f)$ and x is adjacent to exactly $m(f)$ vertices in D.

Let $\widetilde{D} = \{x \in \overline{D}\colon\ x$ is adjacent to all vertices in $D\}$. For each $x \in \overline{D} - \widetilde{D}$, there exists at least one vertex y in D which is not adjacent to x. Taking $D_1 = \{y\}$, (c1) implies

(c2) $\forall x \in \overline{D} - \widetilde{D}$, $f(x) = m(f)$ and x is adjacent to exactly $m(f)$ vertices in D.

From (d) we then have

(d1) $\forall y \in D$, y is adjacent to at least $k - 1$ vertices in $\overline{D} - \widetilde{D}$.

Note that (c2) and (d1) imply $m(f) \geqslant 1$. For each $y \in D$ denote by W^y the set of vertices in $\overline{D} - \widetilde{D}$ which are not adjacent to y. From (b) and (d1) we have

(d2) $\forall y \in D$, any maximal independent set W_1^y of $G[W^y]$ contains exactly $k - 1$ neighbours of any other vertex in D, and $W_1^y \cup \{y\}$ is a maximal independent set of G with the minimum cardinality $i(G)$.

In summary we know the equality in (1) holds only if one of the following two sets of conditions is satisfied:

(i) $i(G) = \gamma_f(G)$, $m(f) = k - 1$, each minimum f-dominating set D of G is a maximal independent set with the minimum cardinality $i(G)$, and each vertex in $\overline{D}$ is adjacent to at most k vertices in D;

(ii) any minimum f-dominating set D induces a complete graph and G has the following structure: $V(G) = D \cup \widetilde{D} \cup (\bigcup\limits_{y \in D} S^y)$, where $\widetilde{D}, S^y$ satisfy

(ii1) $\widetilde{D} \subseteq \overline{D}$, each vertex in $\widetilde{D}$ is adjacent to all vertices in D, and for each $x \in \overline{D} - \widetilde{D}$, $f(x) = m(f)$ and x is adjacent to exactly $m(f)$ vertices in D; and

(ii2) $S^y = N(y) \cap (\overline{D} - \widetilde{D})$, any maximal independent set of $G[\overline{D} - \widetilde{D} - S^y]$ contains exactly $k-1$ neighbors of each vertex of D different from y and such an independent set together with y consists of an independent set of G with cardinality $i(G)$.

Conversely, one can check that if (i) or (ii) is satisfied, then G is an extremal graph for (1). Note that if the maximum independence number of the subgraphs induced by the minimum f-dominating sets of G is $b(G)$, then from the proof of (1) we actually have

$$m(f) i(G) \leqslant (k - 1)\gamma_f(G) - (k - m(f) - 1) b(G),$$

which could be much better than (1) in some cases. Theorem 1 implies the following

Corollary 1. *If G is $K_{1,k+1}$-free and $k \geqslant \max\{2, n + 1\}$, then*

$$i(G) \leqslant \frac{1}{n}\{(k - 1)\gamma_n(G) - (k - n - 1)\}.$$

In particular, we have

Corollary 2. (Bollobás and Cockayne [2]) *If G is $K_{1,k+1}$-free ($k \geqslant 2$), then*

$$i(G) \leqslant (k-1)\gamma(G) - (k-2).$$

Now we give more inequalities concerning $\gamma_f(G)$ and $i(G)$. It is easy to see that $\gamma_f(G) = p$ holds if and only if $f(x) > d(x)$ for all vertices x of G. In the remainder of this section we suppose this is not the case. So we have $\gamma_f(G) < p$. For $U \subseteq V(G)$, denote $\delta(U) = |U| - |N(U)|$. For a minimum f-dominating set D of G, define $\delta(G, \overline{D}) = \max\limits_{U \subseteq \overline{D}} \delta(U)$ and $\delta(G, D) = \max\limits_{U \subseteq D} \delta(U)$. Then we have

Theorem 2. *Let D be a minimum f-dominating set of G. Then*

$$i(G) \leqslant \gamma_f(G) + \delta(G, \overline{D}) - \left\lceil \frac{f(\overline{D})}{\gamma_f(G)} \right\rceil + 1, \tag{2}$$

$$i(G) \leqslant \gamma_f(G) + \delta(G, \overline{D}) - \left\lceil \frac{f(\overline{D})}{p - \gamma_f(G)} \right\rceil + 1. \tag{3}$$

P r o o f. Let G_D be the bipartite graph with bipartition $(D, \overline{D})$ whose edges are those of G with one end-vertex in D and the other in $\overline{D}$. Since $\varepsilon(D, \overline{D}) \geqslant \sum\limits_{x \in \overline{D}} f(x) = f(\overline{D})$, there exists a vertex $z \in D$ which is adjacent to at least $\left\lceil \frac{f(\overline{D})}{\gamma_f(G)} \right\rceil$ vertices in $\overline{D}$. Let $A = N(z) \cap \overline{D}$ and X be a maximal independent set of G containing z. Then $X \subseteq V(G) - A$. For a maximum matching M of G_D and each edge $xy \in M$, at least one of x, y is not in X. So there are at least $|M| - 1$ vertices of $V(G) - A$ which are not in X. Thus, we have

$$i(G) \leqslant |X| \leqslant p - |A| - (|M| - 1) \leqslant p - \beta'(G_D) - \left\lceil \frac{f(\overline{D})}{\gamma_f(G)} \right\rceil + 1, \tag{4}$$

where $\beta'(G_D) = |M|$ is the edge independence number of G_D. By the Main Matching Theorem of [6, pp. 127], we have $\beta'(G_D) = |\overline{D}| - \delta(G, \overline{D})$. Pluging this into the right-hand side of (4) and noting that $|\overline{D}| = p - \gamma_f(G)$, we get (2). Similarly, there exists a vertex $z \in \overline{D}$ which is adjacent to at least $\left\lceil \frac{f(\overline{D})}{p - \gamma_f(G)} \right\rceil$ vertices in D. Let $B = N(z) \cap D$ and Y be a maximal independent set of G containing z. Then $Y \subseteq V(G) - B$. By an analogous argument as above we get $i(G) \leqslant |Y| \leqslant p - |B| - \beta'(G_D) + 1 \leqslant p - \beta'(G_D) - \left\lceil \frac{f(\overline{D})}{p - \gamma_f(G)} \right\rceil + 1$, which implies (3). □

Dually, if we use $\beta'(G_D) = |D| - \delta(G, D)$ (see [6, pp. 127]) in the proof above, then we get

Theorem 3. *Let D be a minimum f-dominating set of G. Then*

$$i(G) \leqslant p - \gamma_f(G) + \delta(G,D) - \left\lceil \frac{f(\overline{D})}{\gamma_f(G)} \right\rceil + 1, \tag{5}$$

$$i(G) \leqslant p - \gamma_f(G) + \delta(G,D) - \left\lceil \frac{f(\overline{D})}{p - \gamma_f(G)} \right\rceil + 1. \tag{6}$$

If $f(x) \leqslant \frac{d(x)+1}{2}$ for all $x \in V(G)$, then $\gamma_f(G) \leqslant p - \beta_f(G)$ as shown in [10]. In such a case, (2) and (5) are better than (3) and (6), respectively. On the other hand, one can find examples for which (3) and (6) are better than (2) and (5), respectively.

Now we give two ways for estimating $\delta(G,D)$ and $\delta(G,\overline{D})$. If $U \subseteq \overline{D}$, then $|N(U)| \geqslant \max_{x\in U} f(x)$ and hence $\delta(G,\overline{D}) \leqslant \max_{U\subseteq\overline{D}}(|U| - \max_{x\in U} f(x))$ (note that the right-hand side of this inequality is nonnegative). Now suppose $m(f) \geqslant 1$, $U \subseteq D$, and G contains no isolated vertices. Let G' be the graph obtained from G by deleting, for each $x \in \overline{D}$, $|N(x)\cap D| - f(x)$ edges connecting x and the vertices in $N(x)\cap D$ such that the number of edges of G' incident with the vertices in U is as large as possible. Then $x \in \overline{D}$ has exactly $f(x)$ neighbours in G' and $U = \bigcup_{x\in N_{G'}(U)} (N_{G'}(x)\cap U)$. So we have $|U| \leqslant \sum_{x\in N_{G'}(U)} |N_{G'}(x)\cap U| \leqslant M(f)|N_{G'}(U)|$, which implies $|N(U)| \geqslant |N_{G'}(U)| \geqslant \left\lceil \frac{|U|}{M(f)} \right\rceil$. Thus, $\delta(G,D) \leqslant \max_{U\subseteq D}(|U| - \left\lceil \frac{|U|}{M(f)} \right\rceil) = |D| - \left\lceil \frac{|D|}{M(f)} \right\rceil$. By using these estimations we get the following two corollaries of Theorems 2 and 3.

Corollary 3. *Let D be a minimum f-dominating set of G. Then*

$$i(G) \leqslant \gamma_f(G) + \max_{U\subseteq\overline{D}}(|U| - \max_{x\in U} f(x)) - \left\lceil \frac{f(\overline{D})}{\gamma_f(G)} \right\rceil + 1, \tag{7}$$

$$i(G) \leqslant \gamma_f(G) + \max_{U\subseteq\overline{D}}(|U| - \max_{x\in U} f(x)) - \left\lceil \frac{f(\overline{D})}{p - \gamma_f(G)} \right\rceil + 1. \tag{8}$$

Corollary 4. *Suppose that G has no isolated vertices and $m(f) \geqslant 1$, and let D be a minimum f-dominating set of G. Then*

$$i(G) \leqslant p - \left\lceil \frac{\gamma_f(G)}{M(f)} \right\rceil - \left\lceil \frac{f(\overline{D})}{\gamma_f(G)} \right\rceil + 1, \tag{9}$$

$$i(G) \leqslant p - \left\lceil \frac{\gamma_f(G)}{M(f)} \right\rceil - \left\lceil \frac{f(\overline{D})}{p - \gamma_f(G)} \right\rceil + 1. \tag{10}$$

We can use $f(\overline{D}) \geqslant m(f)(p - \gamma_f(G))$ or $f(\overline{D}) = f(V(G)) - f(D) \geqslant f(V(G)) - M(f)\gamma_f(G)$ to slacken the right-hand sides of (9)–(10) and get inequalities which do not depend on D. Corollary 3 implies

Corollary 5. *If $n + \gamma_n(G) \leqslant p$, then*

$$i(G) \leqslant p - \left\lceil \frac{np}{\gamma_n(G)} \right\rceil + 1$$

and

$$i(G) \leqslant p - 2n + 1.$$

If $n + \gamma_n(G) > p$, then

$$i(G) \leqslant \gamma_n(G) - \left\lceil \frac{np}{\gamma_n(G)} \right\rceil + n + 1$$

and

$$i(G) \leqslant \gamma_n(G) - n + 1.$$

From Corollary 4, we have

Corollary 6. *If G contains no isolated vertices, then*

(11) $$i(G) \leqslant p - \left\lceil \frac{\gamma_n(G)}{n} \right\rceil - \left\lceil \frac{np}{\gamma_n(G)} \right\rceil + n + 1,$$

(12) $$i(G) \leqslant p - \left\lceil \frac{\gamma_n(G)}{n} \right\rceil - n + 1.$$

Setting $n = 1$ in (11) we get the following

Corollary 7. (Bollobás and Cockayne [2]) *If G has no isolated vertices, then*

$$i(G) \leqslant p - \gamma(G) - \left\lceil \frac{p - \gamma(G)}{\gamma(G)} \right\rceil + 1.$$

This inequality is sharp in some cases, as shown in [2].

3. Inequalities involving $\gamma_{c,f}(G), \gamma_{t,f}(G)$ and $i(G)$

In this section we will prove two inequalities involving the independence domination number and the connected and total f-domination numbers. We suppose without mention in the following that G is a connected graph and f is proper. Thus, both $\gamma_{c,f}(G)$ and $\gamma_{t,f}(G)$ are well-defined.

Theorem 4. *If D is a minimum connected f-dominating set of G, then*

(13) $$i(G) \leqslant \frac{\Delta(G)}{\Delta(G) + 1}\Big(p - \left\lceil \frac{f(\overline{D})}{\gamma_{c,f}(G)} \right\rceil\Big).$$

If D is a minimum total f-dominating set of G, then

$$i(G) \leqslant \frac{\Delta(G)}{\Delta(G)+1}\Big(p - \Big\lceil \frac{f(\overline{D})}{\gamma_{t,f}(G)} \Big\rceil\Big). \tag{14}$$

Proof. The proof uses similar idea as in the proof of Theorem 2. Suppose D is a minimum connected f-dominating set of G. Since the number of edges between D and $\overline{D}$ is no less than $f(\overline{D})$, there exists $z \in D$ which is adjacent to at least $\Big\lceil \frac{f(\overline{D})}{\gamma_f(G)} \Big\rceil$ vertices in $\overline{D}$. Let $A = N(z) \cap \overline{D}$ and X be a maximal independent set of G containing z. Then $X \subseteq V(G) - A$. Since $G[D]$ is connected, $H = G[V(G) - A]$ contains no isolated vertices and hence $\beta'(H) \geqslant \frac{|V(H)|}{\Delta(H)+1} \geqslant \frac{p-|A|}{\Delta(G)+1}$ by [9]. So we have $i(G) \leqslant |X| \leqslant p - |A| - \beta'(H) \leqslant p - |A| - \frac{p-|A|}{\Delta(G)+1} = \frac{\Delta(G)}{\Delta(G)+1}(p - |A|) \leqslant \frac{\Delta(G)}{\Delta(G)+1}\Big(p - \Big\lceil \frac{f(\overline{D})}{\gamma_{c,f}(G)} \Big\rceil\Big)$. In a similar way, one can prove (14). □

It was shown in [11] that for any positive integer k, there exists a tree and a proper function f for T such that $\gamma_{c,f}(T) - \gamma_{t,f}(T) = k$, and that there exists a tree T and a proper f with $\gamma_{t,f}(T) - \gamma_{c,f}(T) = k$. So neither one of (13), (14) is implied by the other. Since $f(\overline{D}) \geqslant m(f)(p - \gamma_{c,f}(G))$ and $f(\overline{D}) = f(V(G)) - f(D) \geqslant f(V(G)) - M(f)\gamma_{c,f}(G)$, we have

$$i(G) \leqslant \frac{\Delta(G)}{\Delta(G)+1}\Big(p + M(f) - \Big\lceil \frac{f(V(G))}{\gamma_{c,f}(G)} \Big\rceil\Big),$$

$$i(G) \leqslant \frac{\Delta(G)}{\Delta(G)+1}\Big(p + m(f) - \Big\lceil \frac{m(f)p}{\gamma_{c,f}(G)} \Big\rceil\Big).$$

For $\gamma_{t,f}(G)$ we have similar results. In the particular case where $f(x) = n$ for all $x \in V(G)$, $\gamma_{c,f}(G)$ and $\gamma_{t,f}(G)$ are called the *connected n-domination number* $\gamma_{c,n}(G)$ and the *total n-domination number* and denoted by $\gamma_{c,n}(G)$ and $\gamma_{t,n}(G)$, respectively. So we have the following

Corollary 8.

$$i(G) \leqslant \frac{\Delta(G)}{\Delta(G)+1}\Big(p + n - \Big\lceil \frac{np}{\gamma_{c,n}(G)} \Big\rceil\Big),$$

$$i(G) \leqslant \frac{\Delta(G)}{\Delta(G)+1}\Big(p + n - \Big\lceil \frac{np}{\gamma_{t,n}(G)} \Big\rceil\Big).$$

In particular, for the total domination number $\gamma_t(G) = \gamma_{t,l}(G)$ we have

Corollary 9.

$$i(G) \leqslant \frac{\Delta(G)}{\Delta(G)+1}\Big(p + 1 - \Big\lceil \frac{p}{\gamma_t(G)} \Big\rceil\Big). \tag{15}$$

We do not mention the similar inequality for the connected domination number $\gamma_c(G) = \gamma_{c,l}(G)$ because it is implied by (15) in view of $\gamma_t(G) \leqslant \gamma_c(G)$. The equality in (15) is attained when, for example, G is the complete bipartite graph $K_{k,k}$.

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