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CHORDAL INTERSECTION GRAPHS OF BANDS

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To Miroslav Fiedler on the occasion of his 70th birthday

A graph G is said to be *chordal* if G does not contain a cycle with n vertices ($n \geqslant 4$) as an induced subgraph. Let S be a semigroup. By $G(S)$ we denote a graph which has as vertices all subsemigroups of S (including S itself) with AB an edge of $G(S)$ if and only if $A \neq B$ and $A \cap B \neq \emptyset$. Bosák [1] began such an investigation in the sixties by considering the graph $G{*}(S) = G(S) \setminus \{S\}$ (of all proper subsemigroups of S).

A *band* is a semigroup in which every element is idempotent. A commutative band is a *semilattice*. Semilattices can be defined as a special type of posets. The relation $\leqslant$ defined on a semilattice S by $a \leqslant b$ if and only $ab = a$ gives S structure of a poset in which every pair of elements has a *greatest lower bound (meet)*. For $a, b \in S$ we put $a < b$ if and only if $a \leqslant b$ and $a \neq b$. Two elements a, b of a semilattice S are said to be *noncomparable* if $a \neq ab \neq b$; we shall writte $a \parallel b$. By $a \operatorname{non} \parallel b$ we denote the fact that a, b are *comparable*, i.e. $a \leqslant b$ or $b \leqslant a$.

In [2] Ackerman, McMoriris and Seif give a characterization of the semilattice S whose graph is chordal.

Theorem S. *Let S be a semilattice. Then $G(S)$ is chordal if and only if S satisfies the following conditions:*

(i) *noncomparable elements of S meet to* 0 *(the zero of S);*

(ii) *S is a tree, i.e. joins of noncomparable elements of S do not exist;*

(iii) *the height of the longest chain in S is less than* 4.

Note that the authors considered the graph $G*(S)$. It is easy to show that $G*(S)$ (including the empty graph) is chordal if and only if $G(S)$ is chordal.

The aim of this paper is to characterize bands whose graphs are chordal.

Let S be a band. Define a relation σ on S by $(a,b) \in \sigma$ if and only if $aba = a$ and $bab = b$ for $a, b \in S$. It is well known (see Proposition II.1.1 of [3]) that σ is the least semilattice congruence on S. Then the quotient semigroup S/σ is a semilattice and each of its classes is a *rectangular band*.

Recall that a band S is said to be rectangular if

(1) $$aba = a \text{ for all } a, b \in S.$$

A semigroup S is a *left (right) zero semigroup* if $ab = a(ab = b)$ for all $a, b \in S$. It is well known (see Lemma II.1,5 of [3]) that

(2) *A semigroup S is a rectangular band if and only if it is isomorphic to the direct product of a left zero semigroup and a right zero semigroup.*

For any element a of a band S by $[a]$ we denote the class of S/σ containing a. Put $\Re(S) = \{(x_1, x_2, x_3, x_4)$, where $x_i \in S$ and $\{x_i, x_{i+1}\}$ are subbands of S for $i \notin I_4\}$. Note that by I we denote the ring of all integers and I_n is the quotient ring I/nI for $n \in I$.

Theorem B. *Let S ba a band. Then the following conditions are equivalent:*

1. *The graph $G(S)$ is chordal.*
2. *If $(e, f, g, h) \in \Re(S)$, then* $\operatorname{card}\{e, f, g, h\} \leqslant 3$.
3. *The band S satisfies the following conditions:*
   - (i) *$G(S/\sigma)$ is chordal;*
   - (ii) $\operatorname{card} Z \leqslant 3$, *where $Z = \min S/\sigma$;*
   - (iii) *if $Z < X \leqslant Y$, then* $\operatorname{card}(X \cup Y) \leqslant 2$, *where $X, Y \in S/\sigma$;*
   - (iv) *if* $\operatorname{card} Z = 3$, *then* $\operatorname{card} xZx = 1$ *for all $x \in S \setminus Z$;*
   - (v) *if* $\operatorname{card} Z = 2 = \operatorname{card} yZy$ *for some $y \in S \setminus Z$, then* $\operatorname{card} xZx = 1$ *for all $x \in S \setminus Z$, $x \neq y$.*
   - (vi) *if $Z < [x] \leqslant [y]$, where $x, y \in S$, $x \neq y$, then $xZx = yZy$ and* $\operatorname{card} xZx = 1$.

Proof. $1 \Rightarrow 2$. Suppose that $S(G)$ is chordal and $(x_1, x_2, x_3, x_4) \in \Re(S)$ with $\operatorname{card}\{x_1, x_2, x_3, x_4\} = 4$. Put $X_i = \{x_i, x_{i+1}\}$ for $i \in I_4$. It is easy to show that X_1, X_2, X_3, X_4 is a cycle of $G(S)$ which is an induced subgraph. Therefore $G(S)$ is not chordal, a contradiction.

$2 \Rightarrow 3$. First we will prove the following lemmas, in which we will suppose that $\operatorname{card}\{e, f, g, h\} \leqslant 3$ whenever $(e, f, g, h) \in \Re(S)$. □

Lemma 1. *If $A \in S/\sigma$, then* $\operatorname{card} A \leqslant 3$ *and so A is a left (or right) zero subsemigroup of S.*

P r o o f. Let $A \in S/\sigma$ and suppose that A is neither a left nor a right zero subsemigroup of S. Then by (2), there are elements $e, f \in A$ such that $\operatorname{card}\{e, f, ef, fe\} = 4$. It follows from (1) that $(e, ef, f, fe) \in \Re(S)$, which is a contradiction. Therefore A is a left or a right zero subsemigroup of S.

By way of contradiction we assume that $\operatorname{card} A \geqslant 4$. If A is a left zero semigroup, then for different elements e, f, g and h from A we have $(e, f, g, h) \in \Re(S)$, a contradiction. Thus $\operatorname{card} A \leqslant 3$. $\square$

Lemma 2. *If $A, B \in S/\sigma$ and $A < B$, then* $\operatorname{card} B \leqslant 2$.

P r o o f. Let $A, B \in S/\sigma$ with $A < B$ and suppose that $e, f, g \in B$ with $\operatorname{card}\{e, f, g\} = 3$. Choose $a \in A$.

If $eae = gag$, then, by Lemma 1, we have $(e, f, g, gag) \in \Re(S)$, which is a contradiction.

If $eae \neq gag$, then eae, $gag \in A$ and by Lemma 1 we obtain $(e, eae, gag, g) \in \Re(S)$, a contradiction.

Therefore $\operatorname{card} B \leqslant 2$. $\square$

Lemma 3. *If $A, B, C \in S/\sigma$ and $A < B < C$, then* $\operatorname{card} C = 1$.

P r o o f. Let $A, B, C \in S/\sigma$ with $A < B < C$ and suppose that $e, f \in C$, $e \neq f$. Choose $a \in A$ and $b \in B$.

If $eae \neq faf$, then eae, $faf \in A$ and by Lemma 1 we have $(e, eae, faf, f) \in \Re(S)$, a contradiction. If $ebe \neq fbf$, then we obtain a contradiction analogously.

Now, we can assume that $eae = faf$ and $ebe = fbf$. According to Lemma 1 we have $(e, eae, f, fbf) \in \Re(S)$, a contradiction. $\square$

Lemma 4. *Then height of the longest chain in S/σ is less than* 4.

P r o o f. Suppose that $A_1 < A_2 < A_3 < A_4$ where $A_i \in S/\sigma$, $i \in I_4$. Choose $a_i \in A_i$, $i \in I_4$, and put $e = a_4$, $f = ea_3e$, $g = fa_2f$ and $h = ga_1g$. Evidently we have $e \in A_4$, $f \in A_3$, $g \in A_2$ and $h \in A_1$.

Case 1. $h = ehe$. Then $(e, f, g, h) \in \Re(S)$, a contradiction.

Case 2. $h \neq ege$. If $ege = fhf$, then according to Lemma 1 we have $(f, g, h, ehe) \in \Re(S)$, a contradiction. If $ehe \neq fhf$, then $(e, f, fhf, ehe) \in \Re(S)$, a contradiction.

Therefore the height of the longest chain in S/σ is less than 4. $\square$

Lemma 5. *The semilattice S/σ is a tree.*

P r o o f. Suppose that $A_1 < A_2 < A_4$, $A_1 < A_3 < A_4$ and $A_2 \parallel A_3$ where $A_i \in S/\sigma$, $i \in I_4$. Choose $a_i \in A_i$, $i \in I_4$ and put $e = a_4$, $f = ea_2e$, $g = ea_3e$ and $h = ea_1e$.

Case 1. $fhf \neq h$. Then $(e, f, fhf, h) \in \Re(S)$, a contradiction.

Case 2. $ghg \neq h$. Analogously to Case 1 we obtain a contradiction.

Case 3. $fhf = h = ghg$. Then $(e, f, g, h) \in \Re(S)$, a contradiction. □

Lemma 6. *The graph $G(S/\sigma)$ is chordal.*

Proof. According to Lemmas 4, 5 and Theorem S, it suffices to show that the meet of two noncomparable elements of S/σ is the infimum of S/σ. On the contrary, suppose that $A_1 < A_2 < A_3$, $A_2 < A_4$ and $A_3 \parallel A_4$ where $A_i \in S/\sigma$, $i \in I_4$. Choose $a_i \in A_i$, $i \in I_4$ and put $e = a_3$, $f = a_4$, $g = ea_2e$ and $h = fa_2f$. If $ea_1e \neq ga_1g$, then by Lemma 1 we have $(e, g, ga_1g, ea_1e) \in \Re(S)$, which is a contradiction. We have $ea_1e = ga_1g$ and analogously we can show that $fa_1f = ha_1h$. According to Lemma 1, we obtain $(g, h, ha_1, ga_1g) \in \Re(S)$ and so $\operatorname{card}\{g, h, ha_1h, ga_1g\} \leqslant 3$.

Case 1. $g \neq h$. Then $ha_1h = ga_1g = ea_1e$ and so $(e, g, h, ha_1h) \in \Re(S)$, a contradiction.

Case 2. $g = h$. Then $fa_1f = ha_1h = ga_1g = ea_1e$ and so $(e, g, f, fa_1f) \in \Re(S)$, a contradiction.

By Z we denote the minimum of S/σ. □

Lemma 7. *If $B, C \in S/\sigma$ and $Z < B < C$, then* $\operatorname{card} B = 1$.

Proof. It follows from Lemma 2 that $\operatorname{card} B \leqslant 2$. Suppose that $\operatorname{card} B = 2$. Choose $h \in Z$, $b \in B$ and $e \in C$ and put $f = ebe$. Then $f \in B$. There is an element g of B such that $g \neq f$. If $ehe \neq fhf$, then by Lemma 1 we have $(e, f, fhf, ehe) \in \Re(S)$, which is a contradiction. Thus we obtain $ehe = fhf$.

Case 1. $ghg \neq ege$. Then by Lemma 1 we have $(f, g, ghg, fhf) \in \Re(S)$, a contradiction.

Case 2. $ghg = ehe$. Then $(e, f, g, ghg) \in \Re(S)$, a contradiction. □

Lemma 8. *If $X, Y \in S/\sigma$ and $Z < X \leqslant Y$, then* $\operatorname{card}(X \cup Y) = 2$.

The proof follows from Lemma 2, 3 and 7. □

Lemma 9. *If* $\operatorname{card} Z = 3$, *then* $\operatorname{card} xZx = 1$ *for all $x \in S \setminus Z$.*

Proof. Suppose that $\operatorname{card} Z = 3$. Let x be an element of $S \setminus Z$ such that $\operatorname{card} xZx \geqslant 2$. Choose $e, f \in xZx$ with $e \neq f$. Then $Z = \{e, f, g\}$ and so, by Lemma 1, we have $(e, g, f, x) \in \Re(S)$, a contradiction. Therefore $\operatorname{card} xZx = 1$ for all $x \in S \setminus Z$. □

Lemma 10. *If* $\operatorname{card} Z = 2 = \operatorname{card} yZy$ *for some* $y \in S \setminus Z$*, then* $\operatorname{card} xZx = 1$ *for all* $x \in S \setminus Z$*,* $x \neq y$*.*

Proof. Suppose that $\operatorname{card} Z = \operatorname{card} xZx = \operatorname{card} yZy = 2$ for some $x, y \in S \setminus Z$, $x \neq y$. Then $Z = xZx = yZy = \{e, f\}$ and so $(e, x, f, y) \in \Re(S)$, which is a contradiction. □

Lemma 11. *If* $Z < [x] \leqslant [y]$*, where* $x, y \in S$*,* $x \neq y$*, then* $xZx = yZy$ *and* $\operatorname{card} xZx = 1$*.*

Proof. Suppose that $Z < [x] \leqslant [y]$, where $x, y \in S$ and $x \neq y$. It follows from Lemma 8 that $\{x, y\}$ is a subband of S. For any pair of elements $e, f \in Z$ Lemma 1 implies that $(x, y, yfy, xex) \in \Re(S)$. Thus we obtain $yfy = xex$ and so $xZx = yZy$ and $\operatorname{card} xZx = 1$. □

Finally, the proof of the implication $2 \Rightarrow 3$ follows from Lemmas 6, 1, 8, 9, 10 and 11.

$3 \Rightarrow 1$. Assume that a band S satisfies (i)–(vi). By way of contradiction we suppose that $B_1, B_2, \ldots, B_n$ ($n \geqslant 4$) is a cycle of $G(S)$, which is an induced subgraph of $G(S)$. This means that $B_i \cap B_j \neq \emptyset$, $i \neq j$, if and only if $i = j + 1$ or $j = i + 1$ for $i, j \in I_n$.

Choose $a_{i+1} \in B_i \cap B_{i+1}$ and if $B_i \cap B_{i+1} \cap Z \neq \emptyset$, then $a_{i+1} \in Z$. It is clear that $a_i \neq a_j$ for $i, j \in I_n$ and $i \neq j$. By A_i we denote the subband of S generated by the set $\{a_i, a_{i+1}\}$. Evidently $A_i \subseteq B_i$ and $A_1, A_2, \ldots, A_n$ is a cycle of $G(S)$ having the following properties:

(3) It is induced subgraph of $G(S)$.

(4) $A_i \cap A_j \neq \emptyset$ ($i \neq j$) if and only if $i = j + 1$ or $j = i + 1$ for $i, j \in I_n$.

(5) If $A_i \cap A_{i+1} \cap Z \neq \emptyset$, then $a_{i+1} \in Z$.

We have the following possibilities:

Case 1. There is an index $i \in I_n$ such that $\{a_i, a_{i+1}, a_{i+2}\} \subseteq Z$. Then by (ii) we have $\{a_i, a_{i+1}, a_{i+2}\} = Z$.

Subcase 1a. $a_{i-1} = a_{i+3}$. Then $n = 4$ and it follows from (iv) that $a_{i+3}Za_{i+3} = \{z\} \subseteq Z$. If $z \in \{\alpha_i, a_{i+1}\}$ then $z \in A_i$ and $z = a_{i+3}a_{i+2}a_{i+3} \in A_{i+2}$, which contradicts with (4).

If $z = a_{i+2}$ then $z \in A_{i+1}$ and $z = a_{i+3}a_i a_{i+3} = a_{i+3}a_{i+4}a_{i+3} \in A_{i+3}$, a contradiction.

Subcase 1b. $a_{i-1} \neq a_{i+3}$ and $[a_{i-1}]$ non $\|$ $[a_{i+3}]$. Then $n \geqslant 5$ and according to (vi), we have $a_{i-1}Za_{i-1} = a_{i+3}Za_{i+3} = \{z\} \subseteq Z$. Therefore $z = a_{i-1}a_i a_{i-1} = a_{i+3}a_{i+2}a_{i+3} \in A_{i-1} \cap A_{i+2}$, which contradicts (4).

Subcase 1c. $[a_{i-1}] \parallel [a_{i+3}]$. Suppose that $[a_{i+3}]$ non $\parallel [a_{i+4}]$, then $a_{i+4} \neq a_{i-1}$ and so $n \geqslant 6$. Therefore a_{i+1}, $a_{i+5} \notin Z$. It follows from (iii) that $[a_{i+4}] \parallel [a_{i+5}]$ and so, by (i) and (i) of Theorem S, we have $A_{i+4} \cap Z \neq \emptyset$. This implies that $A_{i+4} \cap A_i \neq \emptyset$ or $A_{i+4} \cap A_{i+1} \neq \emptyset$, which contradicts (4).

If $[a_{i+3}] \parallel [a_{i+4}]$, then it follows from (i) and (i) of Theorem S that $A_{i+3} \cap Z \neq \emptyset$ and so $A_{i+3} \cap A_i \neq \emptyset$ or $A_{i+3} \cap A_{i+1} \neq \emptyset$, a contradiction.

Case 2. There is an index $i \in I_n$ such that $\{a_i, a_{i+1}\} \subseteq Z$ and a_{i-1}, $a_{i+2} \notin Z$. If $[a_{i-1}]$ non $\parallel [a_{i+2}]$ then, by (vi), we have $a_{i-1}a_ia_{i-1} = a_{i+2}a_{i+1}a_{i+2} \in A_{i-1} \cap A_{i+1}$, which contradicts (4). We can assume that $[a_{i-1}] \parallel [a_{i+2}]$.

Subcase 2a. $a_{i+3} \in Z$. Then according to (iv), we have $a_{i+2}a_{i+1}a_{i+2} = a_{i+2}a_{i+3}a_{i+2} \in A_{i+1} \cap A_{i+2} \cap Z$. It follows from (5) that $a_{i+2} \in Z$, a contradiction.

Subcase 2b. $a_{i-2} \in Z$. Then we obtain a contradiction analogously.

Subcase 2c. a_{i-2}, $a_{i+3} \notin Z$.

If $[a_{i+2}]$non $\parallel [a_{i+3}]$, then according to (iii) we have $a_{i+4} \in Z$ or $[a_{i+4}] \parallel [a_{i+3}]$. This gives in both cases $A_{i+3} \cap Z \neq \emptyset$ and so $\operatorname{card} Z = 3$ because $A_i \cap A_{i+3} = \emptyset$. It follows from (vi) that $a_{i+2}a_{i+1}a_{i+2} = a_{i+3}za_{i+3}$ for $z \in A_{i+3} \cap Z$ and so $A_{i+1} \cap A_{i+3} \neq \emptyset$, which contradicts (4).

Analogously we can show that $[a_{i-2}]$ non $\parallel [a_{i-1}]$ gives a contradiction. Assume that $[a_{i-2}] \parallel [a_{i-1}]$ and $[a_{i+2}] \parallel [a_{i+3}]$. It follows from (i) and (i) of Theorem S that $A_{i-2} \cap Z \neq \emptyset \neq Z \cap A_{i+2}$. According to (4) we have $A_{i-2} \cap A_i = \emptyset = A_{i+2}$ and so $\operatorname{card} Z = 3$ and $A_{i-2} \cap A_{i+2} \neq \emptyset$. Then $n = 4$ or $n = 5$.

If $n = 5$, then $A_{i+2} \cap A_{i+3} \cap Z \neq \emptyset$ and so, by (5), we have $a_{i+3} \in Z$, a contradiction.

If $n = 4$, then $a_{i-1} = a_{i+3}$. According to (iv), (i) and (i) of Theorem S, we have $a_{i-1}a_ia_{i-1} = a_{i-1}(a_{i-1}a_{i-2})a_{i-1} \in A_{i-1} \cap A_{i-2} \cap Z$. Therefore by (5) we have $a_{i-1} \in Z$, a contradiction.

Case 3. There is an index $k \in I_n$ such that $a_k \in Z$ and if $a_i \in Z (i \in I_n)$, then a_{i-1}, $a_{i+1} \notin Z$.

We shall show that

$$\text{(6)} \qquad \text{if } a_i \in Z \text{ and } a_{i-1}, a_{i+1} \notin Z (i \in I_n), \text{ then } [a_{i-1}] \parallel [a_{i+1}].$$

On the contrary, suppose that $[a_{i-1}]$ non $\parallel [a_{i+1}]$. According to (vi), we have $a_{i-1}Za_{i-1} = \{z\} = a_{i+1}Za_{i+1}$ and so $z = a_{i-1}a_ia_{i-1} \in Z \cap A_{i-1}$. If $a_{i+2} \in Z$, then $z = a_{i+1}a_{i+2}a_{i+1} \in A_{i+1}$, which contradicts (4). If $a_{i+2} \notin Z$, then it follows from (iii) that $[a_{i+1}] \parallel [a_{i+2}]$. Hence, by (1) and (i) of Theorem S, we have $u = a_{i+1}a_{i+2} \in Z \cap A_{i+1}$ and so $z = a_{i+1}ua_{i+1} \in A_{i+1}$, a contradiction. Therefore (6) is satisfied.

Subcase 3a. There is an index $i \in I_n$ such that $a_i, a_{i+2} \in Z$. Evidently we have $a_{i-1}, a_{i+1}, a_{i+3} \notin Z$. It follows from (6) that $[a_{i-1}] \parallel [a_{i+1}] \parallel [a_{i+3}]$. If $Z \neq$

$\{a_i, a_{i+2}\}$, then, by (ii) and (iv), we have $a_{i-1}a_ia_{i-1} = a_{i+1}a_{i+2}a_{i+1} \in A_{i-1} \cap A_{i+1}$ which contradicts (4). Thus we obtain that $Z = \{a_i, a_{i+2}\}$.

Subcase 3aα. $[a_{i-1}] \parallel [a_{i+3}]$. Then there is an index $j \in I_n$ such that $[a_j] \parallel [a_{j+1}]$ and $a_j \notin \{a_{i-1}, a_i, a_{i+1}, a_{i+2}\}$. It follows from (i) and (i) of Theorem S that $A_j \cap Z \neq \emptyset$. If $a_i \in A_j$, then $j \in \{i-1, i\}$, a contradiction. If $a_{i+2} \in A_j$, then $j \in \{i+1, i+2\}$, a contradiction.

Subcase 3aβ. $[a_{i-1}]$ non $\parallel$ $[a_{i+3}]$. If $a_{i-1} \neq a_{i+3}$, then $n \geqslant 5$. By (vi) we have $a_{i-1}a_ia_{i-1} = a_{i+3}a_{i+2}a_{i+3} \in A_{i-1} \cap A_{i+2}$, which contradicts (4). We can suppose that $a_{i-1} = a_{i+3}$ and so $n = 4$.

If $a_{i-1}a_ia_{i-1} = a_{i-1}a_{i+2}a_{i-1}$, then $A_{i-1} \cap A_{i+2} \cap Z \neq \emptyset$ and so, by (5), we obtain that $a_{i-1} \in Z$, a contradiction.

If $a_{i+1}a_ia_{i+1} = a_{i+1}a_{i+2}a_{i+1}$, then $A_i \cap A_{i+1} \cap Z \neq \emptyset$ and so, by (5), we have $a_{i+1} \in Z$, a contradiction.

Therefore we have $\operatorname{card} Z = \operatorname{card} a_{i-1}Za_{i-1} = \operatorname{card} a_{i+1}Za_{i+1} = 2$ and so according to (v), we obtain that $a_{i-1} = a_{i+1}$, which is a contradiction.

Subcase 3b. There is an index $i \in I_n$ such that $a_i \in Z$ and so $a_{i-2}, a_{i-1}, a_{i+1}, a_{i+2} \notin Z$. First we shall prove that

$$(7) \qquad [a_{i-2}] \parallel [a_{i-1}] \parallel [a_{i+1}] \parallel [a_{i+2}].$$

On the contrary, suppose that $[a_{i+1}]$ non $\parallel$ $[a_{i+2}]$. It follows from (vi) that $a_{i+1}Za_{i+1} = \{z\} = a_{i+2}Za_{i+2}$ and $z \in A_i$. If $a_{i+3} \in Z$, then $a_{i+2}a_{i+3}a_{i+2} \in A_{i+2}$ and so $z \in A_i \cap A_{i+2}$, which contradicts (4). If $a_{i+3} \notin Z$, then by (iii) we have $[a_{i+2}] \parallel [a_{i+3}]$ and so, by (i) and (i) of Theorem S, we have $A_{i+2} \cap Z \neq \emptyset$. Choose $u \in A_{i+2} \cap Z$. Then we have $z = a_{i+2}ua_{i+2} \in A_i \cap A_{i+2}$, a contradiction. Therefore $[a_{i+1}] \parallel [a_{i+2}]$.

Analogously we can show that $[a_{i-2}] \parallel [a_{i-1}]$. Finally, $[a_{i-1}] \parallel [a_{i+1}]$ follows from (6).

According to (7), (i) and (i) of Theorem S, we have $e = a_{i-2}a_{i-1} \in A_{i-2} \cap Z$ and $f = a_{i+1}a_{i+2} \in A_{i+1} \cap Z$. It follows from (4) and (5) that $e \neq a_i \neq f$. If $e = f$, then $A_{i-2} \cap A_{i+1} \cap Z \neq \emptyset$ and so $n = 4$. By (5) we have $a_{i-2} = a_{i+2} \in Z$, a contradiction. If $e \neq f$, then $\operatorname{card} Z = 3$ (see (ii)). Hence according to (iv), we obtain that $a_{i+1}a_ia_{i+1} = a_{i+1}fa_{i+1} \in A_i \cap A_{i+1} \cap Z$ and so, by (5), we have $a_{i+1} \in Z$, a contradiction.

Subcase 4. $a_i \notin Z$ for each index $i \in I_n$.

Subcase 4a. There is an index $j \in I_n$ such that $[a_j]$ non $\parallel$ $[a_{j+1}]$. It follows from (iii) that $[a_{j-1}] \parallel [a_j]$ and $[a_{j+1}] \parallel [a_{j+2}]$. According to (i) and (i) of Theorem S, we have $e = a_ja_{j-1}a_j \in Z \cap A_{j-1}$ and $f = a_{j+1}a_{j+2}a_{j+1} \in Z \cap A_{j+1}$. From (4) it follows that $e \neq f$. By this yields $e = a_jea_j = a_{j+1}fa_{j+1} = f$, which is a contradiction.

Subcase 4b. $[a_i] \parallel [a_{i+1}]$ for each index $i \in I_n$. Put $e_i = a_i a_{i+1} a_i$. From (i) and (i) of Theorem S it follows that $e_i \in A_i \cap Z$. If $e_i = e_{i+1}$ for an index $i \in I_n$, then $A_i \cap A_{i+1} \cap Z \neq \emptyset$ and so, by (5), we have $a_{i+1} \in Z$, a contradiction. Consequently, we have $e_i \neq e_{i+1}$ for each index $i \in I_n$. By (4) we obtain that $e_i \neq e_{i+2}$ for each index $i \in I_n$. According to (ii), we get that $e_i = e_{i+3}$ and so $A_i \cap A_{i+3} \cap Z \neq \emptyset$. It follows from (4) that $n = 4$ and according to (5), we have $a_i = a_{i+4} \in Z$, which is a contradiction.

Therefore $G(S)$ is chordal.

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