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PSEUDODIMENSION OF RELATIONAL STRUCTURES

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1. Introduction

In the paper [4] we have established the theory of pseudodimension (see [3]) for binary structures, i.e. for sets with a binary relation, and demonstrated the relationship of our theory with the theory of dependence spaces (cf. [6]). In this paper we extend the theory of pseudodimension to relational structures of an arbitrary (finitary) arity. The notation used here coincides with that of [4]; for the reader's convenience we repeat the fundamental concepts and notation.

If G is a set, then the cardinality of G is denoted by $|G|$ and the power set of G is symbolized by $\mathbf{B}(G)$. If G, H are sets, we denote by G^H the set of all mappings of the set H into G. The symbol $\mathbb{N}$ means the set of all natural numbers.

Let $G \neq \emptyset$ be a set, $n \in \mathbb{N}$, $n \geqslant 2$, let $X \subseteq G^n$ be an n-ary relation on G. Then the structure $\mathbf{G} = (G, X)$ will be referred to as an *n-ary structure*. If necessary, we denote by $C(\mathbf{G})$ and $R(\mathbf{G})$ the carrier and the relation of the structure $\mathbf{G}$, respectively, i.e., $C(\mathbf{G}) = G$, $R(\mathbf{G}) = X$. If $\mathbf{G} = (G, X)$, $\mathbf{H} = (H, Y)$ are n-ary structures, then—as usual—a mapping f of the set G into H is called a *homomorphism* of $\mathbf{G}$ into $\mathbf{H}$ whenever for any $(x_1, \ldots, x_n) \in G^n$ the condition $(x_1, \ldots, x_n) \in X$ implies $(f(x_1), \ldots, f(x_n)) \in Y$. The symbol $\operatorname{Hom}(\mathbf{G}, \mathbf{H})$ denotes the set of all homomorphisms of the structure $\mathbf{G}$ into $\mathbf{H}$. A homomorphism $f \in \operatorname{Hom}(\mathbf{G}, \mathbf{H})$ is said to be *strong* if for any $(x_1, \ldots, x_n) \in G^n$ the condition $(x_1, \ldots, x_n) \in X$ is equivalent to the condition $(f(x_1), \ldots, f(x_n)) \in Y$. An injective strong homomorphism of the structure $\mathbf{G}$ into $\mathbf{H}$ is referred to as an *embedding* of $\mathbf{G}$ into $\mathbf{H}$. A bijective strong homomorphism of the structure $\mathbf{G}$ onto $\mathbf{H}$ is called an *isomorphism*. If $\mathbf{G} = (G, X)$, $\mathbf{H} = (H, Y)$ are n-ary structures, then the *power* $\mathbf{G}^{\mathbf{H}}$ is an n-ary structure such that $C(\mathbf{G}^{\mathbf{H}}) = \operatorname{Hom}(\mathbf{H}, \mathbf{G})$ and $R(\mathbf{G}^{\mathbf{H}}) = \{(h_1, \ldots, h_n) \in (\operatorname{Hom}(\mathbf{H}, \mathbf{G}))^n ;\ (h_1(x), \ldots, h_n(x)) \in X$ for any $x \in H\}$. The structure $\mathbf{H}$ is called *discrete* if $R(\mathbf{H}) = \emptyset$; then $C(\mathbf{G}^{\mathbf{H}}) = G^H$.

Let $(G, \leqslant)$ be a complete lattice. Suppose that Θ is an equivalence relation on the set G such that any Θ-block has a greatest element. Then the structure $(G, \leqslant, \Theta)$ is said to be a *dependence space*. Isomorphisms of dependence spaces are defined in the usual way: Let $(G, \leqslant, \Theta)$, $(H, \preceq, \Phi)$ be structures with two binary relations and f a bijection of G onto H. Then f is an isomorphism if the condition $x \leqslant y$ is equivalent to $f(x) \preceq f(y)$ and the condition $x\Theta y$ is equivalent to $f(x)\Phi f(y)$ for any x, y in G. We obtain

Lemma 1.1. *Let $(G, \leqslant, \Theta)$ be a dependence space, $(H, \preceq, \Phi)$ a structure with two binary relations. Suppose that f is an isomorphism of $(G, \leqslant, \Theta)$ onto $(H, \preceq, \Phi)$. Then $(H, \preceq, \Phi)$ is a dependence space.*

P r o o f. It follows from the hypotheses that $(H, \preceq)$ is a complete lattice and that Φ is an equivalence relation on H. Furthermore, to any Φ-block Q there exists a Θ-block P such that $f(P) = Q$, which implies that $f\lceil P$ is an isomorphism of $(P, \leqslant)$ onto $(Q, \preceq)$. If a is the greatest element of P, then $f(a)$ is the greatest element of Q. Hence any Φ-block has a greatest element and $(H, \preceq, \Phi)$ is a dependence space. □

Lemma 1.2. *Let $(G, \leqslant, \Theta)$ be a dependence space, $(H, \preceq)$ a complete lattice, and f an isomorphism of $(G, \leqslant)$ onto $(H, \preceq)$. Then there exists an equivalence relation Φ on H such that $(H, \preceq, \Phi)$ is a dependence space and f is an isomorphism of $(G, \leqslant, \Theta)$ onto $(H, \preceq, \Phi)$.*

P r o o f. For any $u, v \in H$ put $(u, v) \in \Phi$ if and only if $(f^{-1}(u), f^{-1}(v)) \in \Theta$. Then f is an isomorphism of the structure (G, Θ) onto the structure (H, Φ) and, therefore, of the structure $(G, \leqslant, \Theta)$ onto the structure $(H, \preceq, \Phi)$; the assertion follows from Lemma 1.1. □

A dependence space $(G, \leqslant, \Theta)$ is said to be *natural* if there exists a set M such that $G \subseteq \mathbf{B}(M)$ and that the relation $\leqslant$ coincides with the set theoretic inclusion. We may limit our considerations to natural dependence spaces because the following holds.

Theorem 1.1. *Any dependence space is isomorphic to a natural dependence space.*

P r o o f. Let $(G, \leqslant, \Theta)$ be a dependence space. Let f be an embedding of the ordered set $(G, \leqslant)$ into $(\mathbf{B}(G), \subseteq)$ defined by $f(x) = \{t \in G;\ t \leqslant x\}$. It is sufficient to apply Lemma 1.2 to $(G, \leqslant, \Theta)$, f, and $(f(G), \subseteq)$. □

Let $(G, \subseteq, \Theta)$ be a natural dependence space and $x \in G$ an element. Put $c(x, \Theta) = \min\{|y|;\ (x, y) \in \Theta,\ y \subseteq x\}$; this cardinal will be referred to as the Θ-*character* of the element x.

2. Realizer and pseudodimension of an n-ary structure

Let G be a set, $n \in \mathbb{N}$, $n \geqslant 2$, $\mathbf{H} = (H, Y)$ an n-ary structure, and suppose $|G| \geqslant 2$, $|H| \geqslant 2$. Similarly as in [4] we define a mapping S of the set $\mathbf{B}(G^n)$ into $\mathbf{B}(H^G)$ and a mapping T of the set $\mathbf{B}(H^G)$ into $\mathbf{B}(G^n)$ as follows. For an arbitrary set $X \subseteq G^n$, put

$$S(X) = \operatorname{Hom}((G, X), \mathbf{H}).$$

For any set $U \subseteq H^G$, set

$$T(U) = \{(x_1, \ldots, x_n) \in G^n;\ (f(x_1), \ldots, f(x_n)) \in Y \text{ for any } f \in U\}.$$

Clearly, the pair of mappings (S, T) forms a Galois connexion between the complete lattices $(\mathbf{B}(G^n), \subseteq)$ and $(\mathbf{B}(H^G), \subseteq)$. For any sets $U_1, U_2 \in \mathbf{B}(H^G)$, put $(U_1, U_2) \in \Theta(\mathbf{H})$ if $T(U_1) = T(U_2)$. Clearly $\Theta(\mathbf{H})$ is an equivalence relation on the set $\mathbf{B}(H^G)$.

Lemma 2.1. *Let $U_1 \in \mathbf{B}(H^G)$, $U_2 \in \mathbf{B}(H^G)$, $(U_1, U_2) \in \Theta(\mathbf{H})$. Then $(U_1, S(T(U_2))) \in \Theta(\mathbf{H})$ and $U_1 \subseteq S(T(U_2))$.*

Proof. Since $T(U_2) = T(S(T(U_2)))$, we have $(U_2, S(T(U_2))) \in \Theta(\mathbf{H})$; transitivity of $\Theta(\mathbf{H})$ implies $(U_1, S(T(U_2))) \in \Theta(\mathbf{H})$. The hypothesis $(U_1, U_2) \in \Theta(\mathbf{H})$ entails $T(U_1) = T(U_2)$ and, therefore, $S(T(U_1)) = S(T(U_2))$. Since $S \circ T$ is a closure operator, we obtain $U_1 \subseteq S(T(U_1)) = S(T(U_2))$. $\square$

Theorem 2.1. *$(\mathbf{B}(H^G), \subseteq, \Theta(\mathbf{H}))$ is a natural dependence space.*

Proof. $(\mathbf{B}(H^G), \subseteq)$ is a complete lattice and $\Theta(\mathbf{H})$ is an equivalence relation on $\mathbf{B}(H^G)$. Let P be an arbitrary $\Theta(\mathbf{H})$-block; we choose an arbitrary element $U_0 \in P$. By Lemma 2.1, we obtain $S(T(U_0)) \in P$ and $U \subseteq S(T(U_0))$ for an arbitrary $U \in P$. Thus, $S(T(U_0))$ is the greatest element in P. $\square$

Let $X \subseteq G^n$, $U \subseteq H^G$. If $T(U) = X$ holds, then U is called an $\mathbf{H}$-*realizer* of (G, X). An $\mathbf{H}$-realizer of an n-ary structure (G, X) need not exist. But by the same argument as in Theorem 3.1 in [4] we obtain

Theorem 2.2. *Let $X \subseteq G^n$. Then the structure (G, X) has an $\mathbf{H}$-realizer if and only if $T(S(X)) = X$ holds.*

Similarly as Theorem 3.2 in [4] we may prove

Theorem 2.3. *Let $X \subseteq G^n$ and suppose the existence of an $\mathbf{H}$-realizer of the n-ary structure (G, X). A set $U \subseteq H^G$ is an $\mathbf{H}$-realizer of (G, X) if and only if $(U, S(X)) \in \Theta(\mathbf{H})$ holds.*

Suppose $U \subseteq H^G$. By an *evaluation map* for U we mean the mapping e of the set G into H^U such that for any $x \in G$ the mapping $e(x)$ of the set U into H is defined by the condition $e(x)(f) = f(x)$.

By repeating the proof of Theorem 3.3 in [4] we obtain

Theorem 2.4. *Suppose $X \subseteq G^n$, $U \subseteq H^G$. Then the following assertions are equivalent.*

(i) *The set U is an* **H***-realizer of the structure (G, X).*

(ii) *The evaluation map for U is a strong homomorphism of the structure (G, X) into the structure $\mathbf{H}^{\mathbf{U}}$ where $\mathbf{U} = (U, \emptyset)$ is a discrete structure.*

Let $\mathbf{G} = (G, X)$, $\mathbf{H} = (H, Y)$ be n-ary structures, suppose that $V \subseteq \mathrm{Hom}(\mathbf{G}, \mathbf{H})$, $V \neq \emptyset$. The relation X will be referred to as *determined* by the set V if for any elements $x_1, \ldots, x_n$ in G with the property $(x_1, \ldots, x_n) \notin X$ there exists $f \in V$ such that $(f(x_1), \ldots, f(x_n)) \notin Y$.

The following is trivial.

Lemma 2.2. *Let $\mathbf{G} = (G, X)$, $\mathbf{H} = (H, Y)$ be n-ary structures and suppose that $V \subseteq \mathrm{Hom}(\mathbf{G}, \mathbf{H})$, $V \neq \emptyset$. Then the set V is an* **H***-realizer of the structure (G, X) if and only if the relation X is determined by V.*

Theorem 2.5. *Let $X \subseteq G^n$ and suppose that the structure (G, X) has at least one* **H***-realizer. Then the set of all* **H***-realizers of (G, X) forms a complete upper semilattice with respect to set inclusion and $S(X)$ is the greatest element of this semilattice.*

P r o o f. If $\{U_i;\ i \in I\}$ is an arbitrary system of **H**-realizers of the structure (G, X), then Lemma 2.2 implies that $\bigcup_{i \in I} U_i$ is an **H**-realizer of (G, X). Hence the set of all **H**-realizers of the structure (G, X) constitutes a complete upper semilattice with respect to inclusion. By Theorem 2.3, the set $S(X)$ is an **H**-realizer of (G, X). Since the set of all **H**-realizers of the structure (G, X) forms a $\Theta(\mathbf{H})$-block in the dependence space $(\mathbf{B}(H^G), \subseteq, \Theta(\mathbf{H}))$, the greatest element of this $\Theta(\mathbf{H})$-block equals $S(T(U_0))$ where U_0 is an arbitrary element of this block; this follows from the proof of Theorem 2.1. But $T(U_0) = X$ and hence $S(X)$ is the greatest **H**-realizer. $\square$

Theorem 2.6. *Let $X \subseteq G^n$. Suppose that $\mathbf{H} = (H, Y)$ is an n-ary structure. Then the following assertions are equivalent.*

(a) *The structure (G, X) has an* **H***-realizer.*

(b) *The set $S(X)$ is an* **H***-realizer of the structure (G, X).*

(c) *The relation X is determined by the set* $\mathrm{Hom}((G, X), \mathbf{H})$.

(d) *There exists a set $K \neq \emptyset$ and a strong homomorphism of the structure (G, X) into $\mathbf{H}^{\mathbf{K}}$ where $\mathbf{K} = (K, \emptyset)$.*

Proof. (b) implies (a) trivially, (b) follows from (a) by Theorem 2.5. The equivalence of (b) and (c) is a consequence of Lemma 2.2. Furthermore, (d) follows from (a) by Theorem 2.4. We prove that (d) implies (a): Let φ be a strong homomorphism of the structure (G, X) into the structure $\mathbf{H}^{\mathbf{K}}$. For any $k \in K$ we define a mapping f_k of G into H putting $f_k(x) = \varphi(x)(k)$ for any $x \in G$. Put $U = \{f_k;\ k \in K\}$; clearly $U \subseteq H^G$ holds. Let the elements $x_1, \ldots, x_n$ in G be arbitrary. Then any two consecutive conditions in the following sequence are equivalent.

(A) $(x_1, \ldots, x_n) \in X$;

(B) $(\varphi(x_1), \ldots, \varphi(x_n)) \in R(\mathbf{H}^{\mathbf{K}})$;

(C) $(\varphi(x_1)(k), \ldots, \varphi(x_n)(k)) \in Y$ for any $k \in K$;

(D) $(f_k(x_1), \ldots, f_k(x_n)) \in Y$ for any $k \in K$.

The equivalence of (A) and (D) means that the set U is an $\mathbf{H}$-realizer of the structure (G, X) and hence (a) holds. $\square$

Suppose that the structure $\mathbf{G} = (G, X)$ has an $\mathbf{H}$-realizer. We denote by α the type of the structure $\mathbf{H} = (H, Y)$. Then we put

$$\alpha\text{-}\operatorname{pdim} \mathbf{G} = \min\{|U|;\ U \subseteq H^G \text{ and } U \text{ is an } \mathbf{H}\text{-realizer of } \mathbf{G}\}.$$

This cardinal is called the *α-pseudodimension* of the structure (G, X).

The following theorem describes the relationship between the α-pseudodimension of the structure (G, X) and the Θ-character in natural dependence spaces.

Theorem 2.7. *Let $\mathbf{G} = (G, X)$ be an n-ary structure that has an $\mathbf{H}$-realizer where $\mathbf{H} = (H, Y)$ has the type α. Then α-pdim $\mathbf{G} = c(C(\mathbf{H}^{\mathbf{G}}), \Theta(\mathbf{H}))$.*

Proof. By definition, we have $c(C(\mathbf{H}^{\mathbf{G}}), \Theta(\mathbf{H})) = \min\{|U|;\ U \subseteq C(\mathbf{H}^{\mathbf{G}}), (U, C(\mathbf{H}^{\mathbf{G}})) \in \Theta(\mathbf{H})\}$. But $C(\mathbf{H}^{\mathbf{G}}) = \operatorname{Hom}(\mathbf{G}, \mathbf{H}) = S(X)$, which implies that the condition $(U, C(\mathbf{H}^{\mathbf{G}})) \in \Theta(\mathbf{H})$ is equivalent to the condition $(U, S(X)) \in \Theta(\mathbf{H})$. This means $T(U) = T(S(X)) = X$ by Theorem 2.2. But this is equivalent to the condition that U is an $\mathbf{H}$-realizer of the structure (G, X). It follows that $c(C(\mathbf{H}^{\mathbf{G}}), \Theta(\mathbf{H})) = \min\{|U|;\ U \subseteq H^G$ is an $\mathbf{H}$-realizer of $(G, X)\} = \alpha$-pdim $\mathbf{G}$. $\square$

In [4] we have proved the following result. If $\mathbf{H} = (H, Y)$ is a preordered set such that there exist elements $u, v \in H$ with $(u, v) \in Y$, $(v, u) \notin Y$ and if α is the type of $\mathbf{H}$, then for any preordered set $\mathbf{G} = (G, X)$ the pseudodimension α-pdim $\mathbf{G}$ exists. In this section we present particular n-ary relations that have an analogous property. In the whole section we suppose that $G \neq \emptyset$ is a set, $n \in \mathbb{N}$ a natural number such that $n \geqslant 2$, and $X \subseteq G^n$ is an n-ary relation on G.

The relation X is said to be *reflexive* if it contains all constant sequences of length n. The relation X will be referred to as *n-transitive* if it has the following property:

If $(x_1, \ldots, x_n) \in X$, $(y_1, \ldots, y_n) \in X$ hold and if there exist natural numbers i_0, j_0 such that $1 < i_0 \leqslant n$, $1 \leqslant j_0 < n$, $x_{i_0} = y_{j_0}$, then $(x_{i_1}, \ldots, x_{i_k}, y_{j_{k+1}}, \ldots, y_{j_n}) \in X$ for any natural numbers $1 \leqslant k < n$ and $i_1, \ldots, i_k, j_{k+1}, \ldots, j_n$ such that $1 \leqslant i_1 < \ldots < i_k < i_0$, $j_0 < j_{k+1} < \ldots < j_n \leqslant n$.

Example 3.1. For $n = 2$ we obtain that a binary relation is 2-transitive if and only if it is transitive in the usual sense.

Example 3.2. Let $n = 3$. Then a ternary relation X is 3-transitive if and only if it has the following properties.

(i) If $(x, y, z) \in X$, $(y, u, v) \in X$, then $(x, u, v) \in X$.
(ii) If $(x, y, z) \in X$, $(z, u, v) \in X$, then $(x, y, u) \in X$, $(x, y, v) \in X$, $(x, u, v) \in X$, $(y, u, v) \in X$.
(iii) If $(x, y, z) \in X$, $(u, z, v) \in X$, then $(x, y, v) \in X$.

An n-ary relation on a set G that is reflexive and n-transitive will be called an *n-ary preordering* on G. The n-ary structure (G, X) will be referred to as an *n-ary preordered set if* X is an n-ary preordering.

Let X be an n-ary relation on a set G. We define a binary relation $\mathbf{b}[X]$ on G as follows. For any $(x, y) \in G \times G$ we put $(x, y) \in \mathbf{b}[X]$ if there exists $(x_1, \ldots, x_n) \in X$ and natural numbers i, j such that $1 \leqslant i < j \leqslant n$, $x = x_i$, $y = x_j$.

Lemma 3.1. *Let X be an n-ary preordering on a set G. Then $(x, y) \in \mathbf{b}[X]$ holds if and only if $(x, y, \ldots, y) \in X$.*

Proof. If $(x, y, \ldots, y) \in X$, then $(x, y) \in \mathbf{b}[X]$ holds trivially.

Let $(x, y) \in \mathbf{b}[X]$. Then there exists $(x_1, \ldots, x_n) \in X$ and natural numbers i, j such that $1 \leqslant i < j \leqslant n$, $x = x_i$, $y = x_j$. Since $(y, \ldots, y) \in X$, the n-transitivity of X implies $(x, y, \ldots, y) \in X$. $\square$

Theorem 3.1. *Let X be an n-ary preordering on a set G. Then $\mathbf{b}[X]$ is a preordering on G.*

P r o o f. The reflexivity of the relation $\mathbf{b}[X]$ is obvious. Suppose $(x,y)\in\mathbf{b}[X]$, $(y,z)\in\mathbf{b}[X]$. By Lemma 3.1, we obtain $(x,y,\dots y)\in X$, $(y,z,\dots,z)\in X$ and the n-transivity implies $(x,z,\dots,z)\in X$, which entails $(x,z)\in\mathbf{b}[X]$. Thus $\mathbf{b}[X]$ is a transitive relation. □

Theorem 3.2. *Let X be an n-ary preordering on a set G. Then $(x_1,\dots,x_n)\in X$ holds if and only if $(x_i,x_j)\in\mathbf{b}[X]$ is satisfied for any natural numbers i, j with $1\leqslant i<j\leqslant n$.*

P r o o f. If $(x_1,\dots,x_n)\in X$, then $(x_i,x_j)\in\mathbf{b}[X]$ holds for any i, j with the property $1\leqslant i<j\leqslant n$ by the definition of $\mathbf{b}[X]$.

Suppose that $(x_i,x_j)\in\mathbf{b}[X]$ holds for any i, j with $1\leqslant i<j\leqslant n$. Then $(x_1,x_2)\in\mathbf{b}[X]$ implies $(x_1,x_2,\dots,x_2)\in X$ by Lemma 3.1. Similarly, $(x_2,x_3)\in\mathbf{b}[X]$ entails $(x_2,x_3,\dots,x_3)\in X$. The n-transitivity of X implies that $(x_1,x_2,x_3,\dots,x_3)\in X$.

Suppose that we have proved that $(x_1,\dots,x_i,x_i,\dots,x_i)\in X$ for some i with $1<i<n$. Since $(x_i,x_{i+1})\in\mathbf{b}[X]$ holds, we have $(x_i,x_{i+1},\dots,x_{i+1})\in X$ and the n-transitivity of X implies $(x_1,\dots,x_i,x_{i+1},\dots,x_{i+1})\in X$. After n steps we obtain $(x_1,\dots,x_n)\in X$. □

Theorem 3.3. *Let (G,X), (H,Y) be n-ary preordered sets. Then* $\operatorname{Hom}((G,X),(H,Y))=\operatorname{Hom}((G,\mathbf{b}[X]),(H,\mathbf{b}[Y]))$.

P r o o f. (1) Suppose $h\in\operatorname{Hom}((G,X),(H,Y))$, $(x,y)\in\mathbf{b}[X]$. Then there exist $(x_1,\dots,x_n)\in X$ and natural numbers i, j such that $1\leqslant i<j\leqslant n$, $x=x_i$, $y=x_j$. Therefore $(h(x_1),\dots,h(x_n))\in Y$ whence $(h(x),h(y))\in\mathbf{b}[Y]$. It follows that $h\in\operatorname{Hom}((G,\mathbf{b}[X]),(H,\mathbf{b}[Y]))$.

(2) Let $h\in\operatorname{Hom}((G,\mathbf{b}[X]),(H,\mathbf{b}[Y]))$, $(x_1,\dots,x_n)\in X$. Then $(x_i,x_j)\in\mathbf{b}[X]$ holds for any i, j with $1\leqslant i<j\leqslant n$, which implies $(h(x_i),h(x_j))\in\mathbf{b}[Y]$ for any i, j with $1\leqslant i<j\leqslant n$; it follows that $(h(x_1),\dots,h(x_n))\in Y$ by Theorem 3.2. Therefore $h\in\operatorname{Hom}((G,X),(H,Y))$. □

An n-ary preordering X on a set G will be said to be *nontrivial* if there exist elements $x,y\in G$ such that $(x,y)\in\mathbf{b}[X]$, $(y,x)\notin\mathbf{b}[X]$.

Theorem 3.4. *Let (G,X), (H,Y) be n-ary preordered sets where the n-ary preordering Y is nontrivial. Then the relation X is determined by the set* $\operatorname{Hom}((G,X),(H,Y))$.

P r o o f. By hypothesis there are elements $a,b\in H$ such that $(a,b)\in\mathbf{b}[Y]$, $(b,a)\notin\mathbf{b}[Y]$. Suppose that $(x_1,\dots,x_n)\in G^n$, $(x_1,\dots,x_n)\notin X$. By Theorem 3.2 there exist natural numbers i, j such that $1\leqslant i<j\leqslant n$ and $(x_i,x_j)\notin\mathbf{b}[X]$.

We define a mapping h of the set G into H as follows.

$$h(x) = \begin{cases} a & \text{if } (x, x_j) \in \mathbf{b}[X]; \\ b & \text{if } (x, x_j) \notin \mathbf{b}[X]. \end{cases}$$

We prove that $h \in \mathrm{Hom}((G,X),(H,Y))$; by Theorem 3.3 it is sufficient to prove that $h \in \mathrm{Hom}((G,\mathbf{b}[X]),(H,\mathbf{b}[Y]))$. Let $(x,y) \in \mathbf{b}[X]$. If $h(y) = b$, we obtain $(h(x),h(y)) \in \mathbf{b}[Y]$ because the relation $\mathbf{b}[Y]$ is reflexive. If $h(y) = a$, we have $(y,x_j) \in \mathbf{b}[X]$ and the transitivity of the relation $\mathbf{b}[X]$ implies $(x,x_j) \in \mathbf{b}[X]$ whence $h(x) = a$ and we have $(h(x),h(y)) \in \mathbf{b}[Y]$, too. Thus we obtain $h \in \mathrm{Hom}((G,X),(H,Y))$. It follows from the definition of h that $h(x_i) = b$, $h(x_j) = a$ and, therefore, $(h(x_i),h(x_j)) \notin \mathbf{b}[Y]$. By Theorem 3.2, we obtain $(h(x_1),\ldots,h(x_n)) \notin Y$. □

Corollary 3.1. *Let α be the type of a nontrivial n-ary preordered set. Then for any n-ary preordered set (G,X) the pseudodimension α-pdim(G,X) exists.*

The proof follows from Theorem 3.4 and Theorem 2.6.

Corollary 3.2. *Let α be the type of a nontrivial preordered set. Then for any preordered set (G,P) the pseudodimension α-pdim(G,P) exists.*

(Cf. [4], Corollary of Theorem 4.1.)

4. n-ary preordering versus preordering

In the whole section we suppose that $G \neq \emptyset$ is a set and n is a natural number such that $n \geqslant 2$.

Let B be a binary relation on a set G. We define an n-ary relation $\mathbf{n}[B]$ on the set G by putting $(x_1,\ldots,x_n) \in \mathbf{n}[B]$ if $(x_i,x_j) \in B$ for any natural numbers i, j with $1 \leqslant i < j \leqslant n$.

Theorem 4.1. *Let B be a preordering on a set G. Then $\mathbf{n}[B]$ is an n-ary preordering on G.*

P r o o f. Since $(x,x) \in B$ holds for any $x \in G$, we obtain $(x,\ldots,x) \in \mathbf{n}[B]$ for any $x \in G$, i.e. $\mathbf{n}[B]$ is reflexive. Suppose $(x_1,\ldots,x_n) \in \mathbf{n}[B]$, $(y_1,\ldots,y_n) \in \mathbf{n}[B]$, $x_{i_0} = y_{j_0}$ for some natural numbers i_0, j_0 such that $1 < i_0 \leqslant n$, $1 \leqslant j_0 < n$, let $1 \leqslant i_1 < \ldots < i_k < i_0$, $j_0 < j_{k+1} < \ldots < j_n \leqslant n$ for some k with $1 \leqslant k < n$. Then $(x_{i_r}, x_{i_s}) \in B$ holds for any natural numbers r, s with $1 \leqslant r < s \leqslant k$, $(y_{j_p}, y_{j_q}) \in B$ for any natural numbers p, q with $k+1 \leqslant p < q \leqslant n$. Furthermore, we have $(x_{i_s}, x_{i_0}) \in B$, $(y_{j_0}, y_{j_p}) \in B$, $x_{i_0} = y_{j_0}$. The transitivity of B implies that $(x_{i_s}, y_{j_p}) \in B$ for any s with $1 \leqslant s \leqslant k$ and any p with $k+1 \leqslant p \leqslant n$. It follows that $(x_{i_1}, \ldots, x_{i_k}, y_{j_{k+1}}, \ldots, y_{j_n}) \in \mathbf{n}[B]$ and so $\mathbf{n}[B]$ is n-transitive. $\square$

Theorem 3.2 may be reformulated as follows.

Theorem 4.2. *Let X be an n-ary preordering on a set G. Then $\mathbf{n}[\mathbf{b}[X]] = X$.*

We obtain also

Theorem 4.3. *Let B be a preordering on a set G. Then $\mathbf{b}[\mathbf{n}[B]] = B$.*

Proof. If $(x, y) \in B$ then $(x, y, \ldots, y) \in \mathbf{n}[B]$ and, therefore, $(x, y) \in \mathbf{b}[\mathbf{n}[B]]$, which implies $B \subseteq \mathbf{b}[\mathbf{n}[B]]$. On the other hand, if $(x, y) \in \mathbf{b}[\mathbf{n}[B]]$, then Theorem 4.1 and Lemma 3.1 imply that $(x, y, \ldots, y) \in \mathbf{n}[B]$; hence $(x, y) \in B$ and, therefore, $\mathbf{b}[\mathbf{n}[B]] \subseteq B$. $\square$

We denote by $\mathbf{P}_n$ the category whose objects are n-ary preordered sets and whose morphisms are homomorphisms of these structures. It is easy to see that $\mathbf{P}_n$ is a category. For $n = 2$ homomorphisms coincide with isotone mappings.

Theorem 4.4. *Let $n \geqslant 2$. Then the categories $\mathbf{P}_n$ and $\mathbf{P}_2$ are isomorphic.*

Proof. The isomorphism of $\mathbf{P}_n$ onto $\mathbf{P}_2$ will be defined by the object mapping Fo and the morphism mapping Fm. If (G, X) is an object of the category $\mathbf{P}_n$, we put $Fo(G, X) = (G, \mathbf{b}[X])$. If (G, X), (H, Y) are objects of the category $\mathbf{P}_n$ and h a morphism of (G, X) into (H, Y), we put $Fm(h) = h$.

By Theorem 3.1, Fo is a mapping of the class of all objects of the category $\mathbf{P}_n$ into the class of objects in $\mathbf{P}_2$. By Theorems 4.2 and 4.3, the mapping $\mathbf{b}$ is a bijection, which implies that the mapping Fo is a bijection, too.

By Theorem 3.3, the mapping Fm is a bijection of the class of all morphisms of the category $\mathbf{P}_n$ onto the class of all morphisms of the category $\mathbf{P}_2$. Clearly, Fm preserves identity mappings and compositions of morphisms.

It follows that F is an isomorphism. $\square$

Let (H, Y) be an n-ary preordered set of type α. We denote by $\mathbf{b}[\alpha]$ the type of the preordered set $(H, \mathbf{b}[Y])$.

We prove that there exists a very close relationship between the α-pseudodimension of an n-ary preordered set and the $\mathbf{b}[\alpha]$-pseudodimension of the corresponding preordered set.

Lemma 4.1. *Let (G, X), (H, Y) be n-ary preordered sets and suppose $U \subseteq$ $\mathrm{Hom}((G, X), (H, Y))$. Then the set U is an (H, Y)-realizer of the structure (G, X) if and only if it is an $(H, \mathbf{b}[Y])$-realizer of the structure $(G, \mathbf{b}[X])$.*

P r o o f. (1) Let U be an (H,Y)-realizer of the structure (G,X). By Theorem 3.3, we obtain $U \subseteq \mathrm{Hom}((G,\mathbf{b}[X]),(H,\mathbf{b}[Y]))$. Suppose $(x,y) \notin \mathbf{b}[X]$; by Lemma 3.1, we have $(x,y,\ldots,y) \notin X$. Consequently, there exists $h \in U$ such that $(h(x),h(y),\ldots,h(y)) \notin Y$, which entails $(h(x),h(y)) \notin \mathbf{b}[Y]$. It follows that the set U is an $(H,\mathbf{b}[Y])$-realizer of the structure $(G,\mathbf{b}[X])$.

(2) Suppose that U is an $(H,\mathbf{b}[Y])$-realizer of the structure $(G,\mathbf{b}[X])$. Then $U \subseteq \mathrm{Hom}((G,X),(H,Y))$. Let $(x_1,\ldots,x_n) \notin X$; by Theorem 3.2, there exist natural numbers i, j such that $1 \leqslant i < j \leqslant n$, $(x_i,x_j) \notin \mathbf{b}[X]$. Thus there exists $h \in U$ such that $(h(x_i),h(x_j)) \notin \mathbf{b}[Y]$. By Theorem 3.2, we obtain $(h(x_1),\ldots,h(x_n)) \notin Y$. Therefore the set U is an (H,Y)-realizer of the structure (G,X). $\square$

Theorem 4.5. *Let (G,X), (H,Y) be n-ary preordered sets where Y is nontrivial and α is the type of (H,Y). Then α-pdim$(G,X) = \mathbf{b}[\alpha]$-pdim$(G,\mathbf{b}[X])$.*

The proof follows from Lemma 4.1.

Let (H,P) be a preordered set of type β. We denote by $\mathbf{n}[\beta]$ the type of the n-ary preordered set $(H,\mathbf{n}[P])$.

In a similar way we may prove

Theorem 4.6. *Let (G,B) be a preordered set, (H,P) a nontrivial preordered set of type β. Then β-pdim$(G,B) = \mathbf{n}[\beta]$-pdim$(G,\mathbf{n}[B])$.*

5. n-EQUIVALENCE

Suppose that $G \neq \emptyset$ is a set, $n \geqslant 2$ a natural number and X an n-ary relation on G. The relation X will be referred to as *strongly symmetric* if $(x_1,\ldots,x_n) \in X$ implies $(x_{p(1)},\ldots,x_{p(n)}) \in X$ for any permutation p of the set $\{1,\ldots,n\}$. An n-ary relation on the set G that is reflexive, strongly symmetric, and n-transitive (i.e. a strongly symmetric n-ary preordering) will be called an *n-equivalence* on G. (The reader must be warned: The expression "n-equivalence" appears in Section 5 of [4] in a different meaning! In the present paper, we respect the definition presented here.)

Theorem 5.1. *Let X be an n-equivalence on the set G. Then $\mathbf{b}[X]$ is an equivalence relation on G.*

P r o o f. By Theorem 3.1, the relation $\mathbf{b}[X]$ is a preordering on G. If $(x,y) \in \mathbf{b}[X]$ holds then there exist $(x_1,\ldots,x_n) \in X$ and natural numbers i, j such that $1 \leqslant i < j \leqslant n$, $x = x_i$, $y = x_j$. Then $(y_1,\ldots,y_n) \in X$ holds where $y_k = x_k$ for any $k \neq i$, $k \neq j$, $y_i = y$, $y_j = x$. It follows that $(y,x) \in \mathbf{b}[X]$ and the relation $\mathbf{b}[X]$ is symmetric. $\square$

Theorem 5.2. *Let B be an equivalence relation on the set G. Then $\mathbf{n}[B]$ is an n-equivalence on G.*

P r o o f. By Theorem 4.1, $\mathbf{n}[B]$ is an n-ary preordering on G. Let $(x_1, \ldots, x_n) \in \mathbf{n}[B]$ and choose an arbitrary permutation p of the set $\{1, \ldots, n\}$. Then $(x_i, x_j) \in B$ holds for arbitrary natural numbers i, j such that $1 \leqslant i < j \leqslant n$; the reflexivity and symmetry of B imply $(x_s, x_t) \in B$ for arbitrary natural numbers s, $t \in \{1, \ldots, n\}$, which entails $(x_{p(i)}, x_{p(j)}) \in B$ for arbitrary natural numbers i, j with $1 \leqslant i < j \leqslant n$. It follows that $(x_{p(1)}, \ldots, x_{p(n)}) \in \mathbf{n}[B]$ and the relation $\mathbf{n}[B]$ is strongly symmetric. □

Theorem 4.2 and Theorem 4.3 imply, in particular,

Theorem 5.3. *If X is an n-equivalence on a set G, then $\mathbf{n}[\mathbf{b}[X]] = X$. If B is an equivalence relation on a set G, then $\mathbf{b}[\mathbf{n}[B]] = B$.*

We denote by $\mathbf{E}_n$ the subcategory of the category $\mathbf{P}_n$ whose objects are sets with n-equivalences and whose morphisms are homomorphisms. Similarly as Theorem 4.4 we may prove

Theorem 5.4. *The categories $\mathbf{E}_n$ and $\mathbf{E}_2$ are isomorphic.*

An n-equivalence X on a set G will be referred to as *nontrivial* if there exist elements $x, y \in G$ such that $(x, y) \notin \mathbf{b}[X]$.

Theorem 5.5. *Let G, H be sets such that $|G| \geqslant 2$, $|H| \geqslant 2$ and suppose that Y is a nontrivial n-equivalence on the set H. Then an arbitrary n-equivalence X on the set G is determined by the set $\mathrm{Hom}((G, X), (H, Y))$.*

P r o o f. Let $u, v \in H$ be elements such that $(u, v) \notin \mathbf{b}[Y]$. Suppose $(x_1, \ldots, x_n) \in G^n$, $(x_1, \ldots, x_n) \notin X$; by Theorem 3.2, there exist natural numbers i, j such that $1 \leqslant i < j \leqslant n$, $(x_i, x_j) \notin \mathbf{b}[X]$. We define a mapping h of the set G into H as follows:

$$h(x) = \begin{cases} u & \text{if } (x, x_i) \in \mathbf{b}[X]; \\ v & \text{if } (x, x_i) \notin \mathbf{b}[X]. \end{cases}$$

We prove that $h \in \mathrm{Hom}((G, \mathbf{b}[X]), (H, \mathbf{b}[Y]))$. Suppose that $(x, y) \in \mathbf{b}[X]$ holds. If $(x, x_i) \in \mathbf{b}[X]$ then $(y, x_i) \in \mathbf{b}[X]$ because the relation $\mathbf{b}[X]$ is an equivalence relation. It follows that $h(x) = h(y) = u$. If $(x, x_i) \notin \mathbf{b}[X]$, then $(y, x_i) \notin \mathbf{b}[X]$ and, therefore, $h(x) = h(y) = v$. In either case we obtain $(h(x), h(y)) \in \mathbf{b}[Y]$ and hence $h \in \mathrm{Hom}((G, \mathbf{b}[X]), (H, \mathbf{b}[Y]))$. By Theorem 3.3, we have $h \in \mathrm{Hom}((G, X), (H, Y))$. Furthermore, we obtain $h(x_i) = u$, $h(x_j) = v$, which implies that $(h(x_i), h(x_j)) \notin \mathbf{b}[Y]$. We have $(h(x_1), \ldots, h(x_n)) \notin Y$ by Theorem 3.2. □

Corollary 5.1. *Let Y be a nontrivial n-equivalence on a set H, denote by α the type of the structure (H, Y). Then for any set $G \neq \emptyset$ and any n-equivalence X on G the pseudodimension α-$\mathrm{pdim}(G, X)$ exists.*

Corollary 5.2. *Let B be a nontrivial equivalence relation on a set H, denote by α the type of the structure (H, B). Then for any set $G \neq \emptyset$ and any equivalence relation P on G the pseudodimension α-$\mathrm{pdim}(G, P)$ exists.*

Corollary 5.3. *Let P be an equivalence relation on a set $G \neq \emptyset$. Then for any cardinal $m \geqslant 2$ the pseudodimension m-$\mathrm{pdim}(G, X)$ exists where m denotes the type of the structure (H, id_H) with the property $|H| = m$.*

(Cf. [4], Corollary of Theorem 5.1.)

6. Examples

Example 6.1. Let $\mathbf{G} = (G, B)$ be an ordered set where $G = \{x, y, z, u\}$ and $B = \{(x, x), (x, z), (y, y), (y, z), (y, u), (z, z), (u, u)\}$. In Example 1 of [4], we found that $\mathbf{3}$-$\mathrm{pdim}\,\mathbf{G} = 2$, where $\mathbf{3} = (H, \leqslant)$, $H = \{0, 1, 2\}$, $\leqslant = \{(0, 0), (0, 1), (0, 2), (1, 1), (1, 2), (2, 2)\}$.

Put $n = 3$; then $\mathbf{n}[B] = \{(x, x, x), (x, x, z), (x, z, z), (y, y, y), (y, y, z), (y, z, z), (y, y, u), (y, u, u), (z, z, z), (u, u, u)\}$. Similarly, $\mathbf{n}[\leqslant] = \{(0, 0, 0), (0, 0, 1), (0, 1, 1), (0, 0, 2), (0, 2, 2), (1, 1, 1), (1, 1, 2), (1, 2, 2), (0, 1, 2), (2, 2, 2)\}$.
By Theorem 4.2, we obtain $\mathbf{b}[\mathbf{n}[B]] = B$, $\mathbf{b}[\mathbf{n}[\leqslant]] = \leqslant$. By Lemma 4.1, the structure $(G, \mathbf{n}[B])$ has an $(H, \mathbf{n}[\leqslant])$-realizer if and only if (G, B) has an $(H, \leqslant)$-realizer; this $(H, \leqslant)$-realizer has been found in Example 1 of [4]. By Theorem 4.6, we obtain $\mathbf{n}[\mathbf{3}]$-$\mathrm{pdim}(G, \mathbf{n}[B]) = \mathbf{3}$-$\mathrm{pdim}(G, B) = 2$.

Example 6.2. Let $m \geqslant 2$, $n \geqslant 2$ be integers, $G_1, \ldots, G_m$ mutually disjoint finite nonempty sets, let X_i denote the set of all finite sequences of length n formed of elements in G_i for any i with $1 \leqslant i \leqslant m$. Put $G = G_1 \cup \ldots \cup G_m$, $X = X_1 \cup \ldots \cup X_m$. Then (G, X) is an n-ary structure where X is an n-equivalence. Denote by E the equivalence on G whose blocks are the sets G_i $(1 \leqslant i \leqslant m)$.

Suppose that $H = \{a, b\}$, $a \neq b$. Put $Y = \{(a, \ldots, a), (b, \ldots, b)\}$ where the first sequence is formed by n symbols equal to a and the second by n symbols that are equal to b. Then Y is an n-equivalence on H. By Corollary 5.1, the α-$\mathrm{pdim}(G, X)$ exists where α is the type of (H, Y). By Theorem 4.5, we obtain α-$\mathrm{pdim}(G, X) = \mathbf{b}[\alpha]$-$\mathrm{pdim}(G, \mathbf{b}[X]) = 2$-$\mathrm{pdim}(G, E)$.

Let $(G, \bullet)$ be a groupoid (cf., e.g., [2]). We put $X(\bullet) = \{(x, y, \bullet(x, y));\ (x, y) \in G \times G\}$. Then the following lemma is easy to prove (cf. Example 2 of [5]).

Lemma 6.1. *If $(G, \bullet)$, $(H, \circ)$ are groupoids and h is a mapping of the set G into H, then h is a homomorphism of the groupoid $(G, \bullet)$ into $(H, \circ)$ if and only if it is a homomorphism of the ternary structure $(G, X(\bullet))$ into $(H, X(\circ))$.*

Hence, groupoids may be regarded as ternary structures and all concepts introduced for structures may be introduced for groupoids as well. In particular, we may consider realizers of groupoids. The problem of existence of an $(H, \circ)$-realizer of a groupoid $(G, \bullet)$ is solved in the following theorem.

Theorem 6.1. *Let $(G, \bullet)$, $(H, \circ)$ be groupoids, $V \neq \emptyset$ a set of homomorphisms of the groupoid $(G, \bullet)$ into $(H, \circ)$. Then the following two assertions are equivalent.*

(i) *V is an $(H, \circ)$-realizer of $(G, \bullet)$.*

(ii) *If x, y, z in G are arbitrary elements such that $z \neq \bullet(x, y)$, then there exists a homomorphism $f \in V$ such that $f(z) \neq \circ(f(x), f(y))$.*

Proof. Using Lemma 6.1, condition (ii) may be reformulated as follows. If $(x_1, x_2, x_3) \in G^3$ are arbitrary and $(x_1, x_2, x_3) \notin X(\bullet)$, then there exists a homomorphism $f \in V$ of the structure $(G, X(\bullet))$ into $(H, X(\circ))$ such that $(f(x_1), f(x_2), f(x_3)) \notin X(\circ)$. By Lemma 2.2, this is equivalent to the fact that V is an $(H, X(\circ))$-realizer of the structure $(G, X(\bullet))$. Using Lemma 6.1, this can be reformulated in the form of (i). $\square$

Example 6.3. Let $n \geqslant 2$ be an integer. We denote by $\mathbf{Z}_n = (Z_n, +_n)$ the groupoid where $Z_n = \{0, \ldots, n-1\}$ and $+_n$ is the addition modulo n.

Choose $n \geqslant 3$, $(G, \bullet) = (Z_n, +_n)$, $(H, \circ) = (Z_2, +_2)$. Then $1 +_n 1 \neq 0$. If f is an arbitrary homomorphism of $\mathbf{Z}_n$ into $\mathbf{Z}_2$, then $f(0) = 0$ and either $f(1) = 0$ or $f(1) = 1$. In the first case, we obtain $f(1) +_2 f(1) = 0 +_2 0 = 0 = f(0)$, in the other, we have $f(1) +_2 f(1) = 1 +_2 1 = 0 = f(0)$. By Theorem 6.1, there exists no $\mathbf{Z}_2$-realizer of $\mathbf{Z}_n$.

Example 6.4. Let $(G, \vee)$ be a finite upper semi-lattice with a least element 0 and a greatest element 1 (cf. [1], p. 22), let $(H, \cup)$ be a two-element upper semilattice with elements 0 and 1. As usual, put $x \leqslant y$ if and only if $x \vee y = y$ for any x, y in G. It is easy to see that for any homomorphism f of the semilattice $(G, \vee)$ into $(H, \cup)$, there exists an element $g_f \in G$ such that $f(t) = 0$ for any $t \in G$ with the property $t \leqslant g_f$, and $f(t) = 1$ for any $t \in G$ with $t \nleqslant g_f$.

Let a, b, c be elements in G such that $a \vee b \neq c$. If $c \nleqslant a \vee b$, put $f(t) = 0$ for any $t \in G$ with $t \leqslant a \vee b$, and $f(t) = 1$ else. Then $f(a) \cup f(b) = 0 \cup 0 = 0 \neq 1 = f(c)$. Suppose $c \leqslant a \vee b$, i.e., $c < a \vee b$. Define $f(t) = 0$ for any $t \in G$ with $t \leqslant c$ and $f(t) = 1$ else. Then either $a \nleqslant c$ or $b \nleqslant c$ (in the opposite case, we would obtain $a \vee b \leqslant c$) and, therefore, $f(a) \cup f(b) = 1 \neq 0 = f(c)$. Thus, for any a, b, c in G with $a \vee b \neq c$, there exists a homomorphism f of the semilattice $(G, \vee)$ into $(H, \cup)$ such that $f(a) \cup f(b) \neq f(c)$, which implies that the set of all homomorphisms of the semilattice $(G, \vee)$ into $(H, \cup)$ is an $(H, \cup)$-realizer of the semilattice $(G, \vee)$.

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