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COUNTABLE PRODUCTS OF ČECH-SCATTERED SUPERCOMPLETE SPACES

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Abstract. We prove by using well-founded trees that a countable product of supercomplete spaces, scattered with respect to Čech-complete subsets, is supercomplete. This result extends results given in [1], [5], [6], [19], [26] and its proof improves that given in [19].

Keywords: supercomplete, product spaces, Čech-complete, C-scattered, uniform space, paracompact, locally fine

MSC 2000: Primary 54B10, Secondary 54E15

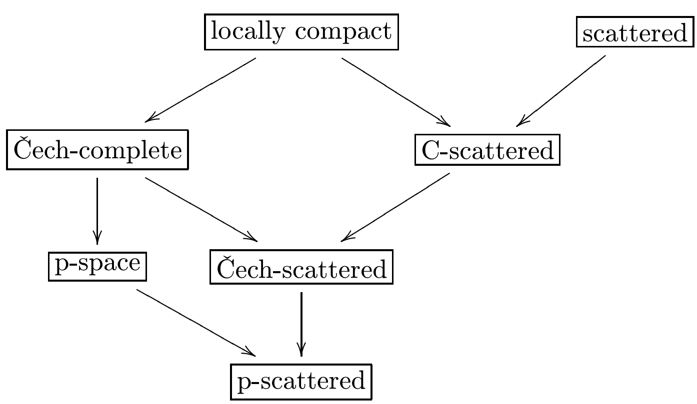
1. Introduction

This article belongs to a series of papers investigating the supercompleteness or paracompactness of product spaces under the structural condition that the factors be recursively decomposable or refinable to compact parts. It was proved by Z. Frolík in [6] that a countable product of locally compact paracompact spaces is paracompact. K. Alster proved in [1] that countable products of C-scattered Lindelöf spaces are again Lindelöf. A similar result for paracompact spaces was given in [5], extending an earlier result of [28] for scattered paracompact spaces. On the other hand, the earlier result of Telgársky [30] on finite products of C-scattered paracompact spaces was extended to the stronger, combinatorial condition of supercompleteness [21] by the first author in [16]. Following this research line ([14], [16], [17], [26]), the first author and J. Pelant gave in [19] the result establishing the supercompleteness of countable products of C-scattered supercomplete spaces, which implies the above-mentioned results of [1], [6], [5], and [28]. However, Frolík [6] had also proved (more generally) that the countable product of Čech-complete paracompact spaces is paracompact.

His result was extended to supercomplete spaces in [16]. In this paper, we continue this research by establishing a *common generalization* of the above results by showing that the countable product of Čech-scattered supercomplete spaces is supercomplete.

By using well-founded trees associated with the Ginsburg-Isbell locally fine coreflection [10], the proof given in [19] was quite compact. (These trees were introduced in [26], and applied in [18].) By using a more refined technique, we extend these results to the case of spaces scattered with respect to Čech-complete subsets. A similar result for Lindelöf spaces is obtained as a corollary to our general theorem on supercompleteness. This case essentially extends that of C-scattered spaces, since Čech-complete paracompacta do not have the property that their products with paracompacta are always paracompact. The main new idea is the *perverse product* of trees. Well-founded subtrees of the perverse products admit induction arguments from bottom to top, making the proof more compact and readable. Instead of exhaustions of $\mathscr{K}$-scattered spaces (as in [16], [17]) we use related decomposition trees of those spaces.

The notion of Čech-scattered spaces is an important *common limit* of the notions C-scattered and Čech-complete.



Indeed, one might conjecture that our main result could be generalized to spaces scattered with respect to sets in a countably productive category of spaces. However, it cannot even be extended to the case of metrizable-scattered paracompacta; the classical Michael line [24] ([4], Ch. 5, Ex. 1.4) is scattered with respect to countable sets and provides a simple counter-example.

1.1. Recursive covers

In general, open covers of products $X \times Y$ cannot be reduced to those of the factors, in particular to covers obtained as products $\mathscr{U} \times \mathscr{V}$, where $\mathscr{U}$ (resp. $\mathscr{V}$) is an open cover of X (resp. Y). However, we may consider product covers in a *recursive* sense by recursively localizing them to the members of the product cover. When the factor covers are chosen from uniformities, this leads to the so-called Ginsburg-Isbell derivatives of product covers, defined and considered later in Section 2.3. It is sufficient here to define recursive open covers of the product space $X \times Y$ as well-founded trees in which every element is an element of the form $U \times V$, and the immediate successors of an element form an open product cover of the corresponding subspace. These covers generalize the notion of simplicial subdivisions, and the condition of well-foundedness corresponds to the requirement that every descending chain of the subdivision is finite. This generalization provides a connection between the product and the factors, and the question of those classes of spaces in which every open cover can be refined by a recursive cover is a problem important for topological arguments based on the factor spaces. This can be done, for example, when both factors are Čech-complete. In this paper we consider the problem in the case of infinitely many factors.

2. Preliminaries

We give here technical preliminary definitions necessary for the proof of Theorem 3.2. This section is divided into four subsections. We adopt many definitions from [10], [19], in particular the definitions of *well-founded tree* and the locally finite (Ginsburg-Isbell) coreflection operator λ. We consider in this paper trees as partially ordered sets T with the underlying set denoted by the same symbol. Here we exclusively consider rooted trees T such that every element has only finitely many predecessors.

2.1. $\mathscr{K}$-scattered spaces

We recall that if $\mathscr{K}$ is a class of topological spaces, then a space X is called $\mathscr{K}$-*scattered* [30], [31] if every non-empty closed subset of X contains a point having a $\mathscr{K}$-neighbourhood in this subset. We can define the $\mathscr{K}$-*derivatives* of a $\mathscr{K}$-scattered space X by transfinite induction as follows. Let $D_{\mathscr{K}}(X)$ denote the subset of all $p \in X$ having no $\mathscr{K}$-neighbourhood. Then $D_{\mathscr{K}}(X)$ is a closed subset of X, and we set

$$D^{(0)}_{\mathscr{K}}(X) = X, \quad D^{(\alpha+1)}_{\mathscr{K}}(X) = D_{\mathscr{K}}(D^{(\alpha)}_{\mathscr{K}}(X)),$$

and

$$D_{\mathscr{K}}^{(\beta)}(X) = \bigcap_{\alpha<\beta} D_{\mathscr{K}}^{(\alpha)}(X)$$

in case β is a limit ordinal. Define

$$\mathrm{Rank}_{\mathscr{K}}(X) = \min\{\alpha \in \mathrm{Ord}\colon\ D_{\mathscr{K}}^{(\alpha)}(X) = \emptyset\}.$$

The ordinal number $\mathrm{Rank}_{\mathscr{K}}(X)$ is called the $\mathscr{K}$-rank of X. In particular, the $\mathscr{K}$-rank of a *locally* $\mathscr{K}$ space is at most one. In this paper, $\mathscr{C}$ denotes the class of all compact spaces and $\check{\mathscr{C}}$ [31] denotes the class of all Čech-complete spaces. If $\mathscr{K}$ is a closed-hereditary class, then (in this paper all spaces are assumed to be at least Tychonoff spaces) we can define the corresponding *$\mathscr{K}$-decomposition tree* $T_{\mathscr{K}}(X)$ of X as follows. (For terminology concerning trees, the reader is kindly referred to [18] or [19].) The elements of $T_{\mathscr{K}}(X)$ are closed subsets of X defined by using the derivative sets $D_{\mathscr{K}}^{(\alpha)}(X)$. Let $\bar{\alpha} = \mathrm{Rank}_{\mathscr{K}}(X)$. Thus, $D_{\mathscr{K}}^{\bar{\alpha}}(X) = \emptyset$, but $D_{\mathscr{K}}^{\alpha}(X) \neq \emptyset$ for all $\alpha < \bar{\alpha}$. The closed subset $\bigcap\{D_{\mathscr{K}}^{\alpha}(X)\colon\ \alpha < \bar{\alpha}\}$ is denoted by the symbol $\mathrm{top}_{\mathscr{K}}(X)$; notice that this is a locally $\mathscr{K}$ subset such that every point p of $X - \mathrm{top}_{\mathscr{K}}(X)$ has a closed neighbourhood $\overline{U}_p$ such that $\mathrm{Rank}_{\mathscr{K}}(\overline{U}_p) < \bar{\alpha}$. Also note that $\bar{\alpha}$ is a limit ordinal if, and only if, $\mathrm{top}_{\mathscr{K}}(X)$ is empty. For $\mathscr{K} = \mathscr{C}$, $\mathrm{top}_{\mathscr{K}}(X)$ is just locally compact, whereas for $\mathscr{K} = \check{\mathscr{C}}$ and X paracompact this is, by a theorem of Frolík [8], itself Čech-complete, which is a convenient feature of $\check{\mathscr{C}}$. We let $\mathrm{Root}(T_{\mathscr{K}}(X)) = X$, and the set of immediate successors of X is defined to consist of the set $\mathrm{top}_{\mathscr{K}}(X)$ and of *all* such closed subsets $\overline{U}$ of $X - \mathrm{top}_{\mathscr{K}}(X)$ with non-empty interior and for which $\mathrm{Rank}_{\mathscr{K}}(\overline{U}) < \bar{\alpha}$. If $\mathrm{Rank}_{\mathscr{K}}(X) = 0$, then $T_{\mathscr{K}}(X) = \{X\}$; otherwise, the sets $\overline{U}$ satisfy $\mathrm{Rank}_{\mathscr{K}}(\overline{U}) < \bar{\alpha}$, and we can (recursively) define the tree $T_{\mathscr{K}}(X)$ by hanging $\mathrm{top}_{\mathscr{K}}(X)$ and the trees $T_{\mathscr{K}}(\overline{U})$ below X. Then $T_{\mathscr{K}}(X)$ is a well-founded tree. Indeed, we can consider the mapping $r\colon T_{\mathscr{K}}(X) \to \mathrm{Ord}$ given by $r(P) = \mathrm{Rank}_{\mathscr{K}}(P)$. If $\mathscr{B}$ were an infinite branch of $T_{\mathscr{K}}(X)$, then $\{r(P)\colon\ P \in \mathscr{B}\}$ would contain an infinite decreasing set of ordinal numbers, which would be impossible. The decomposition trees (for $\mathscr{K} = \mathscr{C}$ and for a metrizable X) were first explicitly mentioned in [17].

2.2. Perverse products of trees

Next we consider special products of trees. If $(T_i\colon\ i \in I)$ is a family of trees, then the direct product $\prod\limits_{i\in I} T_i$ is defined as the direct product of the partially ordered sets T_i. Thus, the direct T product consists of the Cartesian product of the T_i equipped with the direct product order relation: for $p, q \in T$, $p \leqslant q$ if and only if $\pi_i(p) \leqslant \pi_i(q)$ for all $i \in I$. In general, this direct product is not a tree. Another drawback of this product from our point of view is that branches of T do not in general project onto branches in the factor trees T_i. This disadvantage is even shared by the *weak direct product* of the T_i consisting of all $p \in \prod_{i\in I} T_i$ such that $\{i \in I\colon\ \pi_i(p) \neq \mathrm{Root}(T_i)\}$ is finite. For this reason we define special subproducts having this desired property. (In this paper, the term *subproduct* means a subset of the direct product.) A *perversity* is a finite sequence $p = (p_1, \ldots, p_n)$ of integers satisfying $p_1 = 0$ and $p_{i+1} = p_i$ or $p_{i+1} = p_i + 1$ for $i \in [1, n-1]$. (This concept has been used in intersection homology theory, see e.g. [23].) The set of perversities has a natural coordinatewise partial order inherited from the usual ordering of functions $\mathbb{N} \to \mathbb{Z}$.

In this paper, we shall use this concept in a modified sense suitable for our purposes: henceforth a perversity is a decreasing sequence $p\colon\ \mathbb{N} \to \mathbb{Z}$ which is eventually zero. Let $(T_i\colon\ i \in \mathbb{N})$ be a countable family of trees, and let $\mathscr{P}$ be a set of perversities. For a tree T, the level function $\mathrm{Level}_T\colon\ T \to \mathbb{N}$ is defined by letting $\mathrm{Level}_T(p)$ be the number of predecessors of p in T. (We denote by $\mathrm{Level}_n(T)$ the set of all $p \in T$ such that $\mathrm{Level}_T(p) = n$.) Then the perverse product $\prod_{\mathscr{P}} T_i$ is the subset of the Cartesian product $\prod_{i\in I} T_i$ consisting of all elements x such that there is a perversity $p \in \mathscr{P}$ such that

$$\mathrm{Level}_{T_i}(\pi_i(x)) = p(i)$$

for all $i \in \mathbb{N}$. Thus, a perverse product is always a subproduct of the weak direct product of the T_i. However, the perverse product of countably many trees is a tree, provided that the associated family of perversities is itself a tree under its natural partial order. Furthermore, and more importantly for this paper, the perverse product of trees is a partially ordered set such that infinite chains project onto infinite chains in the factors, *provided that the the set of perversities has the same property.*

When $T_i = T$ for each i, then the perverse product of the T_i is called the *perverse power* of T and denoted by $T^{\mathbb{N}}_{\mathscr{P}}$. In the sequel, we will consider 'set-theoretic' trees T consisting of (labelled) subsets of a set X. For set-theoretic trees T_i,$i \in I$, the *set-theoretic direct product* simply consists of the Cartesian products $\prod_{i\in I} P_i$ of the underlying sets P_i of elements of the T_i. Set-theoretic perverse products are defined in the same way. For subsets $A \subset \prod_{i\in\mathbb{N}} X_i$ such that $\{i \in \mathbb{N}\colon\ \pi_i[A] \neq X_i\}$ is finite, we define $\mathrm{Seg}(A) = \max\{i \in \mathbb{N}\colon\ \pi_i[A] \neq X_i\}$. Otherwise, we put $\mathrm{Seg}(A) = \infty$. Thus, the function Seg is finite-valued for all elements of weak direct (and a fortiori, perverse) set-theoretic products.

2.3. Ginsburg-Isbell derivatives

In this paper, the symbol $\mathscr{F}(X)$ denotes the fine uniformity (i.e., the filter of all normal covers) of a Tychonoff space X. The important link between well-founded trees and uniformities is established by the Ginsburg-Isbell derivatives [10] $\mu^{(\alpha)}$, the largest of which is the locally fine coreflection denoted by $\lambda\mu$. This connection (see [26]) has been used in several papers as a technical tool. Let μX be a uniform space, and let $\mathscr{V}$ be a cover of X. Then (we can take this as a definition) $\mathscr{V}$ belongs to $\lambda\mu$ if, and only if there is a well-founded set-theoretic tree T on X satisfying the condition

(∗) *For any element P of T, the immediate successors of P form a uniform cover of P*

and for which the maximal elements of T form a refinement of $\mathscr{V}$. A uniform space μX is called *supercomplete* [21] if every open cover of X belongs to $\lambda\mu$. Because covers refinable by the end elements of a well-founded set-theoretic tree satisfying (∗) have σ-discrete closed refinements, a supercomplete space is always topologically paracompact. In a supercomplete space, the open covers can be obtained by a combinatorial process from uniform covers. (The name "supercomplete" is derived from the original definition: a uniform space μX is supercomplete if its uniform hyperspace of closed subsets is complete. The definition above pertains to the results of [21].) (A form of supercompleteness was already considered in [3].) Thus, the well-founded trees provide a close link between the following three objects:

1) the structure of $\mathscr{K}$-scattered spaces;
2) the open covers of a supercomplete space;
3) the Ginsburg-Isbell locally fine coreflection.

Let us point out the meaning of this connection. The Ginsburg-Isbell derivatives naturally appear when one deals with covers of product spaces. Usually open covers (or even normal covers) of products $X \times Y$ cannot be refined by products $\mathscr{U} \times \mathscr{V}$ of normal covers. In technical notation, this means that

$$\mathscr{F}X \times \mathscr{F}Y \neq \mathscr{F}(X \times Y).$$

However, under suitable conditions (see, for example, [13],[14],[16]) it is possible to refine all normal covers of $X \times Y$ by covers obtained from combinatorial refinements of product covers, i.e., the equation

$$\lambda(\mathscr{F}X \times \mathscr{F}Y) = \mathscr{F}(X \times Y)$$

holds. The supercompleteness of the product $\mathscr{F}X \times \mathscr{F}Y$ means that all open covers of $X \times Y$ are refinable by such combinatorial refinements, producing a link between the open covers and the covers of the factor spaces. Strong theorems of this type can be obtained in a natural way, by the above remarks, for products of $\mathscr{K}$-scattered spaces, where $\mathscr{K}$ is a suitable class of topological spaces.

2.4. Čech-completeness

By using perverse products, we show in this paper that the above link is preserved in countable products if the class $\mathscr{K}$ consists of spaces with a strong completeness property: Čech-completeness. We recall that a Tychonoff space X is called Čech-complete (abbreviated to $\check{\mathscr{C}}$-complete) if X is an absolute G_δ space. It was proved by Frolík in [7] that X is a paracompact Čech-complete space if and only if it has a sequence $(\mathscr{U}_n)$ of locally finite open covers $\mathscr{U}_n$ such that whenever $(F_n\colon\ n \in \mathbb{N})$ is a centered family of closed subsets of X such that for each $n \in \mathbb{N}$, $F_n \subset U_n \in \mathscr{U}_n$ for some U_n, then the intersection $K = \bigcap\{F_n\colon\ n \in \mathbb{N}\}$ is nonempty and compact. (Such a sequence $\mathscr{U}$ is called a complete sequence.) We can assume that the sequence $(\mathscr{U}_n)$ is normal, and that the sets F_n are of the form $\overline{U}_i$. Such compact sets K cover X, and the finite intersections $\bigcap\{\overline{U}_i\colon\ i \in [1, n]\}$ form their countable neighbourhood bases. These facts can be directly obtained from Frolík's characterization of Čech-complete paracompact spaces: X is Čech-complete and paracompact if and only there is a completely metrizable space M and a perfect onto map $f\colon\ X \to M$.

3. Countable products of Čech-scattered paracompacta

We prove here that countable products of $\check{\mathscr{C}}$-scattered supercomplete spaces are supercomplete. We give here specific preliminary definitions for the proof of 3.2. To describe the trees we shall deal with, we start from the $\check{\mathscr{C}}$-decomposition tree $T_{\check{\mathscr{C}}}(X)$ of X. As in [18] and [19], let $\mathrm{End}(T)$ denote the set of all maximal elements of any tree. Recall that for each point P of $T_{\check{\mathscr{C}}}(X)$ the immediate successors of P, $P \notin \mathrm{End}(T_{\check{\mathscr{C}}}(X))$, are the Čech-complete closed subset $\mathrm{top}_{\check{\mathscr{C}}}(P)$ of P and the closed sets $Q \subset P - \mathrm{top}_{\check{\mathscr{C}}}(P)$ such that 1) $\mathrm{Rank}_{\check{\mathscr{C}}}(Q) < \mathrm{Rank}_{\check{\mathscr{C}}}(P)$ and 2) the interiors $\mathrm{int}(Q)$ in P are non-empty. The end elements of $T_{\check{\mathscr{C}}}(X)$ are closed Čech-complete subsets of X. Hence, each $P \in \mathrm{End}(T_{\check{\mathscr{C}}}(X))$ has a complete sequence $(\mathscr{U}_{n,P})$ of normal closed covers. We extend $T_{\check{\mathscr{C}}}(X)$ to a (non-well-founded) finer tree $T(X)$ as follows. First we define for each $P \in \mathrm{End}(T_{\check{\mathscr{C}}}(X))$ a tree T_P. We put $\mathrm{Root}(T_P) = P$; the immediate successors of P are the elements of $\mathscr{U}_{1,P}$; inductively, the immediate successors of an element Q of level n are the non-empty sets $Q \cap U$, where $U \in \mathscr{U}_{n+1,P}$.

We denote the collection of these sets by $\widetilde{\mathscr{U}_{n+1,P}}$. We form $T(X)$ by adding the trees T_P as subtrees below P. In order to choose a suitable subproduct of $\prod_{i \in \mathbb{N}} T$, we define a sequence of perversities (see Section 2) as follows. Let $p_1 = (0, 0, 0, \ldots)$. Suppose that $p_n = (i_1, \ldots, i_m, 0, 0, \ldots)$ has been defined. Put

$$k = \min\{j \in \mathbb{N}\colon\ i_j = i_{j+1}\}.$$

Then define $p_{n+1} = (i_1, \ldots, i_k + 1, i_{k+1}, \ldots, i_m, 0, 0, \ldots)$. The first elements of this sequence (in abbreviated notation) are (),(1),(1,1),(2,1),(2,1,1),(2,2,1) etc. (Notice that $\mathscr{P}$ is linearly ordered, and hence any perverse product of trees with respect to $\mathscr{P}$ is again a tree.) Let $\mathscr{P} = \{p_n\colon\ n \in \mathbb{N}\}$ and let $T' = (\prod_{i \in \mathbb{N}} T)_{\mathscr{P}}$ be the perverse set-theoretic power of $T = T(X)$ with respect to $\mathscr{P}$ as explained in Section 2. The root of T' (the unique element of level 0) is X^ω and the elements of level 3, say, are products

$$P_1 \times P_2 \times X \times X \times \ldots,$$

where P_1 is of level 2 and P_2 is of level 1 in T. Although the tree T' is not well-founded in general, it has the following property of perverse products: if (P_n) is an infinite increasing subset of T' (i.e., an infinite subset of a branch of T') then for each $i \in \mathbb{N}$, the projections $\pi_i[P_n]$ form an infinite increasing subset of T.

We also need a property of the Ginsburg-Isbell locally fine coreflection λ. Let μX be a uniform space. Covers $\mathscr{U} \in \lambda\mu$ are called λ-uniform covers of μX. If A is a subset of X, then V is called a λ-uniform neighbourhood of A, if V is a uniform neighbourhood of A relative to $\lambda\mu$, i.e., there is a cover $\mathscr{U} \in \lambda\mu$ such that $\bigcup \mathrm{St}(A, \mathscr{U}) \subset V$. The proof of the following essential lemma is straightforward.

Lemma 3.1 (The λ-neighbourhood induction lemma). *Let μX be a uniform space, let $A \subset X$, let $\mathscr{V}$ be a λ-uniform cover of the subspace A and for each $V \in \mathscr{V}$ let V' be a λ-uniform neighbourhood of V in X. Then $\bigcup\{V'\colon\ V \in \mathscr{V}\}$ is a λ-uniform neighbourhood of A in X. Moreover, if $\mathscr{G}$ is a λ-uniform cover of each V', then A has a λ-uniform neighbourhood N such that $\mathscr{G}$ is a λ-uniform cover of N.*

Proof. For the first statement, there is a cover $\mathscr{U} \in \lambda\mu$ such that $\mathscr{V} = \mathscr{U} \restriction A$. Write $\mathscr{V} = \{V_i\}$. For each V_i, there is a cover $\mathscr{W}_i = \{W^i_j\} \in \lambda\mu$ such that $\mathrm{St}(V_i, \mathscr{W}_i) \subset V'_i$. Form a cover $\mathscr{L}$ of X by taking all sets of the form $U_i \cap W^i_j$ where $V_i = U_i \cap A$, and additionally all members of $\mathscr{U}$ which do not meet A. Then $\mathscr{L}$ belongs to $\lambda\mu$. Moreover, $\mathrm{St}(A, \mathscr{L})$ is contained in the union of the V', proving the first claim. Indeed, if $U_i \cap W^i_j$ meets A, then W^i_j meets the corresponding V_i, and is contained in V'_i.

For the second statement, there is for each V_i a cover $\mathscr{W}_i = \{W^i_j\} \in \lambda\mu$ such that $\mathscr{W}_i \upharpoonright V'_i \prec \mathscr{G}$. We can assume that $\mathscr{W}_i$ is as in the previous paragraph, and we define $\mathscr{L}$ in the same way. The union N of all members of $\mathscr{L}$ that meet A is a $\lambda\mu$-neighbourhood of A such that $\mathscr{L} \upharpoonright N \prec \mathscr{G}$. Indeed, the elements of this restriction are of the form $W \cap U_i \cap W^i_j$ (where the latter intersection meets A) which is contained in $V'_i \cap W^i_j$ and therefore in a member of $\mathscr{G}$. $\square$

Let us now state our main result.

Theorem 3.2. *Let $(X_i\colon\ i \in \mathbb{N})$ be a countable family of $\check{\mathscr{C}}$-scattered paracompact spaces. Then the product space $\prod\limits_{i\in\mathbb{N}} \mathscr{F} X_i$ is supercomplete.*

P r o o f. We can replace the factors X_i by their discrete sum denoted by X. (Then $\prod\limits_{i\in\mathbb{N}} X_i$ is a closed subproduct of X^ω.) Let us, however, denote the ith factor of X^ω by X_i. We show that for every open cover $\mathscr{G}$ of X^ω, the cover $\mathscr{G}^{<\omega}$ consisting of all finite unions of elements of $\mathscr{G}$ belongs to $\lambda(\mathscr{F}(X)^\omega)$. (As noted in [27], it is enough to consider $\mathscr{G}^{<\omega}$ instead of $\mathscr{G}$.) We construct a well-founded tree $T_{\mathscr{G}}$ of subsets of X^ω and show by induction from $\operatorname{End}(T_{\mathscr{G}})$ to $\operatorname{Root}(T_{\mathscr{G}})$ that $\mathscr{G}^{<\omega}$ is a λ-uniform cover of each element of $T_{\mathscr{G}}$. We consider the perverse power T' as given above, and we construct the desired tree $T_{\mathscr{G}}$ as a well-founded subtree T'' of T'. We let T'' consist of all $P \in T'$ which do not have a predecessor Q satisfying the following condition (let $s = \operatorname{Seg}(Q)$):

(#) there exist open sets $M_1, \ldots, M_s \subset X$, $N_1, \ldots, N_s \subset X$ and a finite subfamily $\mathscr{G}'$ of $\mathscr{G}$ such that

$$Q \subset \bigcap_{i=1}^{s} \pi_i^{-1}[M_i] \subset \bigcap_{i=1}^{s} \pi_i^{-1}[\overline{M_i}] \subset \bigcap_{i=1}^{s} \pi_i^{-1}[N_i] \subset \bigcap_{i=1}^{s} \pi_i^{-1}[\overline{N_i}] \subset \bigcup(\mathscr{G}').$$

(Obviously, T'' is a filter of T' and hence a subtree.) We will prove that T'' is a well-founded tree. To see this, assume that T'' contains an infinite increasing sequence $(P_n\colon\ n \in \mathbb{N})$. Since T'' is a subtree of T', for each i, the projections $\pi_i[P_n]$ form an infinite increasing sequence in T_i, where T_i denotes the ith factor of T'. Since $T_{\check{\mathscr{C}}}(X_i)$ is well-founded, there are (for each i) n_i and $P^{(i)} \in \operatorname{End}(T_{\check{\mathscr{C}}}(X_i))$ such that $(\pi_i[P_n]\colon\ n \geqslant n_i)$ is an infinite increasing sequence in $T_{P^{(i)}}$. This implies that there are elements $U_{i,n} \in \mathscr{U}_{n,P^{(i)}}$, $n \geqslant n_i$, with $\pi_i[P_n] = \widetilde{U_{i,n}}$ and

$$\bigcap_{n=n_i}^{s} U_{i,n} = \bigcap_{n=n_i}^{s} \widetilde{U_{i,n}} \neq \emptyset$$

for all $s \geqslant n_i$. Since $(\mathscr{U}_{n,P^{(i)}})$ is a complete sequence for $P^{(i)}$), it follows that the set

$$K_i = \bigcap_{n\in\mathbb{N}} \pi_i[P_n])$$

is non-empty and compact. Since the elements P_n are products, clearly

$$K = \bigcap_{n\in\mathbb{N}} P_n = \prod_{i\in\mathbb{N}} K_i$$

and thus there is a finite subfamily $\mathscr{G}'$ of $\mathscr{G}$ such that $K \subset \bigcup(\mathscr{G}')$. By Wallace's lemma (general form) (proved by Frolík [6], Lemma 3), we can find open sets $W_1, \ldots, W_s$ such that

$$\bigcap_{i=1}^{s} \pi_i^{-1}[K_i] \subset \bigcap_{i=1}^{s} \pi_i^{-1}[W_i] \subset \bigcup(\mathscr{G}').$$

Let us recall that the sets $\pi_i[P_n]$ have been chosen so that they form a countable neighbourhood base of the corresponding compact set K_i in the subspace $P^{(i)}$ (see Section 2). Thus we can find for each $i \in [1, s]$ a number n_i such that $\pi_i[P_{n_i}] \subset W_i$. As a consequence, by taking $n = \max(n_1, \ldots, n_s)$ and keeping in mind that $\pi_i[P_n] \subset \pi_i[P_{n_i}]$ for all $i \in [1, s]$, we see that

$$P_n \subset \left(\prod_{i=1}^{s} \pi_i[P_n]\right) \times \left(\prod_{i>s} X_i\right) \subset \bigcap_{i=1}^{s} \pi_i^{-1}[W_i] \subset \bigcup(\mathscr{G}'),$$

which easily yields a contradiction with $P_{n+1} \in T''$. Thus, T'' is well-founded.

After constructing T'' we use it to prove that $\mathscr{G}^{<\omega}$ belongs to $\lambda(\mathscr{F}(X)^\omega)$. We prove this by induction on the partial order of T''. We shall prove that for each $P \in T''$, there is a λ-uniform neighbourhood V_P of P in X^ω such that $\mathscr{G}^{<\omega}$ is a λ-uniform cover of V_P. Let $P \in \mathrm{End}(T'')$. Then there are open sets $M_1, \ldots, M_s \subset X$, $N_1, \ldots, N_s \subset X$ such that

$$P \subset \bigcap_{i=1}^{s} \pi_i^{-1}[M_i] \subset \bigcap_{i=1}^{s} \pi_i^{-1}[\overline{M_i}] \subset \bigcap_{i=1}^{s} \pi_i^{-1}[N_i] \subset \bigcap_{i=1}^{s} \pi_i^{-1}[\overline{N_i}] \subset \bigcup(\mathscr{G}')$$

for some finite subfamily $\mathscr{G}'$ of $\mathscr{G}$. It easily follows that P has a uniform neighbourhood V_P such that $\mathscr{G}^{<\omega}$ is a *uniform* cover of V_P. Now let $P \in T''$ and assume that the claim is true for each successor Q of P in T''. By the definition of the perverse order of T'', inherited from that of T', there is $i \in \mathbb{N}$ such that for each immediate successor Q of P in T', we have $\pi_j[P] = \pi_j[Q]$ for all $j \neq i$, whereas the elements

$\pi_i[Q]$ are immediate successors of $P_i = \pi_i[P]$ in the tree $T(X)$. There are three possibilities:

Case 1: P_i belongs to $T_{\check{\mathscr{C}}}(X) - \mathrm{End}(T_{\check{\mathscr{C}}}(X))$, and hence the immediate successors of P_i are elements of $T_{\check{\mathscr{C}}}(X)$;

Case 2: P_i belongs to $\mathrm{End}(T_{\check{\mathscr{C}}}(X))$, and the immediate successors of P_i are elements of $\widetilde{\mathscr{U}_{1,P_i}}$ or

Case 3: P_i belongs to $\widetilde{\mathscr{U}_{k,Z}}$ for some $k \in \mathbb{N}$ and $Z \in \mathrm{End}(T_{\check{\mathscr{C}}}(X))$, and the immediate successors are elements of $\widetilde{\mathscr{U}_{k+1,Z}}$.

We consider these three cases separately. In Case 3, the elements of $\widetilde{\mathscr{U}_{k+1,Z}}$ form a uniform cover (with respect to the fine uniformity of X) of P_i, and hence the immediate successors of P form a uniform cover of P, and by the induction hypothesis each of them has a λ-uniform neighbourhood V such that $\mathscr{G}^{<\omega}$ is a λ-uniform cover of V. By Lemma 3.1, P itself has such a neighbourhood. Case 2 is similar. In Case 1, exactly one of the immediate successors of P_i has the form $\mathrm{top}_{\check{\mathscr{C}}}(P_i)$; let $Q_0 = \pi_i^{-1}[\mathrm{top}_{\check{\mathscr{C}}}(P_i)] \cap P$ be the corresponding immediate successor of P. By the induction hypothesis, Q_0 has a λ-uniform neighbourhood V_0 such that $\mathscr{G}^{<\omega}$ is a λ-uniform cover of V_0. Then $P - V_0$ has a closed λ-uniform neighbourhood W_0 such that $W_0 \cap Q_0 = \emptyset$, and such that the binary cover $\{V_0, W_0\}$ is a λ-uniform cover of P. Indeed, there is a λ-uniform cover $\mathscr{W}$ of X^ω such that $\mathrm{St}(P - V_0, \mathscr{W}) \cap Q_0 = \emptyset$. We can assume that $\mathscr{W}$ consists of the closures of basic open sets. Then each element W of $\mathscr{W} \restriction P$ that meets $P - V_0$ satisfies $\pi_i[W] \cap \pi_i[Q_0] = \emptyset$, which implies that W is an immediate successor of P. But then by the inductive hypothesis, each $W \in \mathscr{W} \restriction P$ with $W \cap (P - V_0) \neq \emptyset$ has a λ-uniform neighbourhood V_W such that $\mathscr{G}^{<\omega}$ is λ-uniform cover of V_W. It easily follows that P has a λ-uniform neighbourhood V_P satisfying the condition of our claim. (Indeed, by 3.1 the set V_0 has a λ-uniform neighbourhood V_0' such that $\mathscr{G}^{<\omega}$ is a λ-uniform cover of V_0'. The sets W such that $W \cap (P - V_0) \neq \emptyset$ together with V_0 form a λ-uniform cover of P. As $\mathscr{G}^{<\omega}$ is λ-uniform over the sets V_W and V_0', the set P has by a new application of 3.1 a λ-uniform neighbourhood V_P with the desired property.)

This completes the inductive step in the proof of our claim. Since T'' is well-founded, it follows that the claim holds for the root of T'', which is X^ω, and consequently $\mathscr{G}^{<\omega}$ is a λ-uniform cover of X^ω, as required. $\square$

Corollary 3.3. *Let $(\mu_n X_n \colon n \in \mathbb{N})$ be a countable family of Čech-scattered supercomplete spaces. Then $\prod_{n\in\mathbb{N}} \mu_n X_n$ is supercomplete.*

P r o o f. Indeed, for each n we have $\lambda\mu_n = \mathscr{F}(X_n)$. By 3.2, $\prod\limits_{n\in\mathbb{N}} \mathscr{F}X_n$ is supercomplete. Hence, $\lambda \prod\limits_{n\in\mathbb{N}} \mathscr{F}(X_n)$ contains every open cover of the topological product of the X_n. But then so does

$$\lambda \prod_{n\in\mathbb{N}} \mu_n = \lambda \prod_{\in\mathbb{N}} \lambda\mu_n = \lambda \prod_{n\in\mathbb{N}} \mathscr{F}(X_n),$$

which implies the result. □

Corollary 3.4. *Let $(X_n\colon\ n \in \mathbb{N})$ be a countable family of Čech-scattered paracompact spaces. Then $\prod\limits_{n\in\mathbb{N}} X_n$ is paracompact.*

Corollary 3.5. *Let $(X_n\colon\ n \in \mathbb{N})$ be a countable family of Čech-scattered Lindelöf spaces. Then $\prod\limits_{n\in\mathbb{N}} X_n$ is a Lindelöf space.*

P r o o f. Here we have a non-trivial application of supercompleteness. Let $c(X)$ be the uniformity of X generated by all countable cozero-covers. Then for each X_i, cX_i is supercomplete, because X_i is Lindelöf. By Corollary 3.3 the product $\prod\limits_{i\in\mathbb{N}} cX_i$ is supercomplete. Thus, $\lambda \prod\limits_{i\in\mathbb{N}} cX_i$ contains every open cover of the product. But $\prod\limits_{i\in\mathbb{N}} cX_i$ has a base consisting of countable covers, and so does its locally fine coreflection $\lambda \prod\limits_{i\in\mathbb{N}} cX_i$. (The reader is advised to think in terms of well-founded trees; a countably branching well-founded tree is countable.) It follows that every open cover of the product space has a countable refinement, and hence a countable subcover, implying that the product is Lindelöf. □

Remark 1. Theorem 3.2 also readily implies that the countable product of ultraparacompact Čech-scattered spaces is ultraparacompact. This corollary is proved in the same way as 3.5, but the uniformity generated by countable cozero-covers is replaced by the uniformity generated by clopen partitions.

Remark 2. One can easily show that supercomplete products are rectangular in the sense of Pasynkov [25]. Hence, Theorem 3.2 yields the additional corollary that finite products of Čech-scattered paracompacta are rectangular, and by [25] such products satisfy the logarithmic inequality for the covering dimension.

Remark 3. Theorem 3.2 can be generalized to spaces which are countable unions of closed paracompact Čech-scattered parts. This readily follows from our proof using decomposition trees. For a space $X = \bigcup\{F_n\colon\ n \in \mathbb{N}\}$, where the sets F_n are closed and $\mathscr{K}$-scattered, the decomposition tree $T_{\mathscr{K}}(X)$ is defined as the tree obtained by hanging the trees $T_{\mathscr{K}}(F_n)$ below the root X. A cover $\mathscr{V}$ of a uniform space μX is called *σ-uniform* if there is a countable collection $(F_n\colon n \in \mathbb{N})$ of closed subspaces of X such that $X = \bigcup\{F_n\colon n \in \mathbb{N}\}$ and for each n, $\mathscr{V} \restriction F_n$ is a uniform cover of the subspace F_n. We note that the uniformity generated by all σ-uniform open covers of μX is the metric-fine coreflection $m\mu$ (see [11], [9]). (Recall here that every uniform cover has a σ-uniformly discrete refinement, hence so does every σ-uniform cover.) The proof of 3.2 applies here, but the immediate successors of some elements in the tree T'' are divided into countably many parts, corresponding to the sets F_n, and one has to replace the product of the fine uniformities by $m\Big(\prod\limits_{n\in\mathbb{N}} \mathscr{F}(X_n)\Big)$. The conclusion is that the locally fine coreflection of $m(\mathscr{F}(X)^\omega)$ contains all the open covers of X^ω, which implies that every open cover of X^ω is normal and thus that X^ω is paracompact. We also obtain the following generalization of 3.3.

Theorem 3.6. *Let $(\mu_n X_n\colon n \in \mathbb{N})$ be a countable family of supercomplete spaces, each of which is a countable union of closed Čech-scattered subsets. Then $m\Big(\prod\limits_{n\in\mathbb{N}} \mu_n X_n\Big)$ is supercomplete.*

Remark 4. Since uniformly continuous perfect pre-images of supercomplete spaces are again supercomplete ([16]), one can ask whether the problem of countable products could be reduced to the metrizable case. Unfortunately, there are $\check{\mathscr{C}}$-scattered paracompacta which are not perfect pre-images of metrizable spaces; for example, we can take the 1-point Lindelöfization of an uncountable discrete space. Indeed, it turns out that $\check{\mathscr{C}}$-scattered metrizable spaces are, in fact, Čech-complete. We give here a sketch of proof. Let us proceed by transfinite induction. Let (X, ϱ) be a $\check{\mathscr{C}}$-scattered metric space with $\mathrm{Rank}_{\check{\mathscr{C}}}(X) = \alpha$ and suppose that the claim is true for all metrizable $\check{\mathscr{C}}$-scattered spaces of lesser rank. If $\mathrm{top}_{\check{\mathscr{C}}}(X)$ is empty, then X has an open cover by sets E such that $\mathrm{Rank}_{\check{\mathscr{C}}}(\overline{E}) < \alpha$. Then by the inductive hypothesis, X is locally Čech-complete and since X is paracompact, a result of Frolík [8] implies that X is itself Čech-complete. On the other hand, if $\mathrm{top}_{\check{\mathscr{C}}}(X)$ is non-empty, then—since X is metrizable—$\mathrm{top}_{\check{\mathscr{C}}}(X)$ is a G_δ subset of X, in fact, we can write $\mathrm{top}_{\check{\mathscr{C}}}(X) = \bigcap\{G_n\colon n \in \mathbb{N}\}$, where $G_n = \{p \in X\colon \varrho(p, E) < 1/n\}$. As $\mathrm{top}_{\check{\mathscr{C}}}(X)$ is locally Čech-complete and paracompact, it is by [8] Čech-complete, and therefore has a complete (compatible) metric σ. By the Hausdorff extension theorem [12], we extend σ to a compatible metric σ' of X. For each n, $F_n = X - G_n$ is a closed subset with $\mathrm{Rank}_{\check{\mathscr{C}}}(F_n) < \alpha$ and by the induction hypothesis has a complete metric σ_n. We extend each σ_n to a compatible metric ${\sigma_n}'$ of X. Define

$$\bar{\sigma}(x, y) = \sigma'(x, y) + \sum_{n\in\mathbb{N}} 2^{-n}(1 \wedge {\sigma_n}'(x, y))$$

for all $x, y \in X$; then it is not difficult to check that $\bar{\sigma}$ is a complete compatible metric for X. Thus, X is completely metrizable or, equivalently, Čech-complete. This finishes the inductive step. It is not difficult to see how to extend the above argument to spaces X in which for every closed subspace $A \subset X$, a complete sequence of open covers of A can be extended to a complete open family in X. Then an inductive argument similar to the above one enables us to give a complete sequence of covers for the whole space.

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