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Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 1, 163–173

Persistent URL: http://dml.cz/dmlcz/127476

Czechoslovak Mathematical Journal, 49 (124) (1999), 163–173

SUBDIRECT PRODUCT DECOMPOSITIONS OF MV-ALGEBRAS

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(Received September 9, 1996)

Each MV-algebra $\mathcal{A}$ can be represented by means of an appropriate abelian lattice ordered group G with a strong unit u. (Cf. [4], [5], [7].)

We denote by $\operatorname{Con}\mathcal{A}$ and $\operatorname{Con}G$ the system of all congruence relations of $\mathcal{A}$ or of G, respectively. Both $\operatorname{Con}\mathcal{A}$ and $\operatorname{Con}G$ are partially ordered in the usual way. In the present paper it will be shown that there exists an isomorphism of $\operatorname{Con}\mathcal{A}$ onto $\operatorname{Con}G$.

This result will be applied for characterizing the relations between subdirect product decompositions of $\mathcal{A}$ and those of G.

To each direct product decomposition of G there corresponds a direct product decomposition of $\mathcal{A}$ (cf. [5]). Let us remark that each direct product decomposition of G has only a finite number of nonzero direct factors; on the other hand, $\mathcal{A}$ can have direct product decompositions with an infinite number of nonzero direct factors.

The mentioned result from [5] concerning direct product decompositions will be sharpened.

Some notions making possible to clasify subdirect product decompositions of lattice ordered groups are contained in [9]. We show that these notions can be adapted for the case of MV-algebras.

In [3], congruence relations on and subdirect product decompositions of MV-algebras have been applied in the context of Priestley duality. In [8], congruence relations on MV-algebras were dealt with by using the results of the theory of $DR\ell$-semigroups.

For the terminology and undefined notions concerning MV-algebras cf. [2], [4], [5].

Supported by grant GA SAV 95/5305/471

Let $\mathcal{A}$ and G be as in the introduction above. For $\varrho \in \operatorname{Con} G$ we denote by $\psi(\varrho)$ the equivalence on A (= the underlying set of $\mathcal{A}$) defined by $a_1\psi(\varrho)a_2$ iff $a_1\varrho a_2$. Since the operations of $\mathcal{A}$ are defined by means of the operations $+$, $-$, $\wedge$ and $\vee$ of G (cf., e.g., [5], Propos. 13) we infer

1.1. Lemma. *For each $\varrho \in \operatorname{Con} G$, $\psi(\varrho)$ belongs to* $\operatorname{Con}\mathcal{A}$.

Let $\varrho_1 \in \operatorname{Con}\mathcal{A}$. For $a \in A$ we denote $a(\varrho_1) = \{a' \in A \colon a\varrho_1 a'\}$. The convex ℓ-subgroup of G generated by the set $0(\varrho_1)$ will be denoted by X_0. Since G is abelian, X_0 is an ℓ-ideal of G; let ϱ' be the congruence relation on G whose kernel is X_0. For $g \in G$ let $g(\varrho')$ be the class in ϱ' containing g.

1.2. Lemma. *Let $0 < g \in G$. The following conditions are equivalent:*

(i) $g \in X_0$;

(ii) *there are elements $a_1, a_2, \ldots, a_n \in 0(\varrho_1)$ such that $g \leqslant a_1 + a_2 + \ldots + a_n$.*

The proof is simple, it will be omitted.

1.3. Lemma. *ϱ_1 is a congruence relation with respect to the operations $\vee$ and $\wedge$ on A. In particular, $a(\varrho_1)$ is a convex sublattice of A for each $a \in A$.*

P r o o f. This is a consequence of the fact that the operations $\vee$ and $\wedge$ on A are defined by means of the basic operations of $\mathcal{A}$ (cf., e.g. [5], Lemma 1.2). □

1.4. Lemma. $0(\varrho_1) = A \cap X_0$.

P r o o f. The relation $0(\varrho_1) \subseteq A \cap X_0$ is obvious. Let $g \in A \cap X_0$. Thus the condition (ii) from 1.2 is valid. This yields that there are elements $a_i' \in [0, a_i]$ $(i = 1, 2, \ldots, n)$ such that $g = a_1' + a_2' + \ldots + a_n'$. Then $g = a_1' \oplus a_2' \oplus \ldots \oplus a_n'$ and according to 1.3 we have $a_i' \in 0(\varrho_1)$ for $i = 1, 2, \ldots, n$. Therefore $g \in 0(\varrho_1)$. □

1.5. Lemma. *For each $a \in A$, $a(\varrho_1) = A \cap a(\varrho')$.*

P r o o f. a) Let $a_1 \in a(\varrho_1)$. Put $a_2 = a \wedge a_1, a_3 = a \vee a_1$. According to 1.3, both a_2 and a_3 belong to $a(\varrho_1)$. There is $t \in G$ such that $a_2 + t = a_3$. By a simple calculation we obtain (cf. also [6], Lemma 1.10)

$$t = \neg(a_2 \oplus \neg a_3).$$

Hence $t \in A$ and

$$t\varrho_1\neg(a_2 \oplus \neg a_2),$$

thus $t \in 0(\varrho_1)$. By applying 1.4 we infer that $a_1 \in a + X_0 = a(\varrho')$. Hence $a(\varrho_1) \subseteq A \cap a(\varrho')$.

b) Let $a_1 \in A \cap a(\varrho')$ and let a_2, a_3, t be as above. Then $a_2, a_3 \in a(\varrho')$, whence in view of $a_2 + t = a_3$ we obtain that $t \in X_0$. Moreover, $t \in A$. Thus 1.4 yields that $t \in 0(\varrho_1)$. There are $t_2, t_3 \in G$ such that $a_2 + t_2 = a$ and $a + t_3 = a_3$. Then $0 \leqslant t_2 \leqslant t$, $0 \leqslant t_3 \leqslant t$, hence $t_2, t_3 \in A$. According to 1.3, both t_2 and t_3 belong to $0(\varrho_1)$. Moreover, $a_2 \oplus t_2 = a$ and $a \oplus t_3 = a_3$. Thus $a_2 \varrho_1 a$ and $a \varrho_1 a_3$. By the convexity of $a(\varrho_1)$ we get $a_1 \in a(\varrho_1)$. □

1.6. Corollary. *$\psi(\varrho') = \varrho_1$ and ψ is an epimorphism.*

Under the notation as above we put $\varphi(\varrho_1) = \varrho'$ for each $\varrho_1 \in \operatorname{Con} \mathcal{A}$.

1.7. Lemma. *Let $\varrho \in \operatorname{Con} G$ and let X_0 be the ℓ-ideal of G generated by the set $0(\varrho) \cap A$. Then $X_0 = 0(\varrho)$.*

P r o o f. The relation $0(\varrho) \cap A \subseteq 0(\varrho)$ yields that $X_0 \subseteq 0(\varrho)$. Let $g \in 0(\varrho)$. There exists a positive integer n such that $|g| \leqslant nu$. Hence there are $a_1, a_2, \ldots, a_n$ in G such that $0 \leqslant a_i \leqslant u$ for $i = 1, 2, \ldots, n$ and $|g| = a_1 + a_2 + \ldots + a_n$. Thus all a_i belong to $0(\varrho) \cap A$ and hence $|g| \in X_0$. Therefore $g \in X_0$ and so $0(\varrho) \subseteq X_0$. □

1.8. Lemma. *Let $\varrho \in \operatorname{Con} G$ and put $\psi(\varrho) = \varrho_1$. Let ϱ' be as above. Then $\varrho' = \varrho$.*

P r o o f. This is a consequence of 1.7 and of the fact that each congruence relation on G is determined by the corresponding kernel. □

Now, 1.6 and 1.8 yield

1.9. Lemma. *φ is an epimorphism and $\psi = \varphi^{-1}$.*

1.10. Theorem. *φ is an isomorphism of the lattice $\operatorname{Conv} \mathcal{A}$ onto the lattice $\operatorname{Con} G$.*

P r o o f. It is obvious that both the mappings φ and ψ are monotone. Hence the assertion follows from 1.9. □

1.11. Proposition. *Let $\varrho_1, \varrho_2 \in \operatorname{Con} \mathcal{A}$, $a \in A$, $a(\varrho_1) = a(\varrho_2)$. Then $\varrho_1 = \varrho_2$.*

P r o o f. By way of contradiction, suppose that $\varrho_1 \neq \varrho_2$. There are $\varrho^1, \varrho^2 \in \operatorname{Con} G$ such that $\psi(\varrho^i) = \varrho_i$ for $i = 1, 2$. In view of 1.10, $\varrho^1 \neq \varrho^2$. Next, according to 1.7, $0(\varrho_1) \neq 0(\varrho_2)$. Thus without loss of generality we can suppose that there is $a_1 \in 0(\varrho_1) \setminus 0(\varrho_2)$.

Put $a_1 \vee a = a_2$, $a_1 \wedge a = a_3$. Then $0 \leqslant a_1 - a_3 \leqslant a_1$, whence $a_1 - a_3 \in 0(\varrho_1)$. We have

$$a_2 - a = a_1 - a_3,$$

hence $a_2 - a \in 0(\varrho_1)$ and thus $a_2 - a \in 0(\varrho^1)$ yielding $a_2 \varrho^1 a$. Therefore $a_2 \varrho_1 a$ and so, by the assumption, $a_2 \varrho_2 a$. Thus $(a_2 - a)\varrho^2 0$.

If $a_3 \in 0(\varrho_2)$, then

$$a_1 = a_3 + (a_1 - a_3) = (a_3 + (a_2 - a))\varrho^2 0,$$

whence $a_1 \varrho_2 0$, which is a contradiction. Hence a_3 does not belong to $0(\varrho_2)$.

Clearly $a_3 \in 0(\varrho_1)$ and $0 < a_3 \leqslant a$. We have

$$a - a_3 \in A, \quad (a - a_3)\varrho^1 a,$$

thus $(a - a_3)\varrho_1 a$. Since $a(\varrho_1) = a(\varrho_2)$ we get $(a - a_3)\varrho_2 a$. Hence $(a - a_3)\varrho^2 a$ giving $-a_3 \varrho^2 0$ and thus $a_3 \varrho^2 0$. Therefore $a_3 \varrho_2 0$, which is a contradiction. □

1.12. Proposition. *Let $\varrho_1, \varrho_2 \in \operatorname{Con} \mathcal{A}$. Then ϱ_1 and ϱ_2 are permutable.*

P r o o f. Let ϱ^1 and ϱ^2 be as in the proof of 1.11. It is well-known that ϱ^1 and ϱ^2 are permutable. Let $a_1, a_2, a_3 \in A$ and suppose that $a_1 \varrho_1 a_2 \varrho_2 a_3$. Hence $a_1 \varrho^1 a_2 \varrho^2 a_3$. Thus there is $g \in G$ with $a_1 \varrho^2 g \varrho^1 a_3$. This yields that

$$a_1 = (a_1 \wedge u)\varrho^2 (g \wedge u)\varrho^1 (a_2 \wedge u) = a_2.$$

Since $g \wedge u \in A$ we obtain

$$a_1 \varrho_2 (g \wedge u) \varrho_1 a_2.$$

□

2. Subdirect product decompositions

For fixing the notation concerning subdirect product decompositions we recall some basic facts.

Let $\mathfrak{A}$ and $\mathfrak{A}_i$ $(i \in I)$ be algebras of the same type. If

$$\varphi_1 \colon \mathfrak{A} \longrightarrow \prod_{i \in I} \mathfrak{A}_i$$

is an isomorphism of $\mathfrak{A}$ into the direct product of algebras $\mathfrak{A}_i$ such that, for each $i \in I$ and each $a^i \in \mathfrak{A}_i$ there is $a \in \mathfrak{A}$ with $(\varphi_1(a))_i = a^i$, then φ_1 is said to be a subdirect product decomposition of $\mathfrak{A}$.

In such a case we define, for each $i \in I$, a binary relation $\varrho_i(\varphi_1)$ on $\mathfrak{A}$ as follows: for a and a' in $\mathfrak{A}$ we put $a\varrho_i(\varphi_1)a'$ if

$$(\varphi_1(a))_i = (\varphi_1(a'))_i.$$

We obtain a set $\{\varrho_i(\varphi_1)\}_{i\in I}$ of congruence relations on $\mathfrak{A}$ which will be denoted by $\chi(\varphi_1)$. Obviously, $\bigwedge_{i\in I} \varrho_i(\varphi_1) = Id$, where Id is the identity relation on $\mathfrak{A}$.

If φ_1 and φ_2 are subdirect product decompositions of $\mathfrak{A}$ such that $\chi(\varphi_1) = \chi(\varphi_2)$, then φ_1 is said to be equivalent with φ_2.

We say that a subdirect product decomposition

$$\sigma\colon \mathfrak{A} \longrightarrow \prod_{i\in I} \mathfrak{A}'_i$$

is determined by a system $\{\varrho^i\}_{i\in I}$ of congruence relations on $\mathfrak{A}$ if the following conditions are satisfied:

(i) $\bigwedge_{i\in I} \varrho^i$ is the identity relation on $\mathfrak{A}$;

(ii) $\mathfrak{A}'_i = \mathfrak{A}/\varrho^i$ for each $i \in I$;

(iii) for each $a \in A$ and each $i \in I$, $\sigma(a)_i = a(\varrho^i)$.

In view of the well-known Birkhoff's theorem (cf., e.g., [1], Chap. VI) each system $\{\varrho^i\}_{i\in I} \subseteq \operatorname{Con}\mathfrak{A}$ satisfying the condition (i) determines a subdirect product decomposition of $\mathfrak{A}$, and each subdirect product decomposition φ_1 of $\mathfrak{A}$ is equivalent to some subdirect product decomposition σ of $\mathfrak{A}$ which is determined by a system of congruence relations on $\mathfrak{A}$.

We denote by $S(\mathfrak{A})$ the set of all subdirect product decompositions σ of $\mathfrak{A}$ such that σ is determined by a system of congruence relations of $\mathfrak{A}$.

As above, let $\varrho \in \operatorname{Con} G$ and $\varrho_1 \in \operatorname{Con}\mathcal{A}$. Consider the corresponding factor structures, i.e., the lattice ordered group G/ϱ, and the MV-algebra $\mathcal{A}/\varrho_1$. It is easy to verify that $u(\varrho)$ is a strong unit of G/ϱ, hence we can construct the MV-algebra $\mathcal{A}_\varrho = \mathcal{A}_0(G/\varrho, u(\varrho))$.

Suppose that $\varrho_1 = \psi(\varrho)$. We define a mapping $\psi_\varrho\colon \mathcal{A}_\varrho \longrightarrow \mathcal{A}/\varrho_1$ as follows. For each $g(\varrho) \in \mathcal{A}_\varrho$ we put

$$\psi_\varrho(g(\varrho)) = g(\varrho) \cap A.$$

Then we obviously have

2.1. Lemma. *ψ_ϱ is a one-to-one mapping of $\mathcal{A}_\varrho$ onto $\mathcal{A}/\varrho_1$.*

2.2. Lemma. *ψ_ϱ is a homomorphism with respect to the operations $\wedge$ and $\vee$.*

P r o o f. Let $g_1(\varrho)$ and $g_2(\varrho)$ be elements of $\mathcal{A}_\varrho$. We have

$$g_1(\varrho) \wedge g_2(\varrho) = (g_1 \wedge g_2)(\varrho).$$

There exist $g'_1 \in g_1(\varrho) \cap A$ and $g'_2 \in g_2(\varrho) \cap A$. Then

$$\begin{aligned}(g_1(\varrho) \cap A) \wedge (g_2(\varrho) \cap A) &= (g'_1(\varrho_1) \wedge g'_2(\varrho_1)) \\ &= (g'_1 \wedge g'_2)(\varrho_1) = (g_1 \wedge g_2)(\varrho) \cap A.\end{aligned}$$

Hence ψ_ϱ is a homomorphism with respect to the operation $\wedge$. The case of the operation $\vee$ is analogous. □

2.3. Lemma. *ψ_ϱ is a homomorphism with respect to the operations $\oplus$ and $\neg$.*

P r o o f. Let $g_1(\varrho), g_2(\varrho), g'_1$ and g'_2 be as in the proof of 2.2. Then

$$\begin{aligned}g_1(\varrho) \oplus g_2(\varrho) &= (g_1(\varrho) + g_2(\varrho)) \wedge u(\varrho) = (g'_1(\varrho) + g'_2(\varrho)) \wedge u(\varrho) \\ &= ((g'_1 + g'_2) \wedge u)(\varrho); \\ (g_1(\varrho) \cap A) \oplus (g_2(\varrho) \cap A) &= (g'_1(\varrho) \cap A) \oplus (g'_2(\varrho) \cap A) \\ &= g'_1(\varrho_1) \oplus g'_2(\varrho_1) = (g'_1(\varrho_1) + g'_2(\varrho_1)) \wedge u(\varrho_1) \\ &= ((g'_1 + g'_2) \wedge u)(\varrho_1) = ((g'_1 + g'_2) \wedge u)(\varrho) \cap A,\end{aligned}$$

which proves the assertion concerning the operation $\oplus$. Next we have

$$\begin{aligned}\neg g_1(\varrho) &= \neg g'_1(\varrho) = u(\varrho) - g'_1(\varrho) = (u - g'_1)(\varrho), \\ \neg(g_1(\varrho) \cap A) &= \neg g'_1(\varrho_1) = u(\varrho_1) - g'_1(\varrho_1) = (u - g'_1)(\varrho_1) = (u - g'_1)(\varrho) \cap A,\end{aligned}$$

which completes the proof. □

2.4. Proposition. *Let $\varrho \in \operatorname{Con} G$ and $\varrho_1 = \psi(\varrho)$. Then ψ_ϱ is an isomorphism of $\mathcal{A}_\varrho$ onto $\mathcal{A}/\varrho_1$.*

P r o o f. This is a consequence of 2.1, 2.2 and 2.3. □

2.5. Theorem. *Let G be a lattice ordered group with a strong unit u and let $\mathcal{A} = \mathcal{A}_0(G, u)$.*

If σ is a subdirect product decomposition of G which is determined by a system $\{\varrho^i\}_{i\in I} \subseteq \operatorname{Con} G$, then

(i) *there exists a subdirect product decomposition $\sigma_1 = \psi^*(\sigma)$ of $\mathcal{A}$ which is determined by the system $\{\psi(\varrho^i)\}_{i\in I}$;*

(ii) *for each $i \in I$, the factor algebra $\mathcal{A}/\psi(\varrho^i)$ is isomorphic to the MV-algebra $\mathcal{A}_0(G/\varrho^i, u(\varrho^i))$.*

If $\sigma'_1 \in S(\mathcal{A})$, then there exists $\sigma' \in S(G)$ such that $\psi^*(\sigma') = \sigma'_1$.

P r o o f. This follows from 1.10 and 2.4. □

3. On some types of subdirect product decompositions

In this section we deal with certain conditions concerning subdirect product decompositions of lattice ordered groups which have been introduced in [9], and we investigate analogous conditions for MV-algebras.

Let $\mathcal{A}$ and G be as above. A subdirect product decomposition

$$\varphi_1\colon\ G \longrightarrow \prod_{i\in I} G_i$$

of G is said to be completely subdirect (cf. [9]) if for each $i \in I$ and each $g^i \in G_i$ there exists $g \in G$ such that

(i) $\varphi_1(g)_i = g^i$,

(ii) $\varphi_1(g)_{i(1)} = 0$ for each $i(1) \in I \setminus \{i\}$.

By analogous conditions we define a completely subdirect product decomposition for MV-algebras.

It is obvious that if φ_1 is a completely subdirect product decomposition and if I is finite, then φ_1 is a direct product decomposition. A similar result is valid for MV-algebras. Next, each direct product decomposition (of G or of $\mathcal{A}$) is a completely subdirect product decomposition.

In view of the results of Section 2 we can suppose, without loss of generality, that the subdirect product decompositions φ_1 and φ'_1 belong to $S(G)$ or to $S(\mathcal{A})$, respectively.

Thus, for $g \in G$, φ_1 is the mapping (under the notation as above)

$$\varphi_1(g) = (g(\varrho_i))_{i\in I} \quad \text{for each} \quad g \in G;$$

similarly, φ'_1 is the mapping

$$\varphi'_1(a) = (a(\varrho^i))_{i\in I} \quad \text{for each} \quad a \in A.$$

In view of 2.4 we have also a subdirect product decomposition

$$\varphi''_1\colon\ \mathcal{A} \longrightarrow \prod_{i\in I} \mathcal{A}_{\varrho_i},$$

where

$$\varphi_1''(a) = (a(\varrho_i))_{i\in I} \quad \text{for each} \quad a \in A.$$

It is obvious that φ_1'' does not essentially differ from φ_1'. Clearly $\prod_{i\in I} \mathcal{A}_{\varrho_i} \subset \prod_{i\in I} G_i$. If $i \in I$ and $g^i \in G_i$, then g^i will be identified with the element $g \in G$ such that $\varphi_1(g)_i = g^i$ and $\varphi_1(g)_j = 0$ for each $j \in I \setminus \{i\}$.

3.1. Lemma. *Let $\varphi_1 \in S(G)$. Then φ_1 is a completely subdirect product decomposition if and only if φ_1'' is a completely subdirect product decomposition.*

Proof. a) Assume that φ_1 is a completely subdirect product decomposition of G. Let $i \in I$ and $a^i \in \mathcal{A}_{\varrho_i}$. Hence $a^i \in G_i$. Thus there exists $g \in G$ such that $\varphi_1(g)_i = a^i$ and $\varphi_1(g)_{i(1)} = 0$ whenever $i(1) \in I \setminus \{i\}$. This yields that $g \leqslant u$, hence $g \in A$; moreover, $\varphi_1'(g)_i = a^i$ and $\varphi_1'(g)_{i(1)} = 0$ for each $i(1) \in I \setminus \{i\}$.

b) Let φ_1'' be a completely subdirect product decomposition of $\mathcal{A}$. Let $i \in I$ and $g^i \in G_i$. Put $g^0 = g^i \vee 0$, $u_i = (\varphi_1''(u))_i$. We have $u_i = (\varphi_1(u))_i$, hence u_i is a strong unit of G_i. Thus there is a positive integer n such that $g^0 \leqslant nu_i$. This yields that there are elements $x_1, \dots, x_n$ in G_i with $g^0 = x_1 + x_2 + \dots + x_n$, $0 \leqslant x_j \leqslant u_i$ for $j = 1, 2, \dots, n$. Therefore all x_j belong to $\mathcal{A}_{\varrho_i}$. Thus there are $a_j \in A$ such that $\varphi_1''(a_j)_i = x_j$ and $\varphi_1''(a_j)_{i(1)} = 0$ whenever $i(1) \in I \setminus \{i\}$. In both these relations φ_1'' can be replaced by φ_1.

Put $a_1 + a_2 + \dots + a_n = g$. Then $\varphi_1(g)_i = g^0$ and $(\varphi_1(g))_{i(1)} = 0$ for each $i(1) \in I \setminus \{i\}$.

Analogously we can verify that there exists $g' \in G$ such that $\varphi_1(g')_i = -(g^i \wedge 0)$ and $\varphi_1(g')_{i(1)} = 0$ for each $i(1) \in I \setminus \{i\}$. Put $g'' = g - g'$. Then $(\varphi_1(g''))_i = g^i$ and $\varphi_1(g'')_{i(1)} = 0$ for each $i(1) \in I \setminus \{i\}$.

Therefore φ_1 is a completely subdirect product decomposition. □

The previous lemma immediately yields:

3.2. Proposition. *Let $\varphi_1 \in S(G)$ and let φ_1' be the corresponding element of $S(\mathcal{A})$. Then the following conditions are equivalent.*

(i) *φ_1 is a completely subdirect product decomposition.*
(ii) *φ_1' is a completely subdirect product decomposition.*

3.3. Corollary. *Let φ_1 and φ_1' be as in 3.2. Assume that I is finite. Then the following conditions are equivalent:*

(i) *φ_1 is a direct product decomposition of G.*
(ii) *φ_1' is a direct product decomposition of $\mathcal{A}$.*

P r o o f. Let (ii) be valid. Hence φ_1' is a completely subdirect product decomposition of $\mathcal{A}$. In view of 3.1, φ_1 is a completely subdirect product decomposition of G. Hence, because I is finite, φ_1 is a direct product decomposition of G. The proof of the implication (i)$\Rightarrow$(ii) is analogous. □

Let us remark that the implication (i)$\Rightarrow$(ii) can be obtained also as a consequence of results of [5].

Again, let us consider the subdirect product decomposition φ_1 and let $i \in I$. The element i will be said to be of type α if there exists $g^i \in G_i$ and $g \in G$ such that

$$g^i \neq 0,\ \varphi_1(g)_i = g^i,\ \varphi(g)_{i(1)} = 0 \quad \text{for each} \quad i(1) \in I \setminus \{i\}.$$

If all elements $i \in I$ are of type α, then φ_1 is called an α-subdirect product decomposition. If $i \in I$ and if it is not of type α, then it is said to be of type β; if all $i \in I$ are of type β, then φ_1 is called a β-subdirect product decomposition.

These notions have been introduced and studied in [9] for the particular case when all G_i were assumed to be linearly ordered.

If φ_1' is as above, then in the same way we can define the indices of type α or β with respect to φ_1'; similarly as in the case of φ_1 we say that φ_1' is an α- or β-subdirect product decompositions if all $i \in I$ are of type α or of type β, respectively.

3.4. Proposition. *Let φ_1 and φ_1' be as in* 3.1. *Let $i \in I$. Then the following conditions are equivalent:*

(a) *i is of type α with respect to φ_1;*
(b) *i is of type α with respect to φ_1'.*

P r o o f. Analogously as in 3.1 we can consider φ_1'' instead of φ_1'; in this case it suffices to apply similar steps as in the proof of 3.1. □

3.5. Corollary. *Let φ_1 and φ_1' be as in* 3.1. *Then the following conditions are equivalent:*

(a) *φ_1 is of type α;*
(b) *φ_1' is of type α.*

Also, type α in (a) and (b) can be replaced by type β.

The subdirect product decomposition φ_1 of G is called reduced if, whenever $i(1)$ and $i(2)$ are distinct elements of I, then there exists $g \in G$ such that $\varphi_1(g)_{i(1)} < 0$, $0 < \varphi_1(g)_{i(2)}$. (Cf. [9].)

3.6. Lemma. *Let φ_1 be a subdirect product decomposition of G. Then the following conditions are equivalent:*

(i) *φ_1 is reduced.*

(ii) *Whenever* $i(1)$ *and* $i(2)$ *are distinct elements of* G*, then there are* $g_1, g_2 \in G$ *such that*

$$\varphi_1(g_1)_{i(1)} > 0, \quad \varphi_1(g_1)_{i(2)} = 0, \quad \varphi_1(g_2)_{i(2)} > 0, \quad \varphi_1(g_2)_{i(1)} = 0.$$

P r o o f. Let $i(1)$ and $i(2)$ be distinct elements of I. Assume that φ_1 is reduced and let g be as above. Put $g_1 = g \vee 0$ and $g_2 = -(g \wedge 0)$. Then the conditions from (ii) are satisfied for these g_1 and g_2.

Conversely, suppose that (ii) holds. Let g_1 and g_2 be as in (ii); we put $g = g_2 - g_1$. Then $\varphi_1(g)_{i(1)} < 0$ and $0 < \varphi_1(g)_{i(2)}$. □

Now let φ_2 be a subdirect product decomposition of $\mathcal{A}$. If φ_2 satisfies the condition (ii) from 3.6, then it is said to be reduced.

3.7. Proposition. *Let* φ_1 *and* φ_1' *be as above. Then* φ_1 *is reduced if and only if* φ_1' *is reduced.*

P r o o f. It suffices to prove the assertion for the case when φ_1' is replaced by φ_1''. Suppose that φ_1 is reduced. Hence the condition (ii) from 3.6 is satisfied; consider the corresponding elements g_1 and g_2. Since u is a strong unit in G there are a positive integer n and elements $a_1, a_2, \ldots, a_n$ in A such that $g_1 = a_1 + a_2 + \ldots + a_n$. Without loss of generality we can suppose that $\varphi_1(a_1)_{i(1)} > 0$. We have $\varphi_1''(a_1)_{i(1)} = \varphi_1(a_1)_{i(1)}$. Clearly $\varphi_1''(a_1)_{i(2)} = \varphi_1(a_1)_{i(2)} = 0$. Similarly we can verify that there is $a_1' \in A$ such that $\varphi_1''(a_1')_{i(2)} > 0$ and $\varphi_1''(a_1')_{i(1)} = 0$. Thus φ_1'' is reduced.

Conversely, suppose that φ_1'' is reduced. Hence there are $a_1, a_2 \in A$ satisfying analogous conditions as in 3.6 (ii) with φ_1 replaced by φ_1''. Now it suffices to put $g_1 = a_1$, $g_2 = a_2$. □

In [2] it has been proved that every MV-algebra can be expressed subdirectly by means of linearly ordered MV-algebras. The following proposition contains a stronger result.

3.8. Proposition. *Let* $\mathcal{A}$ *be an* MV*-algebra,* $A \neq \{0\}$*. Then* $\mathcal{A}$ *possesses a reduced subdirect product decomposition all subdirect factors of which are linearly ordered.*

P r o o f. Let G be as above. Then $G \neq \{0\}$. It is well-known that each abelian lattice ordered group has a subdirect product decomposition all subdirect factors of which are linearly ordered. Hence according to [9] there exists a subdirect product decomposition φ_1 of G such that φ_1 is reduced and (under the notation as above) all G_i are linearly ordered. Let φ_1' be as in 3.1. According to 3.7, φ_1' is reduced. In view of 2.5, all subdirect factors in φ_1' are linearly ordered. □

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