Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 50 (2000), No. 1, 121–133

Persistent URL: http://dml.cz/dmlcz/127556

Czechoslovak Mathematical Journal, 50 (125) (2000), 121–133

A THEOREM FOR AN AXIOMATIC APPROACH TO METRIC PROPERTIES OF GRAPHS

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(Received May 20, 1997)

0. By a graph we mean here a finite undirected graph without loops and multiple edges (i.e. a graph in the sense of [2], for example). Studying graphs we will investigate sets of ordered triples of vertices. For the sake of brevity, the ordered triple (u, v, x) of any objects u, v and x will be denoted by uvx.

Let G be a connected graph, and let d_G denote its distance function. Obviously, the vertex set $V(G)$ of G together with d_G create a metric space. Following [6], by a *step* in G we mean an ordered triple $uvx \in (V(G))^3$ such that

$$d_G(u,v) = 1 \quad \text{and} \quad d_G(v,x) = d_G(u,x) - 1.$$

The set of all steps in G will be referred to as the *step set* of G. The step set of a connected graph is the central notion of the present paper.

Let H be a graph, and let $M \subseteq (V(H))^3$. Following [7], we say that M is associated with H if

u and v are adjacent in H if and only if there exists
a vertex x of H such that either $uvx \in M$ or $vux \in M$

for all distinct vertices u and v of H.

Proposition. *Let G be a connected graph, and let M denote the step set of G. Then M is associated with G and the following Axioms $Y0(M)$–$Y5(M)$ and $Y^*(M)$ hold (for arbitrary $u, v, x, y \in V(G)$):*

$Y0(M)$ $\quad uvx \in M \Rightarrow vuu \in M,$

$Y1(M)$ $\quad \{uvx, vuy\} \subseteq M \Rightarrow x \neq y,$

$Y2(M)$ $\quad \{uvx, xyv\} \subseteq M \Rightarrow xyu \in M,$

* Research supported by the Grant Agency of the Czech Republic, grant No. 405/95/1554.

$Y3(M)$ $\{uvx, xyv\} \subseteq M \Rightarrow uvy \in M$,
$Y4(M)$ $\{uvx, xyy\} \subseteq M \Rightarrow \{xyu, yxv, uvy\} \cap M \neq \emptyset$,
$Y5(M)$ $u \neq x \Rightarrow \exists\, z \in V(G)\ (uzx \in M)$,
$Y^*(M)$ $\{uvx, vuy, xyy\} \subseteq M \Rightarrow xyu \in M$.

Proof is easy and can be found in [6] (see Part One of the proof of Theorem 1 there).

Let G be a connected graph, let $M \subseteq (V(G))^3$, and let M be associated with G. In [6] the present author proved that M is the step of G if and only if M fulfils Axioms $Y0(M) - Y5(M)$, $Y^*(M)$ and the following Axiom $Y6(M)$ (for arbitrary $u, v, x, y \in V(G)$):

$Y6(M)$ $\{uvx, uyv\} \subseteq M \Rightarrow y = v$.

This result will be improved in Theorem 3. As we will see, Axiom $Y6(M)$ is not necessary for characterizing the step set of a connected graph. The proof of Theorem 3 will be based on new arguments. The most important of them will be presented in Theorem 1.

Remark 1. Let G be a connected graph. Then d_G is a metrics on $V(G)$. The step set of G is an important notion for studying metric properties of G (with respect to d_G). There are two other notions important for this study: the set of all shortest paths (geodesics) in G and the interval function of G in the sense of Mulder [3]. (Cf. the notion of a finite graphic interval space in the sense of Bandelt, van de Vel and Verheul [1]). The set of all shortest paths in G was characterized in [4] and the interval function of G was characterized in [5].)

1. In the rest of the paper, the letters $f, g, \dots$, and n will be reserved for denoting integers.

In this section, we will assume that a nonempty set U is given. The results of the following two observations and of Lemmas A and B can be found in [6] or [7]. We will need them for proving Theorem 1.

Observation 1 (see [6]). Let $M \subseteq U^3$, and let M fulfil Axioms $Y0(M)$ and $Y1(M)$. It is clear that

$$\text{if } rst \in M, \quad \text{then } s \neq r \neq t.$$

Observation 2 (see [6]). Let $M \subseteq U^3$, and let M fulfil Axioms $Y2(M)$ and $Y3(M)$. Let $u_0, u_1, v_1, v_2, \dots, v_h \in U$, where $h \geqslant 2$, and let

(1) $$v_1 v_2 u_0, \dots, v_{h-1} v_h u_0 \in M.$$

Assume that $u_1u_0v_1 \in M$. Using induction, we can easily prove that

$$v_gv_{g+1}u_1, u_1u_0v_{g+1} \in M \text{ for each } g,\ 1 \leqslant g \leqslant h-1.$$

Lemma A (see [7]). *Let $M \subseteq U^3$, and let M fulfil Axioms $Y0(M)$, $Y2(M)$ and $Y3(M)$. Let $w_0, \ldots, w_h \in U$, where $h \geqslant 1$, and let*

$$w_fw_{f-1}w_0 \in M \text{ for each } f,\ 1 \leqslant f \leqslant h.$$

Then

$$w_{g-1}w_gw_h \in M \text{ for each } g,\ 1 \leqslant g \leqslant h.$$

Outline of the proof. We proceed by induction on h. The case when $h = 1$ follows from Axiom $Y0(M)$. Let $h \geqslant 2$. By virtue of the induction hypothesis,

$$w_0w_1w_{h-1}, \ldots, w_{h-2}w_{h-1}w_{h-1} \in M.$$

Since $w_hw_{h-1}w_0 \in M$, Observation 2 and Axiom $Y0(M)$ imply the desired result. □

Lemma B (see [6]). *Let $M \subseteq U^3$, and let M fulfil Axioms $Y2(M)$ and $Y3(M)$. Let $u_0, \ldots, u_{k-1}, v_0, \ldots, v_k \in U$, where $k \geqslant 2$, let*

$$(2_0) \qquad v_0v_1u_0, \ldots, v_{k-1}v_ku_0 \in M,$$

and let

$$u_1u_0v_1, \ldots, u_{k-1}u_{k-2}v_{k-1} \in M.$$

Then

$$(2_i) \qquad \begin{aligned} &v_iv_{i+1}u_i, \ldots, v_{k-1}v_ku_i \in M \text{ and} \\ &u_iu_{i-1}v_{i+1}, \ldots, u_iu_{i-1}v_k \in M \end{aligned}$$

for each i, $1 \leqslant i \leqslant k-1$.

Outline of the proof. We will prove that (2_i) holds for each i, $0 \leqslant i \leqslant k-1$. We proceed by induction on i. The case when $i = 0$ is obvious. Let $1 \leqslant i \leqslant k-1$. Clearly, $u_iu_{i-1}v_i \in M$. If we combine the induction hypothesis with Observation 2, we get (2_i). □

For proving Theorem 1, we will need one more lemma. This lemma is a modification of Lemma B:

Lemma B'. *Let $M \subseteq U^3$, and let M fulfil Axioms $Y0(M)$, $Y2(M)$ and $Y3(M)$. Let $w_0, w_1, \ldots, w_{m+k-1} \in U$, where $k \geqslant 2$ and $m \geqslant 1$, let*

$$w_0 w_1 w_m, \ldots, w_{m-1} w_m w_m \in M, \tag{3_0}$$

and let

$$w_{m+1} w_m w_1, \ldots, v_{m+k-1} w_{m+k-2} w_{k-1} \in M.$$

Then

$$\begin{aligned} &w_i w_{i+1} w_{m+i}, \ldots, w_{m+i-1} w_{m+i} w_{m+i} \in M \text{ and} \\ &w_{m+i} w_{m+i-1} w_i, \ldots, w_{m+i} w_{m+i-1} w_{m+i-1} \in M \end{aligned} \tag{3_i}$$

for each i, $1 \leqslant i \leqslant k-1$.

Proof. The case when $m = 1$ is obvious. Let $m \geqslant 2$. We will prove that (3_i) holds for each i, $0 \leqslant i \leqslant k-1$. We proceed by induction on i. If $i = 0$, then (3_i) holds trivially. Let $1 \leqslant i \leqslant k-1$. By virtue of the induction hypothesis,

$$w_i w_{i+1} w_{m+i-1}, \ldots, w_{m+i-2} w_{m+i-1} w_{m+i-1} \in M.$$

Clearly, $w_{m+1} w_{m+i-1} w_i \in M$. Observation 2 implies that

$$\begin{aligned} &w_i w_{i+1} w_{m+i}, \ldots, w_{m+i-2} w_{m+i-1} w_{m+i} \in M \text{ and} \\ &w_{m+i} w_{m+i-1} w_{i+1}, \ldots, w_{m+i} w_{m+i-1} w_{m+i-1} \in M. \end{aligned}$$

Recall that $w_{m+i} w_{m+i-1} w_i \in M$. As follows from Axiom $Y0(M)$, $w_{m+i-1} w_{m+i} w_{m+i} \in M$. Thus, we get (3_i). □

We now state the main result of the present paper. Its wording is rather long:

Theorem 1. *Let $x_0, \ldots, x_{g+h} \in U$, where $\min(g, h) \geqslant 1$, and let $Q, T \subseteq U^3$. Assume that*

$$x_0 x_1 x_1, \ldots, x_{g+h-1} x_{g+h} x_{g+h} \in Q \cap T, \tag{4}$$

$$x_0 x_1 x_g, \ldots, x_{g-1} x_g x_g \in Q, \tag{5}$$

$$x_g x_{g+1} x_0, \ldots, x_{g+h-1} x_{g+h} x_0 \in T \tag{6}$$

and if $x_{g+h} \neq x_0$*, then* $x_h x_{h-1} x_{g+h} \notin T$*. Define* $j = \max(g,h)$ *if* $x_{g+h} = x_0$ *and* $j = h$ *if* $x_{g+h} \neq x_0$*. If* $x_{g+h} = x_0$*, then put*

$$x_{g+h+1} = x_1, \ldots, x_{g+h+j} = x_j.$$

Next, assume that Q *fulfils Axioms* $Y0(Q) - Y4(Q)$ *and* $Y^*(Q)$ *and* T *fulfils Axioms* $Y0(T) - Y3(T)$ *and* $Y^*(T)$ *(for arbitrary* $u, v, x, y \in U$*). Finally, assume that the following Rules* A_1, A_2, B, C *and* D *hold for each* $m, 0 \leqslant m \leqslant j-1$*:*

A_1 $\quad x_{g+m+1} x_{g+m} x_{m+1} \in Q \cap T \ \&\ x_{m+1} x_{m+2} x_{g+m} \in T \Rightarrow x_{m+1} x_{m+2} x_{g+m} \in Q,$

A_2 $\quad m \leqslant j-2 \ \&\ x_{g+m+1} x_{g+m+2} x_{m+1} \in Q \cap T \ \&\ x_{m+1} x_m x_{g+m+2} \in T \Rightarrow x_{m+1} x_m x_{g+m+2} \in Q,$

B $\quad x_{g+m+1} x_{g+m} x_{m+1} \in Q - T \Rightarrow x_{m+1} x_m x_{g+m+1} \in T,$

C $\quad x_{g+m+1} x_{g+m} x_{m+1} \notin Q \ \&\ x_m x_{m+1} x_{g+m+1} \in Q \Rightarrow x_m x_{m+1} x_{g+m+1} \in T,$

D $\quad x_m x_{m+1} x_{g+m} \in Q \ \&\ x_m x_{m+1} x_{g+m+1} \in T \ \&\ x_{g+m} x_{g+m+1} x_{m+1} \in T \Rightarrow x_{g+m} x_{g+m+1} x_{m+1} \in Q.$

Then $x_g x_{g+1} x_0 \in Q$*.*

Proof. Suppose, to the contrary, that

(7) $$x_g x_{g+1} x_0 \notin Q.$$

We will first prove that

(8) $$\text{either } x_{g+j} x_{g+j-1} x_j \notin Q \text{ or } x_j x_{j-1} x_{g+j} \notin T.$$

Let $x_{g+h} = x_0$ and $g \geqslant h$. Since $x_{g+h} = x_0$, combining (4) and (7) we get $h \geqslant 2$. Further, combining the fact that $x_{g+h} = x_0$ with (6) and Lemma A, we get

$$x_{g+h} x_{g+h-1} x_g,\ x_{g+h-1} x_{g+h-2} x_g, \ldots, x_{g+1} x_g x_g \in T.$$

Recall that $h \geqslant 2$. Using Lemma A again, we get

$$x_g x_{g+1} x_{g+h-1}, \ldots, x_{g+h-2} x_{g+h-1} x_{g+h-1} \in T.$$

Thus, we see that

$$x_{g+h} x_{g+h-1} x_g,\ x_g x_{g+1} x_{g+h-1} \in T.$$

First, assume that $g = h$. We see that $x_{g+j}x_{g+j-1}x_j,\ x_jx_{j+1}x_{g+j-1} \in T$. By (7), $x_jx_{j+1}x_{g+j} \notin Q$. If $x_{g+j}x_{g+j-1}x_j \in Q$, then Rule A_1 implies that $x_jx_{j+1}x_{g+j-1} \in Q$, and thus, by Axiom $Y2(Q)$, $x_jx_{j+1}x_{g+j} \in Q$; a contradiction. Hence $x_{g+j}x_{g+j-1}x_j \notin Q$. Now, let $g > h$. By virtue of (5), $x_{2g-1}x_{2g}x_g \in Q$. As follows from Axiom $A1(Q)$, $x_{2g}x_{2g-1}x_g \notin Q$. Hence $x_{g+j}x_{g+j-1}x_j \notin Q$ again.

Let $x_{g+h} \neq x_0$ or $h > g$. Then $j = h$. If $x_{g+h} \neq x_0$, we get $x_jx_{j-1}x_{g+j} \notin T$. Assume that $x_{g+h} = x_0$. Then $h > g$. As follows from (6), $x_{h-1}x_hx_0 \in T$. By Axiom $Y1(T)$, $x_hx_{h-1}x_0 \notin T$. Hence $x_jx_{j-1}x_{g+j} \notin T$ again.

Thus (8) is proved.

By virtue of (8), there exists k, $1 \leqslant k \leqslant j$, such that

(9) $$\text{either } x_{g+k}x_{g+k-1}x_k \notin Q \text{ or } x_kx_{k-1}x_{g+k} \notin T$$

and

(10) $$x_{g+i}x_{g+i-1}x_i \in Q \text{ and } x_ix_{i-1}x_{g+i} \in T \text{ for each } i,\ 1 \leqslant i \leqslant k-1.$$

Let $k \geqslant 2$. Combining (5) and (10) with Lemma B', we get

(11) $$x_ix_{i+1}x_{g+i} \in Q \text{ for each } i,\ 1 \leqslant i \leqslant k-1.$$

First, assume that $x_{g+h} = x_0$. Then $h \geqslant 2$. Combining (6) and (10) with Lemma B', we get

(12) $$x_{g+i}x_{g+i+1}x_i,\ x_ix_{i-1}x_{g+i+1} \in T \text{ for each } i,\ 1 \leqslant i \leqslant k-1.$$

Now, assume that $x_{g+h} \neq x_0$. Since $j = h$ and $k \geqslant 2$, we see that $h \geqslant 2$. Combining (6) and (10) with Lemma B, we get (12) again.

By virtue of (7), there exists $f, 0 \leqslant f \leqslant k-1$, such that

(13) $$x_{g+f}x_{g+f+1}x_f \notin Q$$

and

(14) $$\text{if } f \leqslant k-2, \text{ then } x_{g+f+1}x_{g+f+2}x_{f+1} \in Q.$$

If $f \geqslant 1$, then it follows from (11) and (12) that

(15) $$x_fx_{f+1}x_{g+f} \in Q \text{ and } x_{g+f}x_{g+f+1}x_f \in T.$$

If $f = 0$, then by (5) and (6) we get (15) again.

We distinguish two cases.

Case 1. Let $x_{g+f+1}x_{g+f}x_{f+1} \in Q$. If $x_{g+f+1}x_{g+f}x_{f+1} \notin T$, then Rule B implies that

$$x_{f+1}x_f x_{g+f+1} \in T. \tag{16}$$

Let $x_{g+f+1}x_{g+f}x_{f+1} \in T$. By (15), $x_{g+f}x_{g+f+1}x_f \in T$. As follows from (4) and Axiom $Y0(T)$, $x_{f+1}x_f x_f \in T$. Thus, Axiom $Y^*(T)$ gives (16) again.

First, let $f = k-1$. Since $x_{g+f+1}x_{g+f}x_{f+1} \in Q$, (9) implies that $x_{f+1}x_f x_{g+f+1} \notin T$, which contradicts (16).

Now, let $f \leqslant k-2$. By (14), $x_{g+f+1}x_{g+f+2}x_{f+1} \in Q$. As follows from (12), $x_{g+f+1}x_{g+f+2}x_{f+1}$, $x_{f+1}x_f x_{g+f+2} \in T$. Rule A_2 implies that $x_{f+1}x_f x_{g+f+2} \in Q$. By Axiom $Y2(Q)$, $x_{f+1}x_f x_{g+f+1} \in Q$. By virtue of (15), $x_f x_{f+1}x_{g+f} \in Q$. According to (4), $x_{g+f}x_{g+f+1}x_{g+f+1} \in Q$. Axiom $Y^*(Q)$ implies that $x_{g+f}x_{g+f+1}x_f \in Q$, which contradicts (13).

Case 2. Let $x_{g+f+1}x_{g+f}x_{f+1} \notin Q$. Recall that (by (15)) $x_f x_{f+1}x_{g+f} \in Q$ and by (13), $x_{g+f}x_{g+f+1}x_f \notin Q$. Since (by (4)) $x_{g+f}x_{g+f+1}x_{g+f+1} \in Q$, Axiom $Y4(Q)$ implies that

$$x_f x_{f+1}x_{g+f+1} \in Q.$$

Since $x_{g+f+1}x_{g+f}x_{f+1} \notin Q$, Rule C implies that

$$x_f x_{f+1}x_{g+f+1} \in T. \tag{17}$$

Since (by (15)) $x_{g+f}x_{g+f+1}x_f \in T$, Axiom $Y3(T)$ implies that $x_{g+f}x_{g+f+1}x_{f+1} \in T$. Recall that $x_f x_{f+1}x_{g+f} \in Q$. Combining these facts with (17) and Rule D, we get

$$x_{g+f}x_{g+f+1}x_{f+1} \in Q.$$

Since $x_f x_{f+1}x_{g+f} \in Q$, Axiom $Y2(Q)$ implies that $x_{g+f}x_{g+f+1}x_f \in Q$, which contradicts (13).

We conclude that $x_g x_{g+1}x_0 \in Q$, which completes the proof. □

Remark 2. The idea of Theorem 1 is partially inspired by the lemma in [8].

In the next two sections of this paper Theorem 1 will be applied. We will utilize it in the proofs of Theorems 2 and 3.

2. In this section we will prove a theorem concerning the step set of a connected graph. For proving this theorem we will also need the following lemma. Its idea was implicitly contained in the proof of Lemma 3 of [6].

Lemma C. *Let U be a finite nonempty set, let $M \subseteq U^3$, and let M fulfil Axioms $Y0(M) - Y3(M)$. Let $n \geqslant 1$. Consider an infinite sequence*

$$u_0, u_1, u_2, \ldots$$

of elements in U such that $u_n u_{n+1} u_0 \in M$. Assume that

$$\text{if } u_{n+g} = u_0, \text{ then } u_{n+g+1} = u_{n+g} \text{ and}$$
$$\text{if } u_{n+g} \neq u_0, \text{ then } u_{n+g} u_{n+g+1} u_0 \in M$$

for each $g \geqslant 1$. Then there exists $h \geqslant 1$ such that either $u_{n+h} = u_0$ or $u_h u_{h-1} u_{n+h} \notin M$.

Proof. Suppose, to the contrary, that $u_{n+f} \neq u_0$ and $u_f u_{f-1} u_{n+f} \in M$ for each $f \geqslant 1$. Therefore $u_{n+f} u_{n+f+1} u_0 \in M$ for each $f \geqslant 0$. Put $j = |U|$ and $m = (j-1)n + 1$. By Lemma B,

$$u_i u_{i-1} u_{n+i}, \ldots, u_i u_{i-1} u_{n+m} \in M \text{ for each } i,\ 1 \leqslant i \leqslant m-1.$$

Thus, according to Observation 1,

$$u_i \neq u_{n+i}, \ldots, u_{n+m} \text{ for each } i,\ 1 \leqslant i \leqslant m-1.$$

This implies that the elements

$$u_1, u_{n+1}, \ldots, u_{jn+1}$$

are mutually distinct. We get $|U| > j$, which is a contradiction. Thus the lemma is proved. □

Let G be a connected graph, and let $M \in V(G)$. For each $n \geqslant 0$, we define

$$M(G, \leqslant n) = \{uvx \in M; u, v, x \in V(G) \text{ and } d_G(u, x) \leqslant n\}.$$

Instead of $M(G, \leqslant n)$ we will shortly write $M(\leqslant n)$.

Theorem 2. *Let G be a connected graph, let $M \subseteq (V(G))^3$, let M be associated with G, and let M fulfil Axioms $Y0(M) - Y3(M)$, $Y5(M)$ and $Y^*(M)$ (for arbitrary $u, v, x, y \in V(G)$). Let S denote the step set of G. Then*

$$(18_n) \qquad S(\leqslant n) \subseteq M(\leqslant n) \Rightarrow S(\leqslant n) = M(\leqslant n)$$

for every $n \geqslant 0$.

P r o o f. Put $d_G = d$. We proceed by induction on n. Since $M(\leqslant 0) = \emptyset$, (18_0) holds. Let $n \geqslant 1$. Assume that $S(\leqslant n) \subseteq M(\leqslant n)$. Then $S(\leqslant n-1) \subseteq M(\leqslant n-1)$. By the induction hypothesis, $S(\leqslant n-1) = M(\leqslant n-1)$. Assume that (18_n) does not hold. Then there exist $r, s, t \in V(G)$ such that

$$rst \in M(\leqslant n) - M(\leqslant n-1) \text{ and } rst \notin S.$$

Since $d(r,t) = n$, we see that there exist $x_0, x_1, \ldots, x_n \in V(G)$ such that $x_0 = t$, $x_n = r$ and

$$x_0 x_1 x_n, \ldots, x_{n-1} x_n x_n \in S.$$

Combining Axiom $Y5(M)$ with Lemma C, we see that there exist $h \geqslant 1$ and $x_{n+1}, \ldots, x_{n+h} \in V(G)$ such that $x_{n+1} = s$,

$$\begin{aligned}&x_n x_{n+1} x_0, \ldots, x_{n+h-1} x_{n+h} x_0 \in M, \text{ and}\\ &\text{if } x_{n+h} \neq x_0, \text{ then } x_h x_{h-1} x_{n+h} \notin M.\end{aligned}$$

Put $Q = S$, $T = M$ and $g = n$. Hence

$$Q(\leqslant g) \subseteq T(\leqslant g). \tag{19}$$

Since $S(\leqslant n-1) = M(\leqslant n-1)$, we have

$$Q(\leqslant g-1) = T(\leqslant g-1). \tag{20}$$

Let j be defined as in Theorem 1. Consider an arbitrary m, $0 \leqslant m \leqslant j-1$. We will show that Rules A_1, A_2, B, C and D are fulfilled. (Recall that $Q = S$.)

(A_1) Let $x_{g+m+1} x_{g+m} x_{m+1} \in Q$. Then $d(x_{g+m}, x_{m+1}) \leqslant g-1$. If $x_{m+1} x_{m+2} x_{g+m} \in T$, then (20) implies that $x_{m+1} x_{m+2} x_{g+m} \in Q$.

(A_2) Let $m \leqslant j-2$, and let $x_{g+m+1} x_{g+m+2} x_{m+1} \in Q$. Then $d(x_{m+1}, x_{g+m+2}) \leqslant g-1$. If $x_{m+1} x_m x_{g+m+2} \in T$, then (20) implies that $x_{m+1} x_m x_{g+m+2} \in Q$.

(B) Obviously, $d(x_{g+m+1}, x_{m+1}) \leqslant g$. By (19), $x_{g+m+1} x_{g+m} x_{m+1} \notin Q - T$.

(C) Let $x_{g+m+1} x_{g+m} x_{m+1} \notin Q$. Then $d(x_{g+m+1}, x_{m+1}) \leqslant d(x_{g+m}, x_{m+1}) \leqslant g-1$. Hence $d(x_m, x_{g+m+1}) \leqslant g$. If $x_m x_{m+1} x_{g+m+1} \in Q$, then (19) implies that $x_m x_{m+1} x_{g+m+1} \in T$.

(D) Let $x_m x_{m+1} x_{g+m} \in Q$. Then $d(x_{g+m}, x_{m+1}) \leqslant g-1$. If $x_{g+m} x_{g+m+1} x_{m+1} \in T$, then (20) implies that $x_{g+m} x_{g+m+1} x_{m+1} \in Q$.

Thus Rules A_1, A_2, B, C and D are fulfilled. Since $Q = S$, the proposition implies that Q fulfils Axioms $Y0(Q)$–$Y4(Q)$ and $Y^*(Q)$. By Theorem 1, $x_g x_{g+1} x_0 \in Q$. Since $x_g = r$, $x_{g+1} = s$ and $x_0 = t$, we have a contradiction.

Thus, we get (18_n), which completes the proof. □

Remark 3. The idea of Theorem 2 has a certain connection to that of Lemma 3 in [9] (but the proofs of these results are deeply distinct).

Corollary. *Let G be a connected graph, let $M \subseteq (V(G))^3$, let M be associated with G, and let M fulfil Axioms $Y0(M)$–$Y3(M)$, $Y5(M)$ and $Y^*(M)$ (for arbitrary $u, v, x, y \in V(G)$). Let S denote the step set of G. If $S \subseteq M$, then $S = M$.*

3. The step set of a connected graph was characterized by the present author in [6]. That characterization will be improved in Theorem 3. For proving Theorem 3 we will need two more observations and two more lemmas.

Observation 3 (see [7]). Let U be a nonempty set, let $M \subseteq U^3$, and let M fulfil Axioms $Y2(M)$ and $Y3(M)$. Let $u_0, u_1, v_1, \ldots, v_h \in U$, where $h \geqslant 2$, and let (1) hold. Assume that $u_0u_1v_h \in M$. Using the induction on $h - g$, we can easily prove that

$$v_gv_{g+1}u_1,\ u_0u_1v_g \in M$$

for each $g, 1 \leqslant g \leqslant h - 1$.

The following lemma was implicitly contained in the proof of Lemma 3 of [7].

Lemma D. *Let U be a nonempty set, let $M \subseteq U^3$, and let M fulfil Axioms $Y2(M) - Y4(M)$. Let $u_0, u_1, w_0, \ldots, w_g \in U$, where $g \geqslant 1$, let $u_0u_1u_1 \in M$, and let*

$$w_0w_1u_0, \ldots, w_{g-1}w_gu_0 \in M.$$

Assume that $w_0 = w_g$. Then

$$w_0w_1u_1, \ldots, w_{g-1}w_gu_1 \in M. \tag{21}$$

P r o o f. Put $w_{g+1} = w_1, \ldots, w_{2g} = w_g$. We distinguish two cases.

Case 1. Assume that there exists f, $0 \leqslant f \leqslant g-1$, such that either (a) $u_1u_0w_{f+1} \in M$ or (b) $u_0u_1w_f \in M$. First, let (a) hold. Since

$$w_{f+1}w_{f+2}u_0, \ldots, w_{f+g}w_{f+g+1}u_0 \in M,$$

Observation 2 implies that

$$w_{f+1}w_{f+2}u_1, \ldots, w_{f+g}w_{f+g+1}u_1 \in M,$$

and thus (21) holds. Now, let (b) hold. Then $u_0u_1w_{f+g} \in M$. Since

$$w_fw_{f+1}u_0, \dots, w_{f+g-1}w_{f+g}u_0 \in M,$$

Observation 3 implies that

$$w_fw_{f+1}u_1, \dots, w_{f+g-1}w_{f+g}u_1 \in M,$$

and thus (21) holds.

Case 2. Assume that $u_1u_0w_{f+1}$, $u_0u_1w_f \notin M$ for each f, $0 \leqslant f \leqslant g-1$. Since $u_0u_1u_1 \in M$, Axiom $Y4(M)$ implies that (21) holds again. Hence the lemma is proved. □

Observation 4 (see [7]). Let G be a connected graph, let $M \subseteq (V(G))^3$, let M be associated with G, and let M fulfil Axioms $Y0(M)$–$Y4(M)$. Let $u_0, v_1, \dots, v_h \in V(G)$, where $h \geqslant 2$, and let (1) hold. Combining Observation 1 with Lemma D, we get $v_1 \neq v_h$.

Lemma E (see [7]). *Let G be a connected graph, let $M \subseteq (V(G))^3$, let M be associated with G, and let M fulfil Axioms $Y0(M) - Y5(M)$. Consider distinct $r, t \in V(G)$. Then there exist $m \geqslant 1$ and $r_0, r_1, \dots, r_m \in V(G)$ such that $r_0 = r$, $r_m = t$ and*

$$r_0r_1t, \dots, r_{m-1}r_mt \in M.$$

Outline of the proof. Since $V(G)$ is finite, it is easy to prove the lemma by combining the result of Observation 4 with Axiom $Y5(M)$. □

Remark 4. Let $n \geqslant 2$, let $x_0, \dots, x_n$, $y_0, \dots, y_n$ and z be mutually distinct elements, and let G be the graph with

$$V(G) = \{x_0, \dots, x_n,\ y_0, \dots, y_n, z\}$$

and with the edge set as follows:

$$\begin{aligned}\{\{x_f, x_g\};\ 0 \leqslant f \leqslant n,\ 0 \leqslant g \leqslant n,\ f \neq g\}\\ \cup\{\{y_h, y_i\};\ 0 \leqslant h \leqslant n,\ 0 \leqslant i \leqslant n,\ h \neq i\}\\ \cup\{\{x_j, z\};\ 0 \leqslant j \leqslant n\} \cup \{\{y_k, z\};\ 0 \leqslant k \leqslant n\}.\end{aligned}$$

Obviously, G is connected. Put $x_{n+1} = x_0$, $y_{n+1} = y_0$. Let $M \subseteq (V(G))^3$ be defined as follows: $uvw \in M$ if and only if either u and v are adjacent in G and $w = v$

or there exist f, $0 \leqslant f \leqslant n$, and g, $0 \leqslant g \leqslant n$, such that

$$\text{either } x_f x_{f+1} y_g = uvw$$
$$\text{or } y_f y_{f+1} x_g = uvw.$$

Obviously, M is associated with G. It is not difficult to see that M fulfils Axioms $Y0(M)$–$Y3(M)$, $Y5(M)$ and $Y^*(M)$ (for arbitrary $u, v, x \in V(G)$) but does not fulfil Axiom $Y4(M)$. We can see that for G and M the result of Lemma E does not hold.

Theorem 3. *Let G be a connected graph, let $M \subseteq (V(G))^3$, and let M be associated with G. Then the following statements* (A) *and* (B) *are equivalent:*

(A) *M is the step set of G,*

(B) *M fulfils Axioms $Y0(M)$–$Y5(M)$ and $Y^*(M)$ (for arbitrary $u, v, x \in V(G)$).*

Proof. Let S denote the step set of G. Put $d = d_G$.

By the proposition, $(A) \Rightarrow (B)$. We will prove that $(B) \Rightarrow (A)$. Suppose, to the contrary, that (A) holds but (B) does not hold. It is easy to see that $S(\leqslant 1) \subseteq M(\leqslant 1)$. Thus, by virtue of Theorem 2, there exists $n \geqslant 2$ such that $S(\leqslant n) - M(\leqslant n) \neq \emptyset$ and $S(\leqslant n-1) = M(\leqslant n-1)$. Therefore, there exist $r, s, t \in V(G)$ such that $d(r,t) = n$, $rst \in S$ but $rst \notin M$. Since $r \neq t$, Lemma E implies that there exist $g \geqslant 1$ and $x_0, \ldots, x_g \in V(G)$ such that $x_0 = r$, $x_g = t$ and

$$x_0 x_1 x_g, \ldots, x_{g-1} x_g x_g \in M.$$

Obviously, there exist $x_{g+1}, \ldots, x_{g+n} \in V(G)$ such that $x_{g+1} = s$, $x_{g+n} = x_0$ and

$$x_g x_{g+1} x_0, \ldots, x_{g+n-1} x_{g+n} x_0 \in S.$$

Put $Q = M$, $T = S$ and $h = n$. Since $S(\leqslant n-1) = M(\leqslant n-1)$, we have

$$T(\leqslant h-1) = Q(\leqslant h-1). \tag{22}$$

Let j be defined as in Theorem 1. Consider an arbitrary m, $0 \leqslant m \leqslant j-1$. We will show that Rules A_1, A_2, B, C and D are fulfilled. (Recall that $T = S$.)

(A_1) Let $x_{g+m+1} x_{g+m} x_{m+1} \in T$. Since $d(x_{g+m+1}, x_{m+1}) \leqslant h$, we have $d(x_{g+m}, x_{m+1}) \leqslant h-1$. If $x_{m+1} x_{m+2} x_{g+m} \in T$, then (22) implies that $x_{m+1} x_{m+2} x_{g+m} \in Q$.

(A_2) Let $m \leqslant j-2$ and let $x_{g+m+1} x_{g+m+2} x_{m+1} \in T$. Since $d(x_{g+m+1}, x_{m+1}) \leqslant h$, we have $d(x_{g+m+2}, x_{m+1}) \leqslant h-1$. If $x_{m+1} x_m x_{g+m+2} \in T$, then (22) implies that $x_{m+1} x_m x_{g+m+2} \in Q$.

(B) Let $x_{g+m+1}x_{g+m}x_{m+1} \in Q - T$. Clearly, $d(x_{g+m+1}, x_{m+1}) \leqslant h$. If $d(x_{g+m+1}, x_{m+1}) \leqslant h - 1$, then (22) leads to a contradiction. Thus $d(x_{g+m+1}, x_{m+1}) = h$. We get $x_{m+1}x_m x_{g+m+1} \in T$.

(C) Clearly, $d(x_m, x_{g+m+1}) \leqslant h - 1$. If $x_m x_{m+1} x_{g+m+1} \in Q$, then (22) implies that $x_m x_{m+1} x_{g+m+1} \in T$.

(D) Let $x_m x_{m+1} x_{g+m+1} \in T$. We get $d(x_{m+1}, x_{g+m+1}) \leqslant h - 2$ and therefore, $d(x_{m+1}, x_{g+m}) \leqslant h - 1$. If $x_{g+m}x_{g+m+1}x_{m+1} \in T$, then (22) implies that $x_{g+m}x_{g+m+1}x_{m+1} \in Q$.

Thus Rules A_1, A_2, B, C and D are fulfilled. By Theorem 1, $x_g x_{g+1} x_0 \in Q$. Since $x_g = r$, $x_{g+1} = s$ and $x_0 = t$, we have a contradiction.

Thus (B) $\Rightarrow$ (A), which completes the proof. □

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