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THE COREGULAR PROPERTY ON γ-SPACES

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1. Introduction

A γ-space stands for a T_1 locally quasi-uniform space with a countable base [LF, p. 234]. A filter $\mathscr{U}$ of *neighbornets* on a topological space (X, τ) is a filter of entourages in $X \times X$ which induces on X the topology τ itself, in which case we write $\tau = \tau(\mathscr{U})$ and say that $\mathscr{U}$ is *compatible* with τ. As always $\mathscr{U}^{-1}$ stands for the *dual* (or *conjugate*) of $\mathscr{U}$ and $\mathscr{U}^*$ for the supremum of $\mathscr{U}$ and $\mathscr{U}^{-1}$. If $\mathscr{U}^{-1}$ also induces a topology on X, then the $\tau(\mathscr{U}^{-1}) \times \tau(\mathscr{U})$-neighborhoods of the diagonal constitute a filter $\mathscr{W}$ of neighbornets which induces on X the topology $\tau(\mathscr{U})$ as well. It is well known that the filter $\mathscr{W}$ in some cases induces on X a local quasi-uniformity even if $\mathscr{U}$ does not.

The *γ-space conjecture* is the conjecture that "every γ-space is quasi-metrizable." Although Fox's counterexample (in [F1]) proved that the conjecture is false, it remains for many topologists a great task to characterize these γ-spaces which are quasi-metrizable. To state some of them we refer to [FL], [FK], [Ku1] [Ku2], [Ku3], [LF], [Ko] etc. Despite the wide variety of views on the subject, it seems that the following characterization of the quasi-metrizability of γ-spaces remains the stronger conclusion (cf. [FL p. 162, th. 2.15], [Ku3, p. 62, th. 5], as well in [Kop, p. 103, th. 2.2]):

(∗) "If in a bitopological space $(X, \mathscr{U}, \mathscr{U}^{-1})$ $\mathscr{U}$ is a T_1 local quasi-uniformity with a countable base and also $\mathscr{U}^{-1}$ is a local quasi-uniformity, then there may be defined a quasi-uniformity with a countable base, hence a quasi-metrizable space, which induces on X the topology induced by $\mathscr{U}$."

Our main purpose in this paper is to weaken the conditions cited in (∗), especially those refering to $\mathscr{U}^{-1}$, and thus to enlarge the category of the spaces which the (∗) theorem determines. To this end we introduce the notion of "*coregularity.*" A locally quasi-uniform space is coregular but not inversely. We also make use of the well known (cf. [W]) *neighborhood property.*

The remark 2.12 gives the suppositions for a coregular $\mathscr{U}^{-1}$ to solve the problem although the so called "property α" is not expressed in simple topological terms. It is worth noting that the omission of the local quasi-uniformity from $\mathscr{U}^{-1}$, causes the space to lose some crucial properties, even if $\mathscr{U}^{-1}$ is coregular; for instance the second countability of $\mathscr{U}$ does not imply the second countability of the filter which forms the $\tau(\mathscr{U}^{-1}) \times \tau(\mathscr{U})$-neighborhoods of the diagonal (a filter which plays a central role in the whole subject) or the relation $V[x] \subseteq U[x]$ does not imply $V^2[x] \subseteq U^2[x]$ etc. Thus there is room for serious changes like those of the theorem 2.10 and the remark 2.12 which are far enough removed from the current properties. Inversely, in paragraph 3 the suppositions of the statements mostly refer to the usual ones with emphasis in the symmetry, but they also have to do with the neighborhood property.

As always, theorems of this kind may be presented into the Nagata-Smirnov metrization theorem's form, as for instance occurs also for the theorems 3.2, 3.3 etc.

Let X be any non void set.

1.1 Definition. We call *generalized quasi-uniformity* (G$Q\mathscr{U}$ in brief) a filter $\mathscr{U}$ of reflexive relations on X which is a filter of neighbornets for a topology on X. We denote the structure by $(X, \mathscr{U})$ and we also call *entourages* the elements of $\mathscr{U}$.

Not any filter $\mathscr{U}$ on $X \times X$ is a $GQ\mathscr{U}$. A sufficient condition to be so is the following:

$$(1) \qquad (\forall\, x \in X)\ \ (\forall U \in \mathscr{U})(\forall\, y \in U[x])(\exists\ V \in \mathscr{U})\ [V[y] \subseteq U[x]].$$

If $\mathscr{U}$ and $\mathscr{U}^{-1}$ are GQ$\mathscr{U}$s, then—as well as in the case of quasi-uniform spaces—the $\tau(\mathscr{U}^{-1}) \times \tau(\mathscr{U})$—open sets in $X \times X$ constitute a base for the family of the neighborhoods of the diagonal.

1.2 Proposition. *If $\mathscr{U}$ is a $GQ\mathscr{U}$, then $\mathscr{U}^2 = \{U \in \mathscr{U} \mid (\exists V \in \mathscr{U})[V^2 \subseteq U]\}$ is too.*

P r o o f. Since $\mathscr{U}$ is $GQ\mathscr{U}$, for any $U \in \mathscr{U}$ and any $x \in X$, the set $(U[x])^\circ = \{y \in X/(\exists V \in \mathscr{U})[V[y] \subseteq U[x]\}$ is a $\tau(\mathscr{U})$-open neighborhood of x. If $W = U \cap V$, then $W^2[y] = W(W[y]) \subseteq W(U[x]) \subseteq U^2[x]$, hence the set $(U^2[x])^\circ = \{y \in X/(\exists W \in \mathscr{U})[W^2[y] \subseteq U^2[x]\}$ is also a $\tau(\mathscr{U}^2)$-open neighborhood of x. $\square$

1.3 Proposition. *If $\mathscr{U}$ and $\mathscr{U}^{-1}$ are $GQ\mathscr{U}$, then $\mathscr{U}^*$ is too.*

Proof. For any $x \in X$, if $U \in \mathscr{U}$ and $V^{-1} \in \mathscr{U}^{-1}$, $(U[x])^\circ$ and $(V^{-1}[x])^\circ$ are $\tau(\mathscr{U})$—and $\tau(\mathscr{U}^{-1})$—neighborhoods of x for $\tau(\mathscr{U})$ and $\tau(\mathscr{U}^{-1})$ respectively, then $(U[x])^\circ \cap (V^{-1}[x])^\circ$ is a $\tau(\mathscr{U}^*)$-neighborhood of x. □

2. Coregularity and the neighborhood property

From now on we denote by $LQ\mathscr{U}$ any locally quasi-uniform space (or any local quasi-uniformity).

2.1 Definition. (Cf. [Ke, p. 73, def. 2.4]). A bitopological space (X, τ_1, τ_2) is called *coregular (with respect to τ_1 as first, and τ_2 as second topology)* if for any $x \in X$ and any τ_1-neighborhood A of x there is another τ_1-neighborhood B of x such that

$$\overline{B}^{\tau_2} \subseteq A,$$

where $\overline{B}^{\tau_2}$ is the closure of B with respect to τ_2.

If we say in particular that the space $(X, \mathscr{U}, \mathscr{U}^{-1})$ is coregular, we mean that $\mathscr{U}$ and $\mathscr{U}^{-1}$ are $GQ\mathscr{U}s$ and that $\tau(\mathscr{U})$ is the first, and $\tau(\mathscr{U}^{-1})$ the second topology.

If the spaces (X, τ_1, τ_2) and (X, τ_2, τ_1) are coregular, then X is called *pairwise regular*, as it is well known.

2.2 Example of a coregular non $LQ\mathscr{U}$ space (Cf. [EL, ex. 4.3, p. 55]).

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{J}$ the set of irrational and $\mathbb{Q}$ that of rational numbers, τ_1 (resp. τ_2) the topology defined by taking as basic neighborhoods of any $x \in \mathbb{R}$ to be the intervals $[x, x+\varepsilon)$ (resp. $(x-\varepsilon, x])$ if $x \in \mathbb{J}$ and $(x-\varepsilon, x]$ (resp. $[x, x+\varepsilon)$) if $x \in \mathbb{Q}$; $\varepsilon > 0$. Since for any τ_1-basic neighborhood $V_\varepsilon[x]$ of a x, say irrational, the set ${V_\varepsilon}^2[x] = \bigcup_{t \in V_\varepsilon[x]} V_\varepsilon[t]$ contains points cited at the left of x, the topology does not arise a τ_1—as well as a τ_2—$LQ\mathscr{U}$. On the other hand, the requirement (1) of § 1 is fulfilled, the τ_2-closure of $V_{\varepsilon_0}[x]$ for $\varepsilon_0 < \varepsilon$, is subset of $[x, x+\varepsilon)$ and the $(\mathbb{R}, \tau_1, \tau_2)$ is coregular with respect to τ_1 as the first, and τ_2 as the second space.

The following theorem is basic in our procedure.

2.3 Theorem. *The topology of an $LQ\mathscr{U}$ space is coregular. Conversely; in a coregular space the set of the neighborhoods of the diagonal induces an $LQ\mathscr{U}$ space compatible with the given topology.*

P r o o f. Let $(X, \mathscr{U})$ be the $LQ\mathscr{U}$ space, $x \in X$ and $U \in \mathscr{U}$. Then, there is a $V_x \in \mathscr{U}$ such that $V_x \circ V_x[x] \subseteq U[x]$. On the other hand (cf. [MN, th. 1.15])

$$\overline{V_x[x]}^{\tau(\mathscr{U}^{-1})} = \bigcap_U \{U(V_x[x]) \colon U \in \mathscr{U}\},$$

hence $\overline{V_x[x]}^{\tau(\mathscr{U}^{-1})} \subseteq V_x(V_x[x]) \subseteq U[x]$.

Conversely; let $(X, \mathscr{U}, \mathscr{U}^{-1})$ be the coregular space, and $\mathscr{W}$ a base for the $\tau(\mathscr{U}^{-1}) \times \tau(\mathscr{U})$ neighborhoods of the diagonal. Then for $U \in \mathscr{W}$, $U[x]$ is a $\tau(\mathscr{U})$-neighborhood of x and there are $\tau(\mathscr{U})$-neighborhoods A_x, B_x and C_x of x such that

$$\overline{C_x}^{\tau(\mathscr{U}^{-1})} \subseteq B_x \subseteq \overline{B_x}^{\tau(\mathscr{U}^{-1})} \subseteq A_x \subseteq U[x].$$

Consider, as x runs through X, the sets

$$V_x = (X \times B_x) \cup [(X \setminus \overline{C_x}^{\tau(\mathscr{U}^{-1})}) \times A_x] \cup [(X \setminus \overline{B_x}^{\tau(\mathscr{U}^{-1})}) \times X].$$

Each V_x is $\tau(\mathscr{U}^{-1}) \times \tau(\mathscr{U})$-open, contains the diagonal and fulfils $V_x^2[x] \subseteq U[x]$. In fact, for the latter, there holds that $V_x^2[x] = V_x(V_x[x]) = V_x[B_x]$. On the other hand, if $t \in \overline{C_x}^{\tau(\mathscr{U}^{-1})}$ ($t \in B \setminus \overline{C_x}^{\tau(\mathscr{U}^{-1})}$), then $V_x(t) = B_x$ ($V_x(t) = A_x$) and $B_x \subseteq A_x \subseteq U[x]$.

2.4 Notation. For a GQ$\mathscr{U}$ $\mathscr{U}$ we put $\mathscr{U}^n = \{U \in \mathscr{U} \mid (\exists V \in \mathscr{U})[V^n \subseteq U]\}and(\mathscr{U}^{-1})^n = \mathscr{U}^{-n}$, for any $n \in \mathbb{N} \setminus \{0, 1\}$ and $\mathbb{N}$ the set of natural numbers.

2.5 Proposition (Cf. [W, th. 1.10]). *For any $LQ\mathscr{U}$ $\mathscr{U}$ (resp. $\mathscr{U}^{-1}$) on a space X and any $n \in \mathbb{N}$, $n \geqslant 2$, $\mathscr{U}^n$ (resp. $\mathscr{U}^{-n}$) is an $LQ\mathscr{U}$ compatible with $\tau(\mathscr{U})$ (resp. $\tau(\mathscr{U}^{-1})$).*

Proof. It suffices to prove the theorem for $\mathscr{U}^2$: firstly, for $x \in X$ and $V \in \mathscr{U}$ there is a $W_x \in \mathscr{U}$ such that $W_x^2[x] \subseteq V[x]$, $W_x^2 \in \mathscr{U}^2$, hence $\tau(\mathscr{U}) \subseteq \tau(\mathscr{U}^2)$. On the other hand, if $W \in \mathscr{U}^2$ and $x \in X$, then there is $V \in \mathscr{U}$ such that $V^2 \subseteq W$, hence $V[x] \subseteq V^2[x] \subseteq W[x]$ and $\tau(\mathscr{U}^2) \subseteq \tau(\mathscr{U})$.

Next, we prove that $\mathscr{U}^2$ is an $LQ\mathscr{U}$: if $U \in \mathscr{U}^2$, there are W_1, W_2, W_3 elements of $\mathscr{U}$ such that $W_3^4[x] \subseteq W_2^2[x] \subseteq W_1[x] \subseteq W_1^2[x] \subseteq U[x]$, hence $W_3^4[x] \subseteq U[x]$. □

2.6 Definition. (Cf. [W, def. 1]). We say that a bitopological space $(X, \mathscr{U}, \mathscr{U}^{-1})$, where $\mathscr{U}$ and $\mathscr{U}^{-1}$ are GQ$\mathscr{U}$s, has the *neighborhood property* if

$$(\forall U \in \mathscr{U})(\forall x \in X)(\exists V_x \in \mathscr{U})[V_x^{-1}(x) \times V_x(x) \subseteq U].$$

In such a case the subsets of the form $\bigcup_x \{V_x^{-1}[x] \times V_x[x]\}$ constitute a base for the neighborhood system of the diagonal.

2.7 Proposition. *If in the space $(X, \mathscr{U}, \mathscr{U}^{-1})$, $\mathscr{U}$ and $\mathscr{U}^{-1}$ are LQ$\mathscr{U}$s, then $\mathscr{U}^2$ has the neighborhood property.*

Proof. Let $U \in \mathscr{U}^2$ and $x \in X$. There are $W \in \mathscr{U}$ such that $W^2 \subseteq U$ and V_{1x}, V_{2x} in $\mathscr{U}$ such that $V_{1x}^2[x] \subseteq W[x]$ and $V_{2x}^{-2}[x] \subseteq W^{-1}[x]$. If $V_x = V_{1x} \cap V_{2x}$, then $V_x^2[x] \subseteq W[x]$ and $V_x^{-2}[x] \subseteq W^{-1}[x]$.

On the other hand for every $W \in \mathscr{U}$ and every $x \in X$, there holds $W^{-1}[x] \times W[x] \subseteq W^2$, since $(t_1, t_2) \in W^{-1}[x] \times W[x]$ implies that $(t_1, x) \in W$, $(x, t_2) \in W$ and thus $(t_1, t_2) \in W^2$, hence $V_x^{-2}[x] \times V_x^2[x] \subseteq W^{-1}[x] \times W[x] \subseteq W^2 \subseteq U$ and $V_x^2 \in \mathscr{U}^2$. □

2.8 Proposition. *If $(X, \mathscr{U})$ is an LQ$\mathscr{U}$ space and the bitopological space $(X, \mathscr{U}^{-1}, \mathscr{U})$ is coregular, then $\mathscr{W}^2$, where $\mathscr{W}$ is the set of all $\tau(\mathscr{U}^{-1}) \times \tau(\mathscr{U})$ neighborhoods of the diagonal, has the neighborhood property.*

Proof. After Theorem 2.3, $\mathscr{W}$ and $\mathscr{W}^{-1}$ are LQ$\mathscr{U}$s and by proposition 2.7, $\mathscr{W}^2$ has the neighborhood property. □

The following proposition is a necessary lemma for the 2.10 Theorem; we shall make use of it in some other case, as well.

2.9 Proposition. *If $\mathscr{U}$ and $\mathscr{U}^{-1}$ are GQ$\mathscr{U}$s the following statements are equivalent:*

(1) *$\mathscr{U}$ and $\mathscr{U}^{-1}$ are LQ$\mathscr{U}$s.*

(2) $(\forall U)(\forall x)(\exists V_x)(\forall \alpha \in V_x^{-1}[x])(\forall \beta \in V_x[x])\ [V_x^{-1}[\alpha] \times V_x[\beta] \subseteq U]$.

Proof. (1) $\Rightarrow$ (2).

Firstly: $(\forall W \in \mathscr{U})(\exists V_x \in \mathscr{U})[V_x^{-2}[x] \subseteq W^{-1}[x] \text{and} V_x^2[x] \subseteq W[x]].(*)$

From the proposition 2.7 we may assume that $\mathscr{U}$ has the neighborhood property. There holds:

$$(**) \qquad (\forall U \in \mathscr{U})(\forall x \in X)(\exists W_x \in \mathscr{U})[W_x^{-1}[x] \times W_x[x] \subseteq U].$$

The relations $(*)$ and $(**)$ imply that:

$$(\forall U \in \mathscr{U})(\forall x \in X)(\exists V_x \in \mathscr{U})[V_x^{-2}[x] \times V_x^2[x] \subseteq U].$$

Let $\alpha \in V_x^{-1}[x]$ and $\beta \in V_x[x]$, then

$$V_x^{-1}[\alpha] \times V_x[\beta] \subseteq V_x^{-1}(V_x^{-1}[x]) \times V_x(V_x[x])) \subseteq V_x^{-2}[x] \times V_x^2[x] \subseteq U.$$

(2) $\Rightarrow$ (1). If $t \in V_x^2[x]$, then there is a λ, such that $(x, \lambda) \in V_x$ and $(\lambda, t) \in V_x$. We have $\lambda \in V_x[x]$ and, because of (2) and the fact that $x \in V_x^{-1}[x]$, we have $V_x^{-1}[x] \times V_x[\lambda] \subseteq U$. Since $t \in V_x[\lambda]$, $(x, t) \in V_x^{-1}[x] \times V_x[\lambda] \subseteq U$, hence $t \in U[x]$, that is $V_x^2[x] \subseteq U[x]$. Similarly we conclude that $\mathscr{U}^{-1}$ is $LQ\mathscr{U}$. □

We now reach the main theorem of this paragraph.

2.10 Theorem. *If $(X,\mathscr{U})$ is an LQ$\mathscr{U}$ space with a countable base, the space $(X,\mathscr{U}^{-1},\mathscr{U})$ is coregular and the space $(X,\mathscr{U}^*)$ is Lindelöf, then the space $(X,\mathscr{U})$ admits a quasi-uniformity with a countable base.*

Proof. Let $\mathscr{A}=\{U_n \mid n\in\mathbb{N}\}$ be a base for $\mathscr{U}$, which may consist of $\tau(\mathscr{U}^{-1})\times\tau(\mathscr{U})$-neighborhoods of the diagonal, and $\mathscr{W}$ be the set of all $\tau(\mathscr{U}^{-1})\times\tau(\mathscr{U})$-neighborhoods of the diagonal. Then $\mathscr{W}$ and $\mathscr{W}^{-1}$ are LQ$\mathscr{U}$s and $\mathscr{W}^2$ has the neighborhood property (see prop. 2.8). On the other hand, since $\mathscr{U}^2$ is an $LQ\mathscr{U}$, and $\tau(\mathscr{U})=\tau(\mathscr{U}^2)$, without loss of the generality, we may assume that $\mathscr{W}$ has the neighborhood property and $\mathscr{U}$ is an LQ$\mathscr{U}$ with a countable base. We also assume that $\mathscr{A}$ is nested.

We define a sequence $\{V_n \mid n\in\mathbb{N}\}$ of entourages with the above cited requirements. Firstly, consider any member of $\mathscr{A}$, say U_1, and put

$$V_1=\bigcup_x\{U_1^{-1}[x]\times U_1[x]\}.$$

Since $V_1\in\mathscr{W}$, $\mathscr{W}$ and $\mathscr{W}^{-1}$ are LQ$\mathscr{U}$s and $\mathscr{W}$ has the neighborhood property, we conclude from proposition 2.9, that for any $x\in X$ there exists a $W_{1x}\in\mathscr{W}$ such that

$$(\forall\alpha\in W_{1x}^{-1}[x])(\forall\beta\in W_{1x}[x])[W_{1x}^{-3}[\alpha]\times W_{1x}^{3}[\underline{]}\subseteq V_1]$$

or

$$(\forall y\in W_{1x}^{-1}[x]\bigcap W_{1x}[x])[W_{1x}^{-3}[y]\times W_{1x}^{3}[y]\subseteq V_1].$$

The latter relation implies that

$$(\forall y\in W_{1x}^{-1}[x]\bigcap W_{1x}[x])[W_{1x}^{-3}[y]\times W_{1x}^{3}[y]\subseteq V_1],$$

an easy result in the case of an $LQ\mathscr{U}$ space whose the dual is an $LQ\mathscr{U}$ as well.

Put $W_{1x}^*=W_{1x}\cap U_2$ and $(W_{1x}^*)^{-1}=W_{1x}^{-1}\cap U_2^{-1}$ and the latter relation comes into:

$$(1)\qquad (\forall y\in (W_{1x}^*)^{-1}[x]\bigcap W_{1x}^*[x])[(W_{1x}^*)^{-3}[y]\times (W_{1x}^*)^{3}[y]\subseteq V_1].$$

On the other hand the class $\{(W_{1x}^*)^{-1}[x]\times W_{1x}^*[x]\mid x\in X\}$ constitutes a covering of the diagonal and the subsets $(W_{1x}^*)^{-1}[x]\cap W_{1x}^*[x]$ a covering of X. Since the space is Lindelöf, the class $\{W_{1x}^*[x]\mid x\in X \text{and} W_{1x}^*[x] \text{ fulfils (1)}\}$ can be refined into a countable subcovering $\{W_{\overline{1n_x}}{}^*[x_{n_x}]\mid \overline{1n_x}, n_x \text{ in } \mathbb{N}\}$.

Let $\overline{\overline{1n_x}} = \min\{\overline{1n_x}\colon\ x \in (W^*_{\overline{1n_x}})^{-1}[x_n] \cap W^*_{\overline{1n_x}}[x_n]\}$. We form from this countable subcovering, a nested family: $W^*_{1n_x} = \bigcap_{\overline{\overline{1k_x}} \leqslant \overline{\overline{1n_x}}} W^*_{\overline{\overline{1k_x}}}$. For this nested family there holds:

$$(W^*_{1n_x})^{-3}[x] \times (W^*_{1n_x})^3[x] \subseteq V_1.$$

Next, we put

$$V_2 = \bigcup_x \{(W^*_{1n_x})^{-1}[x] \times W^*_{1n_x}[x] \mid x \in X, 1n_x \geqslant 2\}.$$

We shall prove that $V_2^{\overset{2}{}} \subseteq V_1$ and $V_2 \subseteq U_2^{\overset{2}{}}$ (2)

Let $(x, y) \in V_2^2$; there is a $z \in X$ such that $(x, z) \in V_2$, $(z, y) \in V_2$. The latter relations mean that there are x_{n_k} and x_{n_λ} such that $x \in (W^*_{1n_k})^{-1}[x_{n_k}]$, $z \in W^*_{1n_k}[x_{n_k}]$ and $z \in (W^*_{1n_\lambda})^{-1}[x_{n_\lambda}]$, $y \in W^*_{1n_\lambda}[x_{n_\lambda}]$.

Assume that $W^*_{1n_k} \subseteq W^*_{1n_\lambda}$.

Then $(x_{n_k}, x) \in (W^*_{1n_k})^{-1}$, $(z, x_{n_k}) \in (W^*_{1n_k})^{-1}$, and $(x_{n_\lambda}, z) \in (W^*_{1n_\lambda})^{-1}$. Thus $(z, x) \in (W^*_{1n_k})^{-2} \subseteq (W^*_{1n_\lambda})^{-2}$, $(x_{n_\lambda}, z) \in (W^*_{1n_\lambda})^{-1}$ or $(x_{n_\lambda}, x) \in (W^*_{1n_\lambda})^{-3}$, that is $x \in (W^*_{1n_\lambda})^{-3}[x_{n_\lambda}]$.

It is $y \in W^*_{1n_\lambda}[x_{n_\lambda}]$ and last $(x, y) \in (W^*_{1n_\lambda})^{-3}[x_{n_\lambda}] \times W^*_{1n_\lambda}[x_{n_\lambda}] \subseteq V_1$.

If $W^*_{1n_\lambda} \subseteq W^*_{1n_k}$, then: $(x_{n_\lambda}, y) \in W^*_{1n_\lambda}$, $(z, x_{n_\lambda}) \in W^*_{1n_\lambda}$, and $(x_{n_k}, z) \in W^*_{1n_k}$, thus $(x_{n_k}, y) \in W^*_{1n_\lambda} \circ W^*_{1n_\lambda} \circ W^*_{1n_k} \subseteq (W^*_{1n_k})^3$ or $y \in (W^*_{1n_k})^3[x_{n_k}]$. Since $x \in (W^*_{1n_k})^{-1}[x_{n_k}]$, we have $(x, y) \in (W^*_{1n_k})^{-1}[x_{n_k}] \times (W^*_{1n_k})^3[x_{n_k}] \subseteq V_1$.

We also have that $(W^*_{1x})^{-1}[x] \times W^*_{1x}[x] \subseteq U_2^{-1}[x] \times U_2[x] \subseteq U_2^2$; consequently $V_2 \subseteq U_2^2$ and the proof of (2) is over.

We proceed to the construction of $(V_n)_{n\in\mathbb{N}}$ inductively: we assume that the finite sequence $V_1, V_2, V_3, \ldots, V_{k+1}$ is normal and that $(W^*_{kn_x})^{-1}[x]$ and $W^*_{kn_x}[x]$ play the respective roles of $U_1^{-1}[x]$ and $U_1[x]$. Then, there is a $W^*_{k+1,x} \in \mathscr{W}$ such that $W^*_{k+1,x} \subseteq W_{k+1,x} \cap U_{k+2} \subseteq U_{k+2}$ and $(W^*_{k+1,x})^{-3}[x] \times (W^*_{k+1,x})^3[x] \subseteq V_{k+1}$ and since the space is Lindelöf the class of $W^*_{k+1,x}$ may be refined into a countable nested family, W^*_{k+1,n_x}.

Put

$$V_{k+2} = \bigcup_x \{(W^*_{k+1,n_x})^{-1}[x] \times W^*_{k+1,n_x}[x] \mid x \in X, (k+1, n_x) \geqslant k+2\}.$$

We have that $V_{k+2}^{\overset{2}{}} \subseteq V_{k+1}$ and $V_{k+2} \subseteq U_{k+2}^{\overset{2}{}}$, (the demonstrations are as in the V_2-case).

It is also evident that $(V_n)_{n\in\mathbb{N}}$ induces in X a topology equivalent to $\tau(\mathscr{U})$ (since for each $n \in \mathbb{N}$, $V_n \subseteq U_n^{\overset{2}{}}$ and thus $\tau(\mathscr{U}) = \tau(\mathscr{U}^2) \subseteq \tau(\mathscr{V})$). $\square$

The proposition which follows refers to a space with a special family of neighborhoods of the diagonal (Künzi has mentioned the case in [Ku3, p. 63]) and its proof can be considered as a spesial case of one of the propositions which are contained in the latter demonstration.

2.11 Proposition. *If the bitopological space $(X, \mathscr{U}, \mathscr{U}^{-1})$ is pairwise regular and if the set $\mathscr{W}$ of the $\tau(\mathscr{U}^{-1}) \times \tau(\mathscr{U})$-neighborhoods of the diagonal admits a countable base, then the space is quasi-metrizable.*

2.12 Remark. Let S_1, S_2 be the categories of the following spaces cited in $(*)$ and in the 2.10 theorems, respectively:

$S_1 = (X, \mathscr{U}, \mathscr{U}^{-1})$, $\mathscr{U}$ and $\mathscr{U}^{-1}$ are LQ$\mathscr{U}$s and $\mathscr{U}$ admits a T_1 countable base.

$S_2 = (X, \mathscr{U}, \mathscr{U}^{-1})$, $\mathscr{U}$ is LQ$\mathscr{U}$ with a T_1 countable base and $\mathscr{U}^{-1}$ is coregular.

In general, every S_1 is quasi-metrizable; S_2 is not. Something more is needed, which for the sake of simplicity, let us name "property α."

(α) "There is a nested base $(W_i)_{i\in I}$ of the diagonal such that for every neighborhood W of the diagonal and any x there is a W_{ix}, element of the base, such that: $W_{ix}^{-3}[x] \times W_{ix}^{3}[x] \subseteq W$."

Property (α) is always fulfilled by the space S_1, is fulfilled by a space S_2 if it is Lindelöf and, of course, in other cases as well (for instance, if a space S_2 has a countable base of $\tau(\mathscr{U}^{-1}) \times \tau(\mathscr{U})$-neighborhoods of the diagonal). We do not know whether there are more adequate topological terms to express property (α), but it is clear that the properties of a S_2 space plus the property (α) establish a quasi-metrizability on the space and weaken the conditions cited in the space S_1.

3. Quasi-metrizability from symmetry and the neighborhood property

As usual we state that $\mathscr{U}^{-1}$ is *point symmetric* if $\mathscr{U}$ and $\mathscr{U}^{-1}$ are *GQ$\mathscr{U}$s* and $\tau(\mathscr{U}^{-1}) \subseteq \tau(\mathscr{U})$ (cf. [FL, p. 36], and [Ku3] under the title *strongly quasi-metrizable spaces.*)

3.1 Lemma. *If $(X, \mathscr{U})$ is an LQ$\mathscr{U}$ and $\mathscr{U}^{-1}$ is point symmetric then there exists a compatible LQ$\mathscr{U}$ $\mathscr{V}$ such that*

$$(**) \qquad (\forall U \in \mathscr{V})(\forall x \in X)(\exists V_x \in \mathscr{V})(\forall y \in V_x[x])[V_x[y] \times V_x[y] \subseteq U].$$

P r o o f. The demanded $LQ\mathscr{U}$ is the $\mathscr{U}^2$ which induces a topology equivalent to $\tau(\mathscr{U})$. Then for any $U \in \mathscr{U}^2$ there is a $V \in \mathscr{U}$ such that $V^2 \subseteq U$ and $V^{-1}[x] \times V[x] \subseteq U$ (by prop. 2.7). Since $\mathscr{U}^{-1}$ is point symmetric, for any $x \in X$ there are $V^* \in \mathscr{U}$

such that $V^*[x] \subseteq V^{-1}[x]$ and W^* such that $W^*[x] \times W^*[x] \subseteq U$ and finally there is a $W \in \mathscr{U}$ such that $W^4[x] \times W^4[x] \subseteq U$. Hence for any $y \in W^2[x]$, there holds $W^2[y] \times W^2[y] \subseteq W^4[x] \times W^4[x] \subseteq U$, where $W^2 \in \mathscr{U}^2$. □

3.2 Theorem. *If $(X, \mathscr{U})$ is a γ-space and $\mathscr{U}^{-1}$ is point symmetric, then $(X, \mathscr{U})$ is quasi-metrizable.*

P r o o f. Let $(U_n)_{n\in\mathbb{N}}$ be a countable base of $\mathscr{U}$, which we always assume as nested.

Put $V_1 = U_1$.

We conclude—by lemma 3.1—that for any $x \in X$ there is U_{n_x} such that

$$(\forall y \in U_{n_x}[x])[U_{n_x}[y] \times U_{n_x}[y] \subseteq V_1].$$

Put $m_x = \min\{n_t \mid U_{n_t}[x] \times U_{n_t}[x] \subseteq V_1\}$, where for any $x \in X$, $m_x \geqslant 2$, and

$$W_1 = \bigcup_x (\{x\} \times U_{m_x}[x]).$$

We show

$$(W_1 \cap W_1^{-1})^2 \subseteq V_1. \tag{1}$$

In fact, let $(x, y) \in (W_1 \cap W_1^{-1})^2$. Then there is a z, such that $(x, z) \in W_1 \cap W_1^{-1}$ and $(z, y) \in W_1 \cap W_1^{-1}$. It is $(z, x) \in W_1$, hence $x \in U_{m_z}[z]$ and $(z, y) \in W_1$, hence $y \in U_{m_z}[z]$. Thus, $(x, y) \in U_{m_z}[z] \times U_{m_z}[z] \subseteq V_1$.

Next, given $x \in X$ and $U_{n_x} \in \mathscr{U}$, we determine a $U_{1n_x} \in \mathscr{U}$ such that for any $y \in U_{1n_x}[x]$, there holds $U_{1n_x}[y] \times U_{1n_x}[y] \subseteq U_{n_x}$ (such a U_{1n_x} exists as shown in lemma 3.1). On the other hand we have for any $y \in U_{1n_x}[x]$,

$$U_{1n_x}[x] \subseteq U_{n_x}[y] \subseteq U_{m_y}[y] = W_1[y]. \tag{2}$$

We show that $U_{1n_x}[x] \times U_{1n_x}[x] \subseteq W_1 \cap W_1^{-1}$. (3).

Let $(t_1, t_2) \in U_{1n_x}[x] \times U_{1n_x}[x]$, then (from (2)) $t_1 \in U_{1n_x}[x]$ and $t_2 \in U_{1n_x}[x]$, hence $U_{1n_x}[x] \subseteq W_1[t_1]$ and $U_{1n_x}[x] \subseteq W_1[t_2]$. Since $t_2 \in U_{1n_x}[x] \subseteq W_1[t_1]$ and $t_1 \in U_{1n_x}[x] \subseteq W_1[t_2]$, we conclude that $(t_1, t_2) \in W_1 \cap W_1^{-1}$. Put

$$V_2 = \bigcup_x (U_{1n_x}[x] \times U_{1n_x}[x]). \tag{4}$$

There holds—by (3)—$V_2 \subseteq W_1 \cap W_1^{-1}$ and by (1)—$V_2^2 \subseteq V_1$ and $V_2[x] \subseteq U_2[x]$.

In fact for the latter relation, since for $x \in U_{1n_y}[y]$, there hold $U_{1n_y}[y] \subseteq U_{n_y}[x] \subseteq U_{m_y}[x] \subseteq U_2[x]$ and $V_2[x] = \bigcup\{U_{1n_y}[y] \mid x \in U_{1n_y}[y]\}$, it is implied that $V_2[x] \subseteq U_2[x]$.

As V_2 has the property $(**)$ of the lemma 3.1 we can inductively proceed to the construction of the V_{k+1} entourage from V_k. In fact, if we assume that V_k has the property $(**)$ and that $V_k[x] \subseteq U_k[x]$, then—as in the V_2 case—we construct V_{k+1} such that $V_{k+1}^2 \subseteq V_k$ and $V_{k+1}[x] \subseteq U_{k+1}[x]$.

Next, it is not difficult to be proved that the class $(V_n)_{n\in\mathbb{N}}$ constitutes a base for an $LQ\mathscr{U}$ $\mathscr{V}$ which induces on X a topology equivalent to $\tau(\mathscr{U})$. In fact, as $V_n \subseteq U_n$ for any $n \in \mathbb{N}$, we conclude that $\tau(\mathscr{U}) \subseteq \tau(\mathscr{V})$ and the inverse implication is immediate. □

We shall now state two conditions which make a γ-space developable and therefore—according to a theorem by Künzi ([Ku3, th. 1], and [F2])—quasi-metrizable.

3.3 Proposition. *If in an $LQ\mathscr{U}$ space $(X, \mathscr{U})$ the following condition is satisfied:*

$$\text{(i)} \quad (\forall U \in \mathscr{U})(\forall x \in X)(\exists V_x \in \mathscr{U})[V_x^{-1} \circ V_x[x] \subseteq U[x]]$$
$$(\textit{or its dual } \text{(i}')\quad (\forall \mathrm{U} \in \mathscr{U})(\forall \mathrm{x} \in \mathrm{X})(\exists \mathrm{V_x} \in \mathscr{U})[\mathrm{V_x} \circ \mathrm{V_x^{-1}}[\mathrm{x}] \subseteq \mathrm{U}[\mathrm{x}]])$$

and $\mathscr{U}^{-1}$ is $GQ\mathscr{U}$, then the space is developable.

Proof. Firstly, since $V_x^{-1}[x] \subseteq V_x^{-1} \circ V_x[x]$ (or $V_x^{-1}[x] \subseteq V_x \circ V_x^{-1}[x]$), $\mathscr{U}$ is point symmetric.

Next, consider $\langle V_n\rangle_{n\in\mathbb{N}}$ a nested base for $\mathscr{U}$, and G an open set containing x. The sequence $\langle \mathscr{G}_n\rangle_{n\in\mathbb{N}}$, where $\mathscr{G}_n = \{V_n[x] \mid x \in X, V_n \in \mathscr{U}\}$, is a sequence of open coverings of X and if V_n and V_m are in $\mathscr{U}$ so that $V_n[x] \subseteq G$ and $V_m^2[x] \subseteq V_n[x]$, then, since $\mathscr{U}$ is point symmetric, there is a $V_k \in \mathscr{U}$ such that $V_k^{-1}[x] \subseteq V_m[x]$ (1). We may assume that $k > m$. We have to prove that the set $\mathrm{St}(x, \mathscr{G}_k) = \cup\{V_k[t] \mid x \in V_k[t]\}$ is contained in G. Let $\lambda \in \mathrm{St}(x, \mathscr{G}_k)$, then $\lambda \in V_k[t]$ for a V_k and $x \in V_k[t]$, hence $t \in V_k^{-1}[x]$ and from (1), $t \in V_m[x]$. Therefore $\lambda \in V_k \circ V_m[x] \subseteq V_m^2[x] \subseteq V_n[x] \subseteq G$. □

It is also evident that:

3.4 Corollary. *A γ-space $(X, \mathscr{U})$ which fulfils the property* (i) *or* (i′) *and whose dual $(X, \mathscr{U}^{-1})$ is a $GQ\mathscr{U}$ space, is quasi-metrizable.*

We give some more cases of quasi-metrizable γ-spaces.

3.5 Lemma. *If in a bitopological space* $(X, \mathscr{U}, \mathscr{U}^{-1})$, $\mathscr{U}$ *and* $\mathscr{U}^{-1}$ *are* $GQ\mathscr{U}$ *and enjoy the following property:*

$$\text{(ii)} \quad (\forall U \in \mathscr{U})(\forall x \in X)(\exists V_x \in \mathscr{U})(\forall y \in V_x[x])[V_x[x] \times V_x[y] \subseteq U],$$

then (1) $\mathscr{U}$ *is an* $LQ\mathscr{U}$, (2) $\mathscr{U}^{-1}$ *is point symmetric, and* (3) *the space* $(X, \mathscr{U})$ *is regular.*

P r o o f. (1) If x, V_x and $\mathscr{U}$ are as above, then $t \in V_x^2[x]$ implies that for a z, $(x, z) \in V_x$ and $(z, t) \in V_x$, hence $z \in V_x[x]$, $t \in V_x[z]$ and $(x, t) \in V_x[x] \times V_x[z]$ (where $z \in V_x[x]$), hence (from (ii)) $(x, t) \in U$, that is $t \in U[x]$.

(2) If $t \in V_x[x]$, then $(t, x) \in V_x[x] \times V_x[x]$, thus $(t, x) \in U$ (from (ii)), hence $t \in U^{-1}[x]$ and $V_x[x] \subseteq U^{-1}[x]$.

(3) $\mathscr{U}$ is coregular, hence for an open A_x (in $\mathscr{U}$) there also exists a B_x open in $\mathscr{U}$ such that $\mathrm{cl}_{\tau(\mathscr{U}^{-1})} B_x \subseteq A_x$. Since $\tau(\mathscr{U}^{-1}) \subseteq \tau(\mathscr{U})$ we have $\mathrm{cl}_{\tau(\mathscr{U})} B_x \subseteq \mathrm{cl}_{\tau(\mathscr{U}^{-1})} B_x$ and the proof is over. □

After the lemma the following theorem is evident.

3.6 Theorem. *If* $\mathscr{U}$ *and* $\mathscr{U}^{-1}$ *in the bitopological space* $(X, \mathscr{U}, \mathscr{U}^{-1})$ *are* $GQ\mathscr{U}$*s and* $\mathscr{U}$ *admits a* T_1 *countable base and enjoys the property* (ii), *then the space* $(X, \mathscr{U})$ *admits a compatible quasi-metric.*

And a last proposition.

3.7 Proposition. *If* $\mathscr{U}$ *and* $\mathscr{U}^{-1}$ *are* $GQ\mathscr{U}$*s and*

$$\text{(iii)} \qquad (\forall U \in \mathscr{U})(\forall x \in X)(\exists V_x \in \mathscr{U})(\forall y \in V_x^{-1}[x])[V_x^{-1}[y] \times V_x[y] \subseteq U],$$

then $\mathscr{U}^{-1}$ *is* $LQ\mathscr{U}$.

P r o o f. Let U, x and V_x be as in (iii). Let also $t \in V_x^{-2}[x]$. Then there is a $\lambda \in X$ such that, $(t, \lambda) \in V_x$ and $(\lambda, x) \in V_x$, thus $t \in V_x^{-1}[\lambda]$, $\lambda \in V_x^{-1}[x]$ and $V_x^{-1}[\lambda] \times V_x[\lambda] \subseteq U$. Since $(t, x) \in V_x^{-1}[\lambda] \times V_x[\lambda]$, it is $(t, x) \in U$. □

We can reform (iii) into a dual condition and conclude a dual proposition of 3.7.

3.8 Remark. It seems that plenty of conditions of the form we deal with in this paragraph, may play a role in the establishment of a quasi-metrizability on a γ-space. For instance, a γ-space $(X, \mathscr{U}, \mathscr{U}^{-1})$ which enjoys (iii) is quasi-metrizable.

Also, if $\mathscr{U}$ and $\mathscr{U}^{-1}$ are $GQ\mathscr{U}$s, fulfil the properties (i) of the Proposition 3.3 and the following

$$\text{(iv)} \quad (\forall U \in \mathscr{U})(\forall x \in X)(\exists V_x \in \mathscr{U})(\forall y \in V_x[x])[V_x^{-1}[y] \times V_x[y] \subseteq U],$$

and moreover $\mathscr{U}$ is T_1 and countable, then the space is quasi-metrizable.

In fact; from prop. 2.7 $\mathscr{U}$ is a $LQ\mathscr{U}$ and from (i) $\mathscr{U}$ is point symmetric, thus developable (from prop. 3.3). Hence (from [Ku3, theor. 1]) the space is quasi-metrizable.

And a last example: If $\mathscr{U}$ and $\mathscr{U}^{-1}$ are $GQ\mathscr{U}$s, get the property:

$$\text{(v)} \quad (\forall U \in \mathscr{U})(\forall x \in X)(\exists V_x \in \mathscr{U})(\forall \alpha \in V_x[x])(\forall \beta \in V_x[x])[V_x^{-1}[\alpha] \times V_x[\beta] \subseteq U]$$

and $\mathscr{U}$ is countable, then the space is quasi-pseudo-metrizable.

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