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SPECTRAL ASYMPTOTICS FOR NONLINEAR MULTIPARAMETER PROBLEMS WITH INDEFINITE NONLINEARITIES

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1. Introduction

In order to explain the basic ideas of this paper, let us consider the nonlinear multiparameter eigenvalue problem with the coefficients $a_k(x)$ $(1 \leqslant k \leqslant n)$ changing sign:

$$
\begin{gathered}
u''(x) + \sum_{k=1}^{n} \mu_k a_k(x) u^{p_k}(x) = \lambda a_0(x) u^q(x), \quad x \in I := (-1,1), \\
u(x) > 0, \quad x \in I := (-1,1), \\
u(-1) = u(1) = 0,
\end{gathered} \tag{1.1}
$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_{i-1}, \mu_i \ldots, \mu_n) \in \overline{\mathbb{R}}_+^{i-1} \times \mathbb{R}_+^{n-i+1}$, $\lambda \in \mathbb{R}$ are parameters,

$$
1 \leqslant q \leqslant p_1 \leqslant p_2 \leqslant \ldots \leqslant p_{i-1} < p_i \leqslant \ldots \leqslant p_n < 2q+3, \tag{1.2}
$$

$a_k \in C^1(\bar{I})$ satisfies $a_k(-x) = a_k(x)$ for $x \in \bar{I}$ and

$$
\begin{gathered}
a_k'(x) \leqslant 0, \quad x \in [0,1] \ \ (1 \leqslant k \leqslant i-1), \\
a_k'(x) \leqslant 0, \quad x \in [0,1], \ \ a_k(0) > 0 \ \ (i \leqslant k \leqslant n), \\
a_0'(x) \geqslant 0, a_0(x) > 0, \quad x \in [0,1].
\end{gathered} \tag{1.3}
$$

We do not assume any sign conditions for $a_k(x)$ $(1 \leqslant n \leqslant i-1)$, and $a_k(x)$ $(i \leqslant k \leqslant n)$ are allowed to change sign in I. A nonlinear term such as $a_k(x)u^{p_k}$ is called an *indefinite nonlinearity*, since for a fixed u, it may change sign as a function of x.

Our main interest is to study the asymptotic behavior of the variational eigenvalue λ, which is obtained by Ljusternik-Schnirelman theory on the general level set as $\mu_i \to \infty$.

Nonlinear multiparameter problems arise in various areas of semilinear elliptic problems, which are derived from some physical and biological models in a bounded and an unbounded domain $D \subset \mathbb{R}^N$:

$$\triangle u + f(x, u) = 0 \quad \text{in } D.$$

It should be especially mentioned that to consider the asymptotic behavior of $\lambda \to \infty$ in our problem is related to singular perturbation problems of the type

$$-\varepsilon^2 \triangle u = f(x, u) \quad \text{in } D,$$

through scaling and change of variables, where $0 < \varepsilon \ll 1$.

Our problems are also motivated by linear multiparameter problems, especially, of finding "asymptotic directions of eigenvalues" (asymptotic behavior of the ratio of two eigenvalues). Related topics for linear problems can be found, for instance, in Faierman [2], Turyn [5]. Our problems are regarded as the nonlinear version of finding "asymptotic directions" of eigenvalues.

Recently, in Shibata [4], the simplest case of (1.1), namely,

$$\begin{aligned} u''(x) + \mu a(x) u^p(x) &= \lambda b(x) u^q(x), \ \ x \in I := (-1, 1), \\ u(x) &> 0, \ \ x \in I := (-1, 1), \\ u(-1) = u(1) &= 0, \end{aligned} \tag{1.4}$$

and $a(x) \equiv b(x) \equiv 1$, the definite type, was treated and the following precise asymptotic formula for $\lambda(\mu, \alpha)$ was obtained by using Ljusternik-Schnirelman (LS) theory on the general level set $M_{\mu,\alpha}$ due to Zeidler [6], where $\alpha > 0$ is a fixed parameter. Assume that $1 < q < p < q + 2$. Then as $\mu \to \infty$,

$$\lambda(\mu, \alpha) = C_1 (\alpha \mu^{\frac{q+3}{2(p-q)}})^{\frac{2(p-q)}{p+3}} + o((\alpha \mu^{\frac{q+3}{2(p-q)}})^{\frac{2(p-q)}{p+3}}) \tag{1.5}$$

where

$$C_1 = \left\{ \left(\frac{q+1}{p+1} \right)^{\frac{q+3}{2(p-q)}} \frac{(p+3)(q+1)(p-q)}{2(2q-p+3)} \sqrt{\frac{2}{\pi(q+1)}} \frac{\Gamma\left(\frac{p+3}{2(p-q)}\right)}{\Gamma\left(\frac{q+3}{2(p-q)}\right)} \right\}^{\frac{2(p-q)}{p+3}}. \tag{1.6}$$

Therefore, for nonlinear problems with several parameters, by adopting this variational approach, whole arguments seem to be nicely developed.

In this paper we shall study the effects of nonlinearities *changing sign* on the asymptotic behavior of eigenvalues for nonlinear multiparameter problems. More precisely, we shall extend the asymptotic formula for variational eigenvalues (1.5) to fairly wide class of nonlinear equations including (1.1). We mention here the main difficulty. If we assume, for instance, a simple condition on μ_i, such as $\mu_i \to \infty$ with $\mu_i \sim \mu_j$, then it turns out that the dominant parameter should be μ_1 automatically, since the maximum norm of the corresponding eigenfunction tends to 0. Therefore we need additional assumptions for μ to obtain the asymptotic formula dominated by μ_i. In this paper we give optimal conditions for μ under which μ_i dominates the asymptotic formula for the variational eigenvalue λ.

2. Main result

In what follows, we consider the nonlinear multiparameter problem

$$
\begin{aligned}
&u''(x) + \sum_{k=1}^{n} \mu_k f_k(x, u(x)) = \lambda g(x, u(x)), \quad x \in I := (-1,1), \\
&u(x) > 0, \quad x \in I := (-1,1), \\
&u(-1) = u(1) = 0,
\end{aligned} \tag{2.1}
$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_{i-1}, \mu_i, \ldots, \mu_n) \in \overline{\mathbb{R}}_+^{i-1} \times \mathbb{R}_+^{n-i+1}, \lambda \in \mathbb{R}$ and $\overline{\mathbb{R}}_+ = [0,\infty)$, $\mathbb{R}_+ := (0,\infty)$. We explain the notation before stating our results. Let $X := W_0^{1,2}(I)$ be the usual real Sobolev space and $L := \overline{\mathbb{R}}_+^{i-1} \times \mathbb{R}_+^{n-i+1} \times \mathbb{R}_+$. For $u \in X$, let

$$
\begin{aligned}
&\|u\|_X^2 := \int_I u'(x)^2 \,\mathrm{d}x, \quad \|u\|_d^d = \int_I |u(x)|^d \,\mathrm{d}x \ \ (d \geqslant 1), \quad (u,v)_2 := \int_I u(x)v(x)\,\mathrm{d}x, \\
&\|u\|_\infty := \max_{x\in I} |u(x)|, \quad F_k(x,u) := \int_0^u f_k(x,s)\,\mathrm{d}s, \quad G(x,u) := \int_0^u g(x,s)\,\mathrm{d}s, \\
&\Phi_k(u) := \int_I F_k(x,u(x))\,\mathrm{d}x, \quad \Psi(u) := \int_I G(x,u(x))\,\mathrm{d}x.
\end{aligned}
$$

Furthermore, let C denote various positive constants independent of $\{(\mu,\alpha)\}$. Especially, for simplicity, all constants, which will appear in the computations, will be denoted by the same character C provided they are independent of $\{(\mu,\alpha)\}$. We assume the following conditions.

(A.1). $f_k(x,u), g(x,u) \in C^1(\bar{I} \times \mathbb{R})$ are even functions with respect to x and odd functions with respect to u.

(A.2).

$$
\begin{aligned}
&g(x,u) > 0 \quad \text{for } (x,u) \in \bar{I} \times \mathbb{R}_+, \\
&\frac{\partial g(x,u)}{\partial x} \geqslant 0 \quad \text{for } (x,u) \in [0,1] \times \mathbb{R}_+,
\end{aligned} \tag{2.3}
$$

and there exists a constant $q \geqslant 1$ such that for $x \in \bar{I}$ and $u \geqslant 0$

$$C^{-1}u^q \leqslant g(x,u) \leqslant Cu^q. \tag{2.4}$$

(A.3). There exist constants $\{p_k\}_{k=1}^n$ satisfying (1.2) and

$$|f_k(x,u)| \leqslant C|u|^{p_k} \quad \text{for } x \in \bar{I},\ \ u \in \mathbb{R}, \tag{2.5}$$

$$\frac{\partial f_k(x,u)}{\partial x} \leqslant 0 \quad \text{for } (x,u) \in [0,1] \times \mathbb{R}_+. \tag{2.6}$$

In addition, there exists a compact interval $J = [-x_1, x_1] \subset I$ $(x_1 > 0)$ such that for $x \in J$ and $u \geqslant 0$

$$f_i(x,u) \geqslant Cu^{p_i},\ \ f_k(x,u) \geqslant 0 \ \ (k > i). \tag{2.7}$$

Furthermore, if $\Phi_i(u_0) \geqslant 0$ for $u_0 \in X$, then

$$(f_i(x,u_0), u_0)_2 - 2\Phi_i(u_0) \geqslant 0. \tag{2.8}$$

(A.4). There exists $a_k(x) \in C^1(\bar{I})$ $(1 \leqslant k \leqslant n)$ such that $a_k(-x) = a_k(x)$ for $x \in I$ and as $u \downarrow 0$,

$$\frac{f_k(x,u)}{u^{p_k}} \to a_k(x),\ \frac{g(x,u)}{u^q} \to a_0(x) \tag{2.9}$$

uniformly for $x \in I$, where $a_k(x)$ satisfies the condition (1.3). In addition, for $x \in [0,1]$ and $0 \leqslant u \ll 1$,

$$\frac{\partial f_{k,0}(x,u)}{\partial x} \leqslant 0,\ \ \frac{\partial g_0(x,u)}{\partial x} \geqslant 0, \tag{2.10}$$

where

$$f_{k,0}(x,u) := f_k(x,u) - a_k(x)u^{p_k},\ \ g_0(x,u) := g(x,u) - b_0(x)u^q. \tag{2.11}$$

It is easy to see that the equation (1.1) under the conditions (1.2) and (1.3) satisfies (A.1)–(A.4). Another example of f_k and g which satisfies (A.1)–(A.4) is

$$f_k(x,u) = \begin{cases} \cos \pi x(|u|^{p_k-1}u + |u|^{p_k-1+\varepsilon}u), & |u| \ll 1, \\ \cos \pi x(|u|^{p_k-1}u + |u|^{p_k-1-\varepsilon}u), & |u| \gg 1, \end{cases}$$
$$g(x,u) = (1+x^2)|u|^{q-1}u,$$

where $0 < \varepsilon \ll 1$ and $\{p_k\}_{k=1}^n, q$ satisfy (1.2).

Now we define the variational eigenvalues. $\lambda = \lambda(\mu, \alpha)$ is called a variational eigenvalue of (2.1) when the associated eigenfunction $u_{\mu,\alpha}(x) \in N_{\mu,\alpha}$ satisfies the following condition

$$\left\{ \begin{array}{l} (\mu, \alpha, \lambda(\mu,\alpha), u_{\mu,\alpha}(x)) \in L \times \mathbb{R} \times N_{\mu,\alpha} \ \text{ satisfies } \ (2.1), \\ \Psi(u_{\mu,\alpha}) = \beta(\mu,\alpha) := \inf\limits_{u \in N_{\mu,\alpha}} \Psi(u), \end{array} \right. \tag{2.12}$$

where

$$N_{\mu,\alpha} := \left\{ u \in X \colon \ E(\mu, u) := \frac{1}{2}\|u\|_X^2 - \sum_{k=1}^n \mu_k \Phi_k(u) = -\alpha \right\},$$

and $\alpha > 0$ is a parameter. $\lambda(\mu,\alpha)$ is obtained as a Lagrange multiplier and explicitly represented as follows:

$$\lambda(\mu,\alpha) = \frac{2\alpha + \sum_{k=1}^n \mu_k \{(f_k(x, u_{\mu,\alpha}), u_{\mu,\alpha})_2 - 2\Phi_k(u_{\mu,\alpha})\}}{(g(x, u_{\mu,\alpha}), u_{\mu,\alpha})_2}. \tag{2.13}$$

Actually, multiplying (2.1) by $u_{\mu,\alpha}(x)$, we obtain by integration by parts that

$$-\|u_{\mu,\alpha}\|_X^2 + \sum_{k=1}^n \mu_k (f_k(x, u_{\mu,\alpha}), u_{\mu,\alpha})_2 = \lambda(\mu,\alpha)(g(x, u_{\mu,\alpha}), u_{\mu,\alpha})_2; \tag{2.14}$$

this along with the fact that $u_{\mu,\alpha} \in N_{\mu,\alpha}$ implies (2.13).

We introduce the condition (B.1) for a sequence $\{(\mu,\alpha)\}$. A sequence $\{(\mu,\alpha)\} \subset L$ is said to satisfy the condition (B.1) if the following conditions hold:

(**B.1**)

$$\alpha \mu_i^{\frac{2}{p_i - 1}} \to \infty, \tag{2.15}$$

$$\alpha \mu_i^{-1/2} \to 0. \tag{2.16}$$

Furthermore, for $k \neq i$,

$$\mu_k \alpha^{\frac{2(p_k - p_i)}{p_i + 3}} \mu_i^{-\frac{p_k + 3}{p_i + 3}} \to 0. \tag{2.17}$$

Now we state our main result.

Theorem 2.1. *Assume* (A.1)–(A.4). *Furthermore, assume that a sequence* $\{(\mu,\alpha)\} \subset L$ *satisfies* (B.1). *Then the following asymptotic formula holds:*

$$\lambda(\mu,\alpha) = C_2 a_0(0)^{-1} a_i(0)^{\frac{q+3}{p_i+3}} \left(\alpha \mu_i^{\frac{q+3}{2(p_i - q)}}\right)^{\frac{2(p_i - q)}{p_i + 3}} + o\left(\left(\alpha \mu_i^{\frac{q+3}{2(p_i-q)}}\right)^{\frac{2(p_i-q)}{p_i+3}}\right) \tag{2.18}$$

where
(2.19)

$$C_2 = \left\{ \left(\frac{q+1}{p_i+1}\right)^{\frac{q+3}{2(p_i-q)}} \frac{(p_i+3)(q+1)(p_i-q)}{2(2q-p_i+3)} \sqrt{\frac{2}{\pi(q+1)}} \frac{\Gamma\left(\frac{p_i+3}{2(p_i-q)}\right)}{\Gamma\left(\frac{q+3}{2(p_i-q)}\right)} \right\}^{\frac{2(p_i-q)}{p_i+3}}.$$

It should be mentioned that the condition (B.1) is not technical but optimal to obtain the formula (2.18). In fact, if $q = p_1$, $a_k(x) \equiv -1$ for $1 \leqslant k \leqslant i-1$, $a_k(x) \equiv 1$ for $i \leqslant k \leqslant n$ and the condition (2.17) fails for $k = 1$, namely, if there exists a constant $\delta > 0$ such that $\mu_1 \alpha^{\frac{2(p_k-p_i)}{p_i+3}} \mu_i^{-\frac{p_k+3}{p_i+3}} \to \delta$, then we can obtain another asymptotic formula depending on δ, which coincides with (2.18) when $\delta = 0$.

The remainder of this paper is organized as follows. In Section 3, the existence of variational eigenvalues is formulated. In Section 4 and Section 5, we will prepare some fundamental tools. Section 6 is devoted to the proof of Theorem 2.1.

3. Existence of variational eigenvalues

In what follows, for a subsequence, we use the same notation as that of original sequence for convenience. Furthermore, let $a_i(0) = a_{i+1}(0) = \ldots = a_n(0) = a_0(0) = 1$ for simplicity. In this section, we assume that $(\mu, \alpha) \in L$ belongs to a sequence of L which satisfies (B.1).

Lemma 3.1. *Let $(\mu, \alpha) \in L$ be fixed. Then $N_{\mu,\alpha} \neq \emptyset$.*

Proof. We fix $\varphi(x) \in C_0^\infty(J)$ satisfying $\varphi \not\equiv 0$ and $\varphi(x) \geqslant 0$ for $x \in J$, where J is a compact interval defined in (A.3). For $t \geqslant 0$ we put

$$m(t) = m(t, \mu, \varphi) := \frac{1}{2}\|t\varphi\|_X^2 - \sum_{k=1}^n \mu_k \Phi_k(t\varphi). \tag{3.1}$$

By (2.5) and (2.7) we obtain that for $t \geqslant 0$

$$\begin{aligned} &\Phi_i(t\varphi) \geqslant Ct^{p_i+1}\|\varphi\|_{p_i+1}^{p_i+1}, \\ &\Phi_k(t\varphi) \geqslant 0 \quad (k > i), \\ &|\Phi_k(t\varphi)| \leqslant Ct^{p_k+1}\|\varphi\|_{p_k+1}^{p_k+1} \quad (1 \leqslant k \leqslant i-1). \end{aligned} \tag{3.2}$$

Therefore, by (3.1) and (3.2), we obtain that as $t \to \infty$

$$
\begin{aligned}
(3.3) \qquad m(t) &\leqslant \frac{1}{2}t^2\|\varphi\|_X^2 - \mu_i\Phi_i(t\varphi) + \sum_{k=1}^{i-1}\mu_k|\Phi_k(t\varphi)| \\
&\leqslant t^2\left\{\frac{1}{2}\|\varphi\|_X^2 - C\mu_i t^{p_i-1}\|\varphi\|_{p_i+1}^{p_i+1} + C\sum_{k=1}^{i-1}\mu_k t^{p_k-1}\|\varphi\|_{p_k+1}^{p_k+1}\right\} \to -\infty,
\end{aligned}
$$

since $p_k < p_i$ for $1 \leqslant k \leqslant i-1$ by (1.2). Since $m(0) = 0$, we obtain that there exists $t = t_0 > 0$ such that $m(t_0) = -\alpha$, namely, $t_0\varphi \in N_{\mu,\alpha}$. Thus the proof is complete. □

Lemma 3.2. *Let $(\mu,\alpha) \in L$ be fixed. Then $\beta(\mu,\alpha) > 0$.*

Proof. We assume that $\beta(\mu,\alpha) = 0$ and derive a contradiction. Let $\{u_l\} \subset N_{\mu,\alpha}$ be a minimizing sequence of (2.12), that is, as $l \to \infty$,

$$(3.4) \qquad \Psi(u_l) \to \beta(\mu,\alpha) = 0.$$

By (2.4), we have for $x \in I$, $s \in \overline{\mathbb{R}}_+$ and $u \in X$

$$(3.5) \qquad C^{-1}s^{q+1} \leqslant G(x,s) \leqslant Cs^{q+1}, \quad C^{-1}\|u\|_{q+1}^{q+1} \leqslant \Psi(u) \leqslant C\|u\|_{q+1}^{q+1}.$$

Furthermore, we know from (2.5) that for $(x,s) \in I \times \mathbb{R}$, $u \in X$ and $1 \leqslant k \leqslant n$

$$(3.6) \qquad |f_k(x,s)| \leqslant C|s|^{p_k}, \ \ |F_k(x,s)| \leqslant C|s|^{p_k+1}, \ \ |\Phi_k(u)| \leqslant C\|u\|_{p_k+1}^{p_k+1}.$$

We recall here the Gagliardo-Nirenberg inequality for $u \in X$:

$$(3.7) \qquad \|u\|_{p_k+1}^{p_k+1} \leqslant C\|u\|_{q+1}^{(1-a_k)(p_k+1)}\|u\|_X^{a_k(p_k+1)},$$

where $a_k = 2(p_k-q)/\{(q+3)(p_k+1)\}$. Then by (3.5)–(3.7) we have

$$
\begin{aligned}
(3.8) \qquad |\Phi_k(u_l)| &\leqslant C\|u_l\|_{p_k+1}^{p_k+1} \leqslant C\|u_l\|_{q+1}^{(1-a_k)(p_k+1)}\|u_l\|_X^{\frac{2(p_k-q)}{q+3}} \\
&\leqslant C\Psi(u_l)^{\frac{(p_k+1)(1-a_k)}{q+1}}\|u_l\|_X^{\frac{2(p_k-q)}{q+3}}.
\end{aligned}
$$

Since $u_l \in N_{\mu,\alpha}$, we obtain by (3.8) that

$$(3.9) \qquad \frac{1}{2}\|u_l\|_X^2 \leqslant \sum_{k=1}^{n}\mu_k|\Phi_k(u_l)| \leqslant C\sum_{k=1}^{n}\mu_k\Psi(u_l)^{\frac{(p_k+1)(1-a_k)}{q+1}}\|u_l\|_X^{\frac{2(p_k-q)}{q+3}}.$$

Since $2(p_k-q)/(q+3) < 2$, we see from (3.4) and (3.9) that $\|u_l\|_X \to 0$ as $l \to \infty$. Hence, $\Phi_k(u_l) \to 0$ as $l \to \infty$ by (3.8). Then as $l \to \infty$,

$$-\alpha = \frac{1}{2}\|u_l\|_X^2 - \sum_{k=1}^{n}\mu_k\Phi_k(u_l) \to 0.$$

This is a contradiction. Thus the proof is complete. □

Lemma 3.3. *Let $(\mu,\alpha)\in L$ be fixed. Furthermore, let $\{u_l\}\subset N_{\mu,\alpha}$ be a minimizing sequence of* (2.12)*, namely, $\Psi(u_l)\to\beta(\mu,\alpha)$ as $l\to\infty$. Then there exists a subsequence of $\{u_l\}$ and $u_\infty\in N_{\mu,\alpha}$ such that $u_l\to u_\infty$ in X as $l\to\infty$.*

Proof. By (3.9) and Lemma 3.2 we see that $\{u_l\}\subset X$ is bounded. Then by Sobolev's embedding theorem, there exists a subsequence of $\{u_l\}$ and $u_\infty\in X$ such that $u_l\to u_\infty$ weakly in X, $u_l\to u_\infty$ in $C(\bar I), L^d(I)$ for any $d\geqslant 1$ as $l\to\infty$. Therefore,

$$\frac{1}{2}\|u_\infty\|_X^2\leqslant\frac{1}{2}\liminf_{l\to\infty}\|u_l\|_X^2=\lim_{l\to\infty}\sum_{k=1}^n\mu_k\Phi_k(u_l)-\alpha=\sum_{k=1}^n\mu_k\Phi_k(u_\infty)-\alpha. \tag{3.10}$$

We shall show that $u_\infty\in N_{\mu,\alpha}$. To this end, by (3.10), we assume that

$$\frac{1}{2}\|u_\infty\|_X^2<\sum_{k=1}^n\mu_k\Phi_k(u_\infty)-\alpha \tag{3.11}$$

and derive a contradiction. We put $m(t)=m(t,\mu,u_\infty)$ for $t\geqslant 0$, where $m(t)$ is a function defined in (3.1). Since $m(0)=0$ and $m(1)<-\alpha$ by (3.11), there exists $0<t_1<1$ such that $m(t_1)=-\alpha$, namely, $t_1u_\infty\in N_{\mu,\alpha}$. Since $g(x,u)>0$ is odd in u, we have $G(x,t_1u_\infty)<G(x,u_\infty)$ for $x\in I$. Then by (2.12)

$$\beta(\mu,\alpha)\leqslant\Psi(t_1u_\infty)<\Psi(u_\infty)=\beta(\mu,\alpha). \tag{3.12}$$

This is a contradiction. Hence (3.11) is impossible, and we see from (3.10) that $u_\infty\in N_{\mu,\alpha}$. Moreover, it follows from (3.10) that there exists a subsequence of $\{u_l\}$ such that

$$\|u_\infty\|_X^2=\lim_{l\to\infty}\|u_l\|_X^2. \tag{3.13}$$

By (3.13), $u_l\to u_\infty$ in X as $l\to\infty$, and $u_\infty\not\equiv 0$, since $0<\beta(\mu,\alpha)=\Psi(u_\infty)$. Thus the proof is complete. □

Lemma 3.4. *For a fixed $(\mu,\alpha)\in L$, there exists $(\lambda(\mu,\alpha),u_{\mu,\alpha}(x))\in\mathbb{R}\times N_{\mu,\alpha}$ which satisfies* (2.12).

Proof. By Lemma 3.3 we see that $u_\infty\in N_{\mu,\alpha}$ is the minimizer of the minimizing problem in (2.12). Then by the Lagrange multiplier theorem there exists $\lambda(\mu,\alpha)\in\mathbb{R}$ such that $(u_\infty,\lambda(\mu,\alpha))$ satisfies the equation in (2.1). Furthermore, by a standard regularity argument we see that $u_\infty\in C^2(\bar I)$. Now we put $u_{\mu,\alpha}(x):=|u_\infty(x)|$. Since f_k,g are odd with respect to u, F_k,G are even with respect to u. Therefore,

$$\Phi_k(u_{\mu,\alpha})=\Phi_k(u_\infty),\Psi(u_{\mu,\alpha})=\Psi(u_\infty)=\beta(\mu,\alpha). \tag{3.14}$$

Furthermore, since $u_\infty \in C^2(\bar{I})$, we have $\|u_\infty\|_X = \|u_{\mu,\alpha}\|_X$. This along with (2.3) and (3.14) implies that $u_{\mu,\alpha} \in N_{\mu,\alpha}$, and $(\mu, \alpha, \lambda(\mu, \alpha), u_{\mu,\alpha}) \in L \times \mathbb{R} \times N_{\mu,\alpha}$ satisfies (2.1) and $u_{\mu,\alpha}$ is the mimimizer of (2.12) with the same Lagrange multiplier $\lambda(\mu, \alpha)$ as that of u_∞. Finally, if there exists $x_0 \in I$ such that $u_{\mu,\alpha}(x_0) = 0$, then $u'_{\mu,\alpha}(x_0) = 0$, since $u_{\mu,\alpha}(x) \geqslant 0$ in I. Then by the uniqueness theorem of ODE, we obtain that $u_{\mu,\alpha} \equiv 0$ in I. However, this is impossible, since $u_{\mu,\alpha} \in N_{\mu,\alpha}$ and $0 \notin N_{\mu,\alpha}$. Thus $u_{\mu,\alpha} > 0$ in I. □

Remark 3.5. To apply the Lagrange multiplier theorem, the fact $E_u(\mu, u_\infty)u_\infty \neq 0$ was used, where E_u is the Fréchet derivative of E with respect to u. This is guaranteed, since $E_u(\mu, u_\infty)u_\infty = E_u(\mu, u_{\mu,\alpha})u_{\mu,\alpha} \neq 0$ by (4.22) in the next section.

4. Fundamental lemmas

We put $\sigma_{\mu,\alpha} := \max_{x \in I} u_{\mu,\alpha}(x)$. By Gidas, Ni and Nirenberg [3] we know that $u_{\mu,\alpha}$ possesses the following properties:

$$
\begin{gathered}
u_{\mu,\alpha}(-x) = u_{\mu,\alpha}(x), u'_{\mu,\alpha}(x) < 0, \quad x \in (0,1), \\
u'_{\mu,\alpha}(0) = 0, \sigma_{\mu,\alpha} = u_{\mu,\alpha}(0).
\end{gathered}
\tag{4.1}
$$

In what follows, for norms of a function defined on $\mathbb{R}$ we use the same notation as that defined at the beginning of Section 2.

Lemma 4.1. *For a fixed $(\mu, \alpha) \in L$, the following equality holds for $x \in \bar{I}$:*

$$
\begin{aligned}
&\frac{1}{2}(u'_{\mu,\alpha}(x))^2 + J(\mu, \alpha, x, u_{\mu,\alpha}(x)) - \sum_{k=1}^{n} \mu_k B_k(\mu, \alpha, x) + \lambda(\mu, \alpha) B_0(\mu, \alpha, x) \\
&\qquad = J(\mu, \alpha, 0, \sigma_{\mu,\alpha}) = \frac{1}{2} u'_{\mu,\alpha}(1)^2 - \sum_{k=1}^{n} \mu_k B_k(\mu, \alpha, 1) + \lambda(\mu, \alpha) B_0(\mu, \alpha, 1),
\end{aligned}
\tag{4.2}
$$

where

$$
J(\mu, \alpha, x, u) := \sum_{k=1}^{n} \mu_k F_k(x, u) - \lambda(\mu, \alpha) G(x, u),
\tag{4.3}
$$

$$
B_k(\mu, \alpha, x) := \int_0^x \left\{ \int_0^{u_{\mu,\alpha}(r)} \frac{\partial f_k(r, s)}{\partial r} \, \mathrm{d}s \right\} \mathrm{d}r \leqslant 0, \quad x \in [0,1],
\tag{4.4}
$$

$$
B_0(\mu, \alpha, x) := \int_0^x \left\{ \int_0^{u_{\mu,\alpha}(r)} \frac{\partial g(r, s)}{\partial r} \, \mathrm{d}s \right\} \mathrm{d}r \geqslant 0, \quad x \in [0,1].
\tag{4.5}
$$

P r o o f. Multiplying (2.1) by $u'_{\mu,\alpha}(x)$ we obtain for $x \in \bar{I}$

$$\{u''_{\mu,\alpha}(x) + \sum_{k=1}^{n} \mu_k f_k(x, u_{\mu,\alpha}(x)) - \lambda(\mu,\alpha) g(x, u_{\mu,\alpha}(x))\} u'_{\mu,\alpha}(x) = 0; \tag{4.6}$$

namely,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \Big\{ & \frac{1}{2}(u'_{\mu,\alpha}(x))^2 + J(\mu,\alpha,x,u_{\mu,\alpha}(x)) \\ & - \sum_{k=1}^{n} \mu_k B_k(\mu,\alpha,x) + \lambda(\mu,\alpha) B_0(\mu,\alpha,x) \Big\} \equiv 0; \end{aligned} \tag{4.7}$$

this implies that for $x \in \bar{I}$

$$\begin{aligned} & \frac{1}{2}(u'_{\mu,\alpha}(x))^2 + J(\mu,\alpha,x,u_{\mu,\alpha}(x)) \\ & - \sum_{k=1}^{n} \mu_k B_k(\mu,\alpha,x) + \lambda(\mu,\alpha) B_0(\mu,\alpha,x) \equiv \text{const.} \end{aligned} \tag{4.8}$$

Now, put $x = 0, 1$ in (4.8). Then (4.2) follows immediately. □

Lemma 4.2. *Assume that* $\{(\mu,\alpha)\} \subset L$ *satisfies* (B.1). *Then*

$$\|u_{\mu,\alpha}\|_{q+1}^{q+1} \leqslant C\alpha^{\frac{2q+3-p_i}{p_i+3}} \mu_i^{-\frac{q+3}{p_i+3}}. \tag{4.9}$$

P r o o f. Let $\eta = \eta_{\mu,\alpha} := (\alpha \mu_i^{\frac{2}{p_i-1}})^{\frac{p_i-1}{p_i+3}}$. Furthermore, let w_η satisfy

$$\begin{aligned} & w''_\eta(s) + w_\eta^{p_i}(s) - w_\eta^q(s) = 0, \quad -\eta_{\mu,\alpha} < s < \eta_{\mu,\alpha}, \\ & w_\eta(s) > 0, \quad -\eta_{\mu,\alpha} < s < \eta_{\mu,\alpha}, \\ & w_\eta(\pm\eta_{\mu,\alpha}) = 0. \end{aligned} \tag{4.10}$$

The existence of w_η is obtained easily, for instance, by variational method. We put

$$U_{\mu,\alpha}(x) := \begin{cases} d_{\mu,\alpha}(\alpha^2\mu_i^{-1})^{\frac{1}{p_i+3}} w_\eta(s), \quad x = \eta_{\mu,\alpha}^{-1} x_1 s, \quad x \in J, \\ 0, \ \ x \in I \setminus J, \end{cases}$$

where $x_1 > 0$ is the number defined in (A.3) and

$$d_{\mu,\alpha} := \inf\{t > 0 \colon \ t(\alpha^2\mu_i^{-1})^{\frac{1}{p_i+3}} w_\eta(\eta x_1^{-1} x) \in N_{\mu,\alpha}\}.$$

By (3.3) we see that $d_{\mu,\alpha} > 0$ exists. We shall show that

$$C^{-1} \leqslant d_{\mu,\alpha} \leqslant C. \tag{4.11}$$

Since $\eta \to \infty$ by (2.15), we know from Shibata [4, Lemma 4.7] that $w_\eta \to w_\infty$ uniformly on any compact subset in $\mathbb{R}$ and $C^{-1} \leqslant \|w_\eta\|_p \leqslant C$ ($p \geqslant q+1$), where w_∞ is the ground state solution of

$$\begin{aligned} w''(t) + w^{p_i}(t) - w^q(t) = 0, &\quad t \in \mathbb{R}, \\ w(t) > 0, &\quad t \in \mathbb{R}, \\ \lim_{|t|\to\infty} w(t) = 0. & \end{aligned} \tag{4.12}$$

First, we assume that there exists a subsequence of $\{(\mu,\alpha)\}$ such that $d_{\mu,\alpha} \to \infty$, and derive a contradiction. Since $\operatorname{supp} U_{\mu,\alpha} \subset J$, we have by (2.5), (2.7), (2.17) and direct calculation that

$$\begin{aligned} -\alpha &= \frac{1}{2}\|U_{\mu,\alpha}\|_X^2 - \sum_{k=1}^{n} \mu_k \Phi_k(U_{\mu,\alpha}) \\ &\leqslant \frac{1}{2}\|U_{\mu,\alpha}\|_X^2 - \mu_i\Phi_i(U_{\mu,\alpha}) + \sum_{k=1}^{i-1} \mu_k |\Phi_k(U_{\mu,\alpha})| \\ &\leqslant \frac{1}{2}\eta x_1^{-1} d_{\mu,\alpha}^2 (\alpha^2\mu_i^{-1})^{\frac{2}{p_i+3}} \|w_\eta\|_X^2 \\ &\quad - \mu_i \eta^{-1} x_1 d_{\mu,\alpha}^{p_i+1} (\alpha^2\mu_i^{-1})^{\frac{p_i+1}{p_i+3}} \|w_\eta\|_{p_i+1}^{p_i+1} \\ &\quad + \sum_{k=1}^{i-1} \mu_k \eta^{-1} x_1 d_{\mu,\alpha}^{p_k+1} (\alpha^2\mu_i^{-1})^{\frac{p_k+1}{p_i+3}} \|w_\eta\|_{p_k+1}^{p_k+1} \\ &\leqslant \alpha d_{\mu,\alpha}^2 \Bigg\{ x_1^{-1} \frac{1}{2} \|w_\eta\|_X^2 - x_1 d_{\mu,\alpha}^{p_i-1} \|w_\eta\|_{p_i+1}^{p_i+1} \\ &\quad + x_1 \sum_{k=1}^{i-1} (\mu_k \alpha^{\frac{2(p_k-p_i)}{p_i+3}} \mu_i^{-\frac{p_k+3}{p_i+3}}) d_{\mu,\alpha}^{p_k-1} \|w_\eta\|_{p_k+1}^{p_k+1} \Bigg\} \\ &\leqslant -C\alpha, \end{aligned} \tag{4.13}$$

where $C \gg 1$ is a constant. This is a contradiction. Hence $d_{\mu,\alpha} \leqslant C$. Then (3.6) implies that

$$\begin{aligned}-\alpha &= \frac{1}{2}\|U_{\mu,\alpha}\|_X^2 - \sum_{k=1}^n \mu_k \Phi_k(U_{\mu,\alpha}) \geqslant \frac{1}{2}\|U_{\mu,\alpha}\|_X^2 - \sum_{k=1}^n \mu_k |\Phi_k(U_{\mu,\alpha})| \\ &\geqslant \eta x_1^{-1} d_{\mu,\alpha}^2 (\alpha^2 \mu_i^{-1})^{\frac{2}{p_i+3}} \|w_\eta\|_X^2 - \sum_{k=1}^n \mu_k \eta^{-1} x_1 d^{p_k+1} (\alpha^2 \mu_i^{-1})^{\frac{p_k+1}{p_i+3}} \|w_\eta\|_{p_k+1}^{p_k+1} \\ &\geqslant \alpha d_{\mu,\alpha}^2 \left\{ x_1^{-1} \frac{1}{2}\|w_\eta\|_X^2 - \sum_{k=1}^n x_1 (\mu_k \alpha^{\frac{2(p_k-p_i)}{p_i+3}} \mu_i^{-\frac{p_k+3}{p_i+3}}) d_{\mu,\alpha}^{p_k-1} \|w_\eta\|_{p_k+1}^{p_k+1} \right\};\end{aligned}$$

this along with (2.17) implies that $d_{\mu,\alpha} \to 0$ does not occur. Therefore, we obtain (4.11). Now by (2.12) and (3.5)

$$\begin{aligned}\|u_{\mu,\alpha}\|_{q+1}^{q+1} &\leqslant C\Psi(u_{\mu,\alpha}) \leqslant C\Psi(U_{\mu,\alpha}) \leqslant C\|U_{\mu,\alpha}\|_{q+1}^{q+1} \\ &\leqslant C(\alpha^2\mu_i^{-1})^{\frac{q+1}{p_i+3}} \eta_{\alpha,\mu}^{-1} \|w_\eta\|_{q+1}^{q+1} \leqslant C\alpha^{\frac{2q+3-p_i}{p_i+3}} \mu_i^{-\frac{q+3}{p_i+3}}.\end{aligned} \tag{4.14}$$

Thus the proof is complete. $\square$

Lemma 4.3. *Assume that* $\{(\mu,\alpha)\} \subset L$ *satisfies* (B.1). *Then*

$$C^{-1}\alpha \leqslant \|u_{\mu,\alpha}\|_X^2 \leqslant C\alpha. \tag{4.15}$$

Proof. By (3.5), (3.8) and (4.9), we obtain that for $1 \leqslant k \leqslant n$

$$\begin{aligned}\mu_k |\Phi_k(u_{\mu,\alpha})| &\leqslant C\mu_k \|u_{\mu,\alpha}\|_{q+1}^{\frac{(p_k+3)(q+1)}{q+3}} \|u_{\mu,\alpha}\|_X^{\frac{2(p_k-q)}{q+3}} \\ &= C\Big(\mu_k \alpha^{\frac{2(p_k-p_i)}{p_i+3}} \mu_i^{-\frac{p_k+3}{p_i+3}}\Big) \alpha^{\frac{2q+3-p_k}{q+3}} \|u_{\mu,\alpha}\|_X^{\frac{2(p_k-q)}{q+3}}.\end{aligned} \tag{4.16}$$

By the inequality

$$ab \leqslant \frac{1}{d_1}a^{d_1} + \frac{1}{d_2}b^{d_2} \quad \Big(a, b \geqslant 0, \frac{1}{d_1} + \frac{1}{d_2} = 1\Big)$$

we have

$$\alpha^{\frac{2q+3-p_k}{q+3}} \|u_{\mu,\alpha}\|_X^{\frac{2(p_k-q)}{q+3}} \leqslant C(\alpha + \|u_{\mu,\alpha}\|_X^2). \tag{4.17}$$

Since $u_{\mu,\alpha} \in N_{\mu,\alpha}$, we obtain by (2.17), (4.16) and (4.17) that
(4.18)

$$\begin{aligned}\alpha + \frac{1}{2}\|u_{\mu,\alpha}\|_X^2 &\leqslant \mu_i \Phi_i(u_{\mu,\alpha}) + \sum_{k=1,\neq i}^{n} \mu_k |\Phi_k(u_{\mu,\alpha})| \\ &\leqslant C\alpha^{\frac{2q+3-p_i}{q+3}} \|u_{\mu,\alpha}\|_X^{\frac{2(p_i-q)}{q+3}} + o(1) \sum_{k=1,\neq i}^{n} \alpha^{\frac{2q+3-p_k}{q+3}} \|u_{\mu,\alpha}\|_X^{\frac{2(p_k-q)}{q+3}} \\ &\leqslant C\alpha^{\frac{2q+3-p_i}{q+3}} \|u_{\mu,\alpha}\|^{\frac{2(p_i-q)}{q+3}} + o(1)(\alpha + \|u_{\mu,\alpha}\|_X^2);\end{aligned}$$

that is,

$$\alpha + \frac{1}{2}\|u_{\mu,\alpha}\|_X^2 \leqslant C\alpha^{\frac{2q+3-p_i}{q+3}} \|u_{\mu,\alpha}\|_X^{\frac{2(p_i-q)}{q+3}}. \tag{4.19}$$

Then (4.15) follows immediately from (4.19). □

Lemma 4.4. *Assume that* $\{(\mu,\alpha)\} \subset L$ *satisfies* (B.1). *Then* $\lambda(\mu,\alpha) > 0$.

Proof. We obtain by (4.18) that

$$0 < \alpha + \|u_{\mu,\alpha}\|_X^2 \leqslant C\mu_i \Phi_i(u_{\mu,\alpha}) + o(1)(\alpha + \|u_{\mu,\alpha}\|_X^2);$$

this implies that

$$0 < \alpha + \|u_{\mu,\alpha}\|_X^2 \leqslant C\mu_i \Phi_i(u_{\mu,\alpha}). \tag{4.20}$$

Furthermore, by (4.16) and Lemma 4.3 we obtain that for $k \neq i$

$$\mu_k |(f_k(u_{\mu,\alpha}), u_{\mu,\alpha})_2|, \ \ \mu_k |\Phi_k(u_{\mu,\alpha})| = o(1)\alpha. \tag{4.21}$$

Then by (2.8), (2.13), (4.21)

$$\begin{aligned}\lambda(\mu,\alpha)(g(x,u_{\mu,\alpha}),u_{\mu,\alpha})_2 &\geqslant 2\alpha + \mu_i\{(f_i(x,u_{\mu,\alpha}),u_{\mu,\alpha})_2 - 2\Phi_i(u_{\mu,\alpha})\} \\ &\quad - C\sum_{k=1,k\neq i}^{n} \mu_k(|(f_k(u_{\mu,\alpha}),u_{\mu,\alpha})_2| + |\Phi_k(u_{\mu,\alpha})|) \\ &\geqslant (2 - o(1))\alpha.\end{aligned} \tag{4.22}$$

Thus the proof is complete. □

Lemma 4.5. *Assume that* $\{(\mu,\alpha)\} \subset L$ *satisfies* (B.1). *Then*

$$C\alpha^{\frac{2(p_i-q)}{p_i+3}}\mu_i^{\frac{q+3}{p_i+3}} \leqslant \lambda(\mu,\alpha). \tag{4.23}$$

Proof. It follows from (2.4), (4.9) and (4.22) that

$$C\alpha \leqslant \lambda(\mu,\alpha)(g(x,u_{\mu,\alpha}),u_{\mu,\alpha})_2 \leqslant C\lambda(\mu,\alpha)\|u_{\mu,\alpha}\|_{q+1}^{q+1} \leqslant C\lambda(\mu,\alpha)\alpha^{\frac{2q+3-p_i}{p_i+3}}\mu_i^{-\frac{q+3}{p_i+3}};$$

this implies (4.23). Thus the proof is complete. □

Lemma 4.6. *Assume that* $\{(\mu,\alpha)\} \subset L$ *satisfies* (B.1). *Then* $\sigma_{\mu,\alpha} \to 0$.

Proof. By (4.2)–(4.5) and Lemma 4.4 we see that $J(\mu,\alpha,0,\sigma_{\mu,\alpha}) > 0$, that is,

$$\lambda(\mu,\alpha)G(0,\sigma_{\mu,\alpha}) < \sum_{k=1}^{n}\mu_k F_k(0,\sigma_{\mu,\alpha});$$

this along with (3.5) and (3.6) implies that

$$\lambda(\mu,\alpha)\sigma_{\mu,\alpha}^{q+1} \leqslant C\sum_{k=1}^{n}\mu_k\sigma_{\mu,\alpha}^{p_k+1}. \tag{4.24}$$

We have by (3.6), (4.2)–(4.5) and (4.24) that

$$\begin{aligned}\frac{1}{2}(u'_{\mu,\alpha}(x))^2 &= J(\mu,\alpha,0,\sigma_{\mu,\alpha}) - J(\mu,\alpha,x,u_{\mu,\alpha}(x))\\ &\quad+\sum_{k=1}^{n}\mu_k B_k(\mu,\alpha,x) - \lambda(\mu,\alpha)B_0(\mu,\alpha,x)\\ &\leqslant J(\mu,\alpha,0,\sigma_{\mu,\alpha}) - J(\mu,\alpha,x,u_{\mu,\alpha}(x))\\ &= \sum_{k=1}^{n}\mu_k(F_k(0,\sigma_{\mu,\alpha}) - F_k(x,u_{\mu,\alpha}(x)))\\ &\quad-\lambda(\mu,\alpha)(G(0,\sigma_{\mu,\alpha}) - G(x,u_{\mu,\alpha}(x)))\\ &\leqslant C\sum_{k=1}^{n}\mu_k\sigma_{\mu,\alpha}^{p_k+1} + C\lambda(\mu,\alpha)\sigma_{\mu,\alpha}^{q+1}\\ &\leqslant C\sum_{k=1}^{n}\mu_k\sigma_{\mu,\alpha}^{p_k+1}.\end{aligned} \tag{4.25}$$

Let $x_1 = x_{1,\mu,\alpha} \in [0,1]$ satisfy $u_{\mu,\alpha}(x_1) = (1-\varepsilon)\sigma_{\mu,\alpha}$, where $0 < \varepsilon \ll 1$ is a constant. By the mean value theorem and (4.25) we have for $y_{1,\mu,\alpha} \in [0, x_{1,\mu,\alpha}]$

$$\frac{\varepsilon\sigma_{\mu,\alpha}}{x_1} = \left|\frac{u_{\mu,\alpha}(0) - u_{\mu,\alpha}(x_1)}{x_1}\right| = |u'_{\mu,\alpha}(y_{1,\mu,\alpha})| \leqslant \sqrt{C\sum_{k=1}^{n}\mu_k\sigma_{\mu,\alpha}^{p_k+1}}. \tag{4.26}$$

Hence, by (2.4), (4.9) and (4.26),

$$\begin{aligned} C\alpha^{\frac{2q+3-p_i}{p_i+3}}\mu_i^{-\frac{q+3}{p_i+3}} &\geqslant C\|u_{\mu,\alpha}\|_{q+1}^{q+1} \geqslant (g(x,u_{\mu,\alpha}),u_{\mu,\alpha})_2 \\ &\geqslant C\int_0^{x_1} g(x,u_{\mu,\alpha}(x))u_{\mu,\alpha}(x)\,\mathrm{d}x \\ &\geqslant C\int_0^{x_1} u_{\mu,\alpha}(x)^{q+1}\,\mathrm{d}x \geqslant C(1-\varepsilon)\sigma_{\mu,\alpha}^{q+1}x_1 \\ &\geqslant C(1-\varepsilon)\varepsilon\sigma_{\mu,\alpha}^{q+2}\left(\sum_{k=1}^{n}\mu_k\sigma_{\mu,\alpha}^{p_k+1}\right)^{-1/2}; \end{aligned} \tag{4.27}$$

this implies that

$$\begin{aligned} \sigma_{\mu,\alpha}^{q+2} &\leqslant C\alpha^{\frac{2q+3-p_i}{p_i+3}}\mu_i^{-\frac{q+3}{p_i+3}}\sum_{k=1}^{n}\mu_k^{1/2}\sigma_{\mu,\alpha}^{\frac{p_k+1}{2}} \\ &= C\sum_{k=1}^{n}\left(\mu_k\alpha^{\frac{2(p_k-p_i)}{p_i+3}}\mu_i^{-\frac{p_k+3}{p_i+3}}\right)^{1/2}(\alpha\mu_i^{-1/2})^{\frac{2q+3-p_k}{p_i+3}}\sigma_{\mu,\alpha}^{\frac{p_k+1}{2}}. \end{aligned} \tag{4.28}$$

Now, our conclusion follows from (2.16), (2.17) and (4.28), since $(p_k+1)/2 < q+2$. □

Lemma 4.7. *Assume that* $\{(\mu,\alpha)\} \subset L$ *satisfies* (B.1). *Then for* $1 \leqslant k \leqslant n$

$$\mu_k\sigma_{\mu,\alpha}^{p_k+1} \leqslant C\mu_i\sigma_{\mu,\alpha}^{p_i+1}. \tag{4.29}$$

P r o o f. For a fixed (μ,α), let $1 \leqslant j(\mu,\alpha) \leqslant n$ satisfy

$$\max_{1\leqslant k\leqslant n}\mu_k\sigma_{\mu,\alpha}^{p_k+1} = \mu_{j(\mu,\alpha)}\sigma_{\mu,\alpha}^{p_{j(\mu,\alpha)+1}}. \tag{4.30}$$

Then there exists a subsequence of $\{(\mu,\alpha)\}$ and $1 \leqslant j \leqslant n$ such that $j = j(\mu,\alpha)$ for this subsequence. We fix and consider this subsequence. Then using (4.28) and (4.30) we obtain

$$\sigma_{\mu,\alpha}^{q+2} \leqslant C\sqrt{\sum_{k=1}^{n}\mu_k\sigma_{\mu,\alpha}^{p_k+1}}\,\alpha^{\frac{2q+3-p_i}{p_i+3}}\mu_i^{-\frac{q+3}{p_i+3}} \leqslant C\mu_j^{1/2}\sigma_{\mu,\alpha}^{\frac{p_j+1}{2}}\alpha^{\frac{2q+3-p_i}{p_i+3}}\mu_i^{-\frac{q+3}{p_i+3}},$$

that is,

$$\sigma_{\mu,\alpha} \leqslant C\Big(\mu_j^{1/2}\alpha^{\frac{2q+3-p_i}{p_i+3}}\mu_i^{-\frac{q+3}{p_i+3}}\Big)^{\frac{2}{2q+3-p_j}}. \tag{4.31}$$

By (4.23), (4.24) and (4.31) we obtain

$$\alpha^{\frac{2(p_i-q)}{p_i+3}}\mu_i^{\frac{q+3}{p_i+3}} \leqslant C\lambda(\mu,\alpha) \leqslant C\sum_{k=1}^{n}\mu_k\sigma_{\mu,\alpha}^{p_k-q} \leqslant C\mu_j\Big(\mu_j^{1/2}\alpha^{\frac{2q+3-p_i}{p_i+3}}\mu_i^{-\frac{q+3}{p_i+3}}\Big)^{\frac{2(p_j-q)}{2q+3-p_j}};$$

namely,

$$\mu_j \geqslant C\alpha^{\frac{2(p_i-p_j)}{p_i+3}}\mu_i^{\frac{p_j+3}{p_i+3}}.$$

This along with (2.17) implies that (4.30) holds for $j = j(\mu,\alpha) = i$ except a finite number of the elements of $\{(\mu,\alpha)\}$. Thus, we obtain (4.29). $\square$

5. Further observations

Let

$$\begin{aligned}
&\xi := \xi_{\mu,\alpha} = (\lambda(\mu,\alpha)/\mu_i)^{\frac{1}{p_i-q}},\ \ \tau := \tau_{\mu,\alpha} = \mu_i^{\frac{1-q}{2(p_i-q)}}\lambda(\mu,\alpha)^{\frac{p_i-1}{2(p_i-q)}},\\
&t := \tau x,\ \ w_{\mu,\alpha}(t) := \xi^{-1}u_{\mu,\alpha}(x).
\end{aligned}$$

Furthermore, let $A_k(t) := a_k(t/\tau)$. Then by (2.1), $w_{\mu,\alpha}(t)$ satisfies the equation
(5.1)

$$\begin{aligned}
w''_{\mu,\alpha}(t) &+ A_i(t)w^{p_i}_{\mu,\alpha}(t) - A_0(t)w^q_{\mu,\alpha}(t) + \sum_{k=1,k\neq i}^{n}\tau^{-2}\mu_k\xi^{p_k-1}A_k(t)w^{p_k}_{\mu,\alpha}(t)\\
&+\sum_{k=1}^{n}\tau^{-2}\mu_k\xi^{-1}f_{k,0}(t/\tau,\xi w_{\mu,\alpha}(t)) - \lambda(\mu,\alpha)\tau^{-2}\xi^{-1}g_0(t/\tau,\xi w_{\mu,\alpha}(t))\\
&= 0,\ \ t\in I_{\mu,\alpha} := (-\tau,\tau),\\
w_{\mu,\alpha}(t) &> 0,\ \ t\in I_{\mu,\alpha},\\
w_{\mu,\alpha}(\pm\tau) &= 0.
\end{aligned}$$

It follows from (2.15) and Lemma 4.5 that

$$\tau_{\mu,\alpha} \geqslant C\mu_i^{\frac{1-q}{2(p_i-q)}}\Big(\alpha^{\frac{2(p_i-q)}{p_i+3}}\mu_i^{\frac{q+3}{p_i+3}}\Big)^{\frac{p_i-1}{2(p_i-q)}} = C\Big(\alpha\mu_i^{\frac{2}{p_i-1}}\Big)^{\frac{p_i-1}{p_i+3}} \to \infty. \tag{5.2}$$

Therefore, we expect that $w_{\mu,\alpha}(t) \to w_\infty(t)$ if $\{(\mu,\alpha)\} \subset L$ satisfies (B.1), where w_∞ is the ground state of (4.12). The goal of this section is to show the following lemma:

Lemma 5.1. *Assume that* $\{(\mu,\alpha)\}\subset L$ *satisfies* (B.1). *Then* $w_{\mu,\alpha}(t)\to w_\infty(t)$ *in* $L^{q+1}(\mathbb{R})$ *and* $L^{p_k+1}(\mathbb{R})$.

To prove Lemma 5.1, we recall some important properties of w_∞. We know that there exists a unique solution w_∞ of (4.12), which is called the ground state solution of (4.12), and satisfies the following properties:

$$w_\infty(0)=\zeta:=\Big(\frac{p_i+1}{q+1}\Big)^{\frac{1}{p_i-q}},\ \ w_\infty(t)=w_\infty(-t),\ \ t\in\mathbb{R},\ \ w_\infty'(t)\leqslant 0,\ \ t\geqslant 0, \tag{5.3}$$

$$\frac{1}{2}(w_\infty'(t))^2+\frac{1}{p_i+1}w_\infty^{p_i+1}(t)-\frac{1}{q+1}w_\infty^{q+1}(t)=0,\ \ t\in\mathbb{R}, \tag{5.4}$$

$$w_\infty(t)\leqslant C\mathrm{e}^{-C|t|}\ \ (q=1),\quad w_\infty(t)\leqslant C(1+|t|)^{-s}\ \Big(q>1, 0<s<\frac{2}{q-1}\Big),\ \ t\in\mathbb{R}. \tag{5.5}$$

We refer to Berestycki and Lions [1] for these properties. Let $\zeta_{\mu,\alpha}:=\max\limits_{t\in I_{\mu,\alpha}} w_{\mu,\alpha}(t)=\xi^{-1}\sigma_{\mu,\alpha}$.

Lemma 5.2. *Assume that* $\{(\mu,\alpha)\}\subset L$ *satisfies* (B.1). *Then* $C^{-1}\leqslant\zeta_{\mu,\alpha}\leqslant C$.

Proof. By (4.24) and Lemma 4.7 we have that

$$\lambda(\mu,\alpha)\leqslant C\sum_{k=1}^{n}\mu_k\sigma_{\mu,\alpha}^{p_k-q}\leqslant C\mu_i\sigma_{\mu,\alpha}^{p_i-q}. \tag{5.6}$$

This implies the first inequality. Next, since (4.31) holds for $j=i$, we see from Lemma 4.5 that

$$\sigma_{\mu,\alpha}^{p_i-q}\leqslant C(\alpha\mu_i^{-1/2})^{\frac{2(p_i-q)}{p_i+3}}\leqslant C\frac{\lambda(\mu,\alpha)}{\mu_i}. \tag{5.7}$$

Thus we obtain the second inequality. $\square$

Lemma 5.3. *Assume that* $\{(\mu,\alpha)\}\subset L$ *satisfies* (B.1). *Then*

$$\lambda(\mu,\alpha)\tau^{-2}\xi^{q-1}=1,\ \ \tau^{-2}\mu_k\xi^{p_k-1}\to 0\quad (k\neq i). \tag{5.8}$$

Proof. The first equality follows from the definition of τ and ξ. We shall show the second inequality. It follows from (5.6) and (5.7) that

$$\lambda(\mu,\alpha)\leqslant C\Big(\alpha\mu_i^{\frac{q+3}{2(p_i-q)}}\Big)^{\frac{2(p_i-q)}{p_i+3}}. \tag{5.9}$$

Then by (2.17), Lemma 4.5 and (5.9) that for $k \neq i$

$$\tau^{-2}\mu_k\xi^{p_k-1} = \lambda(\mu,\alpha)^{\frac{p_k-p_i}{p_i-q}}\mu_k\mu_i^{-\frac{p_k-q}{p_i-q}} \leqslant C\mu_k\alpha^{\frac{2(p_k-p_i)}{p_i+3}}\mu_i^{-\frac{p_k+3}{p_i+3}} \to 0.$$

Thus the proof is complete. □

Lemma 5.4. *Assume that* $\{(\mu,\alpha)\} \subset L$ *satisfies* (B.1). *Then* $\|w_{\mu,\alpha}\|_{q+1} \leqslant C$.

Proof. By Lemma 4.2 and Lemma 4.5 we obtain

$$\|w_{\mu,\alpha}\|_{q+1}^{q+1} = \xi^{-(q+1)}\tau\|u_{\mu,\alpha}\|_{q+1}^{q+1} \leqslant C\Big(\lambda(\mu,\alpha)^{-1}\alpha^{\frac{2(p_i-q)}{p_i+3}}\mu_i^{\frac{q+3}{p_i+3}}\Big)^{\frac{2q+3-p_i}{2(p_i-q)}} \leqslant C. \tag{5.10}$$

□

We regard $w_{\mu,\alpha}(t)$ as a function defined on $\mathbb{R}$ by 0-extension.

Lemma 5.5. *Assume that* $\{(\mu,\alpha)\} \subset L$ *satisfies* (B.1). *Then* $\|w'_{\mu,\alpha}\|_\infty \leqslant C$. *Furthermore,* $w_{\mu,\alpha}$ *satisfies the inequality*

$$w'_{\mu,\alpha}(t) \leqslant -\sqrt{T_1(w_{\mu,\alpha}(t))}, \tag{5.11}$$

where

$$T_1(w) = (w'_{\mu,\alpha}(\tau))^2 + Cw^{q+1} - Cw^{p_i+1} - o(1)\sum_{k=1,k\neq i}^{n} w^{p_k+1}. \tag{5.12}$$

P r o o f. It follows from (2.11) and Lemma 4.1 that

(5.13)

$$\begin{aligned}
&\frac{1}{2}(u'_{\mu,\alpha}(x))^2 + \sum_{k=1}^{n} \mu_k \frac{1}{p_k+1} a_k(x) u_{\mu,\alpha}^{p_k+1}(x) - \sum_{k=1}^{n} \mu_k \int_0^x a'_k(s) u_{\mu,\alpha}^{p_k+1}(s)\,\mathrm{d}s \\
&\quad + \sum_{k=1}^{n} \mu_k \int_0^{u_{\mu,\alpha}(x)} f_{k,0}(x,s)\,\mathrm{d}s - \sum_{k=1}^{n} \mu_k \int_0^x \mathrm{d}y \Big(\int_0^{u_{\mu,\alpha}(y)} \frac{\partial}{\partial z} f_{k,0}(z,s)\,\mathrm{d}s \Big) \\
&\quad - \lambda(\mu,\alpha) \frac{1}{q+1} a_0(x) u_{\mu,\alpha}^{q+1}(x) + \lambda(\mu,\alpha) \int_0^x a'_0(s) u_{\mu,\alpha}^{q+1}(s)\,\mathrm{d}s \\
&\quad - \lambda(\mu,\alpha) \int_0^{u_{\mu,\alpha}(x)} g_0(x,s)\,\mathrm{d}s + \lambda(\mu,\alpha) \int_0^x \mathrm{d}y \Big(\int_0^{u_{\mu,\alpha}(y)} \frac{\partial}{\partial z} g_0(z,s)\,\mathrm{d}s \Big) \\
&= \sum_{k=1}^{n} \mu_k \Big\{ \frac{1}{p_k+1} a_k(0) \sigma_{\mu,\alpha}^{p_k+1} + \int_0^{\sigma_{\mu,\alpha}} f_{k,0}(0,s)\,\mathrm{d}s \Big\} \\
&\quad - \lambda(\mu,\alpha) \Big\{ \frac{1}{q+1} a_0(0) \sigma_{\mu,\alpha}^{q+1} + \int_0^{\sigma_{\mu,\alpha}} g_0(0,s)\,\mathrm{d}s \Big\} \\
&= \frac{1}{2}(u'_{\mu,\alpha}(1))^2 \\
&\quad - \sum_{k=1}^{n} \mu_k \Big\{ \int_0^1 a'_k(s) u_{\mu,\alpha}^{p_k+1}(s)\,\mathrm{d}s + \int_0^1 \mathrm{d}y \Big(\int_0^{u_{\mu,\alpha}(y)} \frac{\partial}{\partial z} f_{k,0}(z,s)\,\mathrm{d}s \Big) \Big\} \\
&\quad + \lambda(\mu,\alpha) \Big\{ \int_0^1 a'_0(s) u_{\mu,\alpha}^{q+1}(s)\,\mathrm{d}s + \int_0^1 \mathrm{d}y \Big(\int_0^{u_{\mu,\alpha}(y)} \frac{\partial}{\partial z} g_0(z,s)\,\mathrm{d}s \Big) \Big\}.
\end{aligned}$$

By the first equality of (5.13), (2.9), (4.29) and (5.6) we have

$$\frac{1}{2}\xi^2\tau^2(w'_{\mu,\alpha}(t))^2 = \frac{1}{2}(u'_{\mu,\alpha}(x))^2 \leqslant C \sum_{k=1}^{n} \mu_k \sigma_{\mu,\alpha}^{p_k+1} + C\lambda(\mu,\alpha)\sigma_{\mu,\alpha}^{q+1} \leqslant \mu_i \sigma_{\mu,\alpha}^{p_i+1};$$

this implies that

$$(w'_{\mu,\alpha}(t))^2 \leqslant C\mu_i \sigma_{\mu,\alpha}^{p_i+1} \tau^{-2} \xi^{-2} \leqslant C. \tag{5.14}$$

Thus, we obtain the first assertion. Next, by the second equality of (5.13) we have

$$\begin{aligned}\frac{1}{2}(u'_{\mu,\alpha}(x))^2 &= \frac{1}{2}(u'_{\mu,\alpha}(1))^2 + \lambda(\mu,\alpha)\frac{1}{q+1}a_0(x)u^{q+1}_{\mu,\alpha}(x)\\
&- \sum_{k=1}^{n}\mu_k\frac{1}{p_k+1}a_k(x)u^{p_k+1}_{\mu,\alpha}(x) - \sum_{k=1}^{n}\mu_k\int_0^{u_{\mu,\alpha}(x)} f_{k,0}(x,s)\,\mathrm{d}s\\
&+ \lambda(\mu,\alpha)\int_0^{u_{\mu,\alpha}(x)} g_0(x,s)\,\mathrm{d}s - \sum_{k=1}^{n}\mu_k\int_x^1 a'_k(s)u^{p_k+1}_{\mu,\alpha}(s)\,\mathrm{d}s\\
&+ \lambda(\mu,\alpha)\int_x^1 a'_0(s)u^{q+1}_{\mu,\alpha}(s)\,\mathrm{d}s - \sum_{k=1}^{n}\mu_k\int_x^1 \mathrm{d}y\Big(\int_0^{u_{\mu,\alpha}(y)}\frac{\partial}{\partial z}f_{k,0}(z,s)\,\mathrm{d}s\Big)\\
&+ \lambda(\mu,\alpha)\int_x^1 \mathrm{d}y\Big(\int_0^{u_{\mu,\alpha}(y)}\frac{\partial}{\partial z}g_0(z,s)\,\mathrm{d}s\Big).\end{aligned}\tag{5.15}$$

Then by (1.3), (2.10) and (5.15) we obtain

$$\begin{aligned}\frac{1}{2}(u'_{\mu,\alpha}(x))^2 \geqslant \frac{1}{2}(u'_{\mu,\alpha}(1))^2 &+ \lambda(\mu,\alpha)\frac{1}{q+1}b_0(x)u^{q+1}_{\mu,\alpha}(x)\\
&- \sum_{k=1}^{n}\mu_k\frac{1}{p_k+1}a_k(x)u^{p_k+1}_{\mu,\alpha}(x)\\
&- \sum_{k=1}^{n}\mu_k\int_0^{u_{\mu,\alpha}(x)} f_{k,0}(x,s)\,\mathrm{d}s + \lambda(\mu,\alpha)\int_0^{u_{\mu,\alpha}(x)} g_0(x,s)\,\mathrm{d}s.\end{aligned}\tag{5.16}$$

By (2.9) and Lemma 4.6 we have for $1 \leqslant k \leqslant n$

$$\begin{aligned}\Big|\int_0^{u_{\mu,\alpha}(x)} f_{k,0}(x,s)\,\mathrm{d}s\Big| &= o(1)u^{p_k+1}_{\mu,\alpha}(x),\\
\Big|\lambda(\mu,\alpha)\int_0^{u_{\mu,\alpha}(x)} g_0(x,s)\,\mathrm{d}s\Big| &= o(1)\lambda(\mu,\alpha)u^{q+1}_{\mu,\alpha}(x).\end{aligned}\tag{5.17}$$

Therefore, by (5.16) and (5.17) we obtain

$$\frac{1}{2}(u'_{\mu,\alpha}(x))^2 \geqslant \frac{1}{2}(u'_{\mu,\alpha}(1))^2 + C\lambda(\mu,\alpha)u^{q+1}_{\mu,\alpha}(x) - C\sum_{k=1}^{n}\mu_k u^{p_k+1}_{\mu,\alpha}(x).\tag{5.18}$$

By (5.8) and (5.18), we obtain

$$
\begin{aligned}
\frac{1}{2}(w'_{\mu,\alpha}(t))^2 &\geqslant \frac{1}{2}(w'_{\mu,\alpha}(\tau))^2 + Cw^{q+1}_{\mu,\alpha}(t) - Cw^{p_i+1}_{\mu,\alpha}(t) \\
&\quad - C\sum_{k=1,k\neq i}^{n} \mu_k \tau^{-2}\xi^{p_k-1} w^{p_k+1}_{\mu,\alpha}(t) \\
&\geqslant \frac{1}{2}(w'_{\mu,\alpha}(\tau))^2 + Cw^{q+1}_{\mu,\alpha}(t) - Cw^{p_i+1}_{\mu,\alpha}(t) \\
&\quad - o(1)\sum_{k=1,k\neq i}^{n} w^{p_k+1}_{\mu,\alpha}(t).
\end{aligned}
\tag{5.19}
$$

Since $w'_{\mu,\alpha}(t) \leqslant 0$ for $0 \leqslant t \leqslant \tau$, we obtain (5.11). Thus the proof is complete. $\square$

Lemma 5.6. *Assume that $\{(\mu,\alpha)\} \subset L$ satisfies* (B.1). *Then $w_{\mu,\alpha} \to w_\infty$ uniformly on any compact subset on $\mathbb{R}$.*

Proof. We see that $\{w_{\mu,\alpha}\}, \{w'_{\mu,\alpha}\}$ and $\{w''_{\mu,\alpha}\}$ are bounded by (5.1), Lemma 5.2 and Lemma 5.5. Hence, by Ascoli-Arzela's theorem, (5.1) and Lemma 5.3, we can choose a subsequence of $\{(\mu,\alpha)\}$ such that $w_{\mu,\alpha} \to w_1$ uniformly on any compact subset on $\mathbb{R}$, where w_1 satisfies the equation in (4.12). Furthermore, by a standard regularity argument, we obtain that $w_1 \in C^2(\mathbb{R})$. Let $t_0 := \inf\{t > 0\colon\ w_1(t_0) > 0\}$. Then by Lemma 5.2, $t_0 > 0$. If $t_0 < \infty$, then $w_1(t_0) = w'_1(t_0) = 0$, since $w_1(t) \geqslant 0$ for $t \in \mathbb{R}$. Then the uniqueness theorem of ODE implies that $w_1 \equiv 0$. This is a contradiction. Hence $t_0 = \infty$, namely, $w_1(t) > 0$ for $t \in \mathbb{R}$. Now by Fatou's lemma and Lemma 5.4

$$
\int_{\mathbb{R}} w_1^{q+1}(t)\,\mathrm{d}t \leqslant \liminf \int_{\mathbb{R}} w^{q+1}_{\mu,\alpha}(t)\,\mathrm{d}t \leqslant C. \tag{5.20}
$$

Since $w_1(t)$ is decreasing for $t > 0$, we see that $w_1(t) \to 0$ as $|t| \to \infty$. Hence w_1 satisfies (4.12), namely, $w_1 \equiv w_\infty$. $\square$

Lemma 5.7. *Assume that $\{(\mu,\alpha)\} \subset L$ satisfies* (B.1). *Then there exists $y(t) \in L^{q+1}(\mathbb{R})$ such that $w_{\mu,\alpha}(t) \leqslant y(t)$ for $t \in \mathbb{R}$.*

Proof. Let $q < r < 2q+3$ be fixed. Furthermore, let $y_1(t) := (t+1)^{-2/(r-1)}$. Then $y_1(t)$ satisfies

$$
\begin{aligned}
&y'_1(t) = -\sqrt{T_0(y_1(t))}, \quad t > 0, \\
&y_1(0) = 1,
\end{aligned}
\tag{5.21}
$$

where $T_0(y) := 4(r-1)^{-2}y^{r+1}$. Moreover, since $r < 2q+3 < 2p_k+3$, it is clear that $y_1(t) \in L^{q+1}(\mathbb{R}_+) \cap L^{p_k+1}(\mathbb{R}_+)$. Next, by (5.11), $w_{\mu,\alpha}(t)$ satisfies

$$\begin{aligned} &w'_{\mu,\alpha}(t) \leqslant -\sqrt{T_1(w_{\mu,\alpha}(t))}, \quad 0 < t < \tau, \\ &w_{\mu,\alpha}(0) = \zeta_{\mu,\alpha}. \end{aligned} \tag{5.22}$$

By (5.12) we obtain that for $0 \leqslant y \leqslant \varepsilon$

$$\begin{aligned} T_1(y) - T_0(y) \geqslant (w'_{\mu,\alpha}(\tau))^2 + Cy^{q+1} - Cy^{p_1+1} \\ - o(1)\sum_{k=1}^{n} y^{p_k+1} - \frac{4}{(r-1)^2}y^{r+1} > 0. \end{aligned} \tag{5.23}$$

We fix $t_0 \gg 1$ satisfying $y_1(t_0) < \varepsilon$. Then by (5.5) and Lemma 5.6 we see that $w_{\mu,\alpha}(t_0) < y_1(t_0)$. Then by the comparison theorem of ODE, we obtain that $w_{\mu,\alpha}(t) \leqslant y_1(t)$ for $t > t_0$. Now define $y(t)$ by

$$y(t) = \begin{cases} C, & |t| \leqslant t_0, \\ y_1(|t|), & |t| > t_0, \end{cases} \tag{5.24}$$

where $C > 0$ is a large constant. This is the desired function. □

Proof of Lemma 5.1. Lemma 5.1 is a direct consequence of Lemma 5.6, Lemma 5.7 and Lebesgue's convergence theorem. Thus the proof is complete. □

6. Proof of Theorem 2.1

We introduce the following lemma before the proof of Theorem 2.1.

Lemma 6.1 ([4, Lemma 4.6]). *Let $w_\infty(t)$ be the ground state of* (4.12). *Then*

$$\int_{\mathbb{R}} w_\infty^{q+1}(t)\,\mathrm{d}t = \frac{2}{p_i - q}\sqrt{\frac{\pi(q+1)}{2}}\zeta^{\frac{q+3}{2}}\frac{\Gamma\left(\frac{q+3}{2(p_i-q)}\right)}{\Gamma\left(\frac{p_i+3}{2(p_i-q)}\right)}. \tag{6.1}$$

Proof of Theorem 2.1. We know from [4, (4.44)] that

$$\|w_\infty\|_{p_i+1}^{p_i+1} = \frac{(p_i+1)(q+3)}{(p_i+3)(q+1)}\|w_\infty\|_{q+1}^{q+1}. \tag{6.2}$$

Furthermore, by (2.17), Lemma 4.5 and (5.9)

$$\begin{aligned}
(6.3) \qquad \mu_i\|u_{\mu,\alpha}\|_{p_i+1}^{p_i+1} &= \lambda(\mu,\alpha)^{\frac{p_i+3}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|w_{\mu,\alpha}\|_{p_i+1}^{p_i+1},\\
\|u_{\mu,\alpha}\|_{q+1}^{q+1} &= \lambda(\mu,\alpha)^{\frac{2q+3-p_i}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|w_{\mu,\alpha}\|_{q+1}^{q+1},\\
\mu_k\|u_{\mu,\alpha}\|_{p_k+1}^{p_k+1} &= \mu_k\lambda(\mu,\alpha)^{\frac{p_k-p_i}{p_i-q}}\mu_i^{-\frac{p_k-q}{p_i-q}}\left\{\lambda(\mu,\alpha)^{\frac{p_i+3}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1}\right\}\\
&\leqslant C\mu_k\alpha^{\frac{2(p_k-p_i)}{p_i+3}}\mu_i^{-\frac{p_k+3}{p_i+3}}\left\{\lambda(\mu,\alpha)^{\frac{p_i+3}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1}\right\}\\
&= o(1)\left\{\lambda(\mu,\alpha)^{\frac{p_i+3}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1}\right\} \quad (k\neq i).
\end{aligned}$$

Since we have Lemma 5.1 and Lemma 5.6, it follows by direct computation that

$$\begin{aligned}
(6.4) \qquad (f_k(x,u_{\mu,\alpha}),u_{\mu,\alpha})_2 &= (1+o(1))a_k(0)\xi^{p_k+1}\|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1},\\
\Phi_k(u_{\mu,\alpha}) &= \frac{1}{p_k+1}(1+o(1))a_k(0)\xi^{p_k+1}\|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1},\\
(g(x,u_{\mu,\alpha}),u_{\mu,\alpha})_2 &= (1+o(1))a_0(0)\xi^{q+1}\|w_{\mu,\alpha}\|_{q+1}^{q+1}.
\end{aligned}$$

Then by (2.17), (6.3) and (6.4) and the assumption $a_i(0)=a_0(0)=1$ we obtain

$$\begin{aligned}
(6.5) \qquad \lambda(\mu,\alpha) &= \frac{2\alpha+\sum_{k=1}^{n}\mu_k\frac{p_k-1}{p_k+1}(1+o(1))a_k(0)\xi^{p_k+1}\|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1}}{(1+o(1))a_0(0)\xi^{q+1}\|w_{\mu,\alpha}\|_{q+1}^{q+1}}\\
&= \frac{2\alpha+\frac{p_i-1}{p_i+1}\lambda(\mu,\alpha)^{\frac{p_i+3}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\{(1+o(1))\|w_{\mu,\alpha}\|_{p_i+1}^{p_i+1}+o(1)\}}{(1+o(1))\lambda(\mu,\alpha)^{\frac{2q+3-p_i}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|w_{\mu,\alpha}\|_{q+1}^{q+1}};
\end{aligned}$$

this implies that

$$(6.6) \qquad \frac{\lambda(\mu,\alpha)}{\alpha^{\frac{2(p_i-q)}{p_i+3}}\mu_i^{\frac{q+3}{p_i+3}}} = \left(\frac{2}{S_{\mu,\alpha}}\right)^{\frac{2(p_i-q)}{p_i+3}} \to \left(\frac{2}{S}\right)^{\frac{2(p_i-q)}{p_i+3}}.$$

Here by (6.2)

$$\begin{aligned}
(6.7) \qquad S_{\mu,\alpha} &= (1+o(1))\|w_{\mu,\alpha}\|_{q+1}^{q+1} - \frac{p_i-1}{p_i+1}((1+o(1))\|w_{\mu,\alpha}\|_{p_i+1}^{p_i+1}+o(1)),\\
S &= \|w_\infty\|_{q+1}^{q+1} - \frac{p_i-1}{p_i+1}\|w_\infty\|_{p_i+1}^{p_i+1} = \frac{2(2q+3-p_i)}{(p_i+3)(q+1)}\|w_\infty\|_{q+1}^{q+1}.
\end{aligned}$$

Then Theorem 2.1 follows from (6.6), (6.7) and Lemma 6.1. Under the general situation of $a_i(0)$ and $a_0(0)$, we also obtain our conclusion by replacing μ_i and $\lambda(\mu,\alpha)$ by $a_i(0)\mu_i$ and $a_0(0)\lambda(\mu,\alpha)$, respectively, and by repeating the same arguments as those used above. Thus the proof is complete. □

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