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ON L-FUZZY IDEALS IN SEMIRINGS II

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Abstract. We study some properties of L-fuzzy left (right) ideals of a semiring R related to level left (right) ideals.

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1. Preliminaries

Fuzzy ideals in semigroups were introduced in [9] and studied further by several authors [6, 8]. Wang-Jing Liu [7] introduced the notion of fuzzy ideals of a ring. Recently Y. B. Jun, J. Neggers and H. S. Kim [2, 3] have also studied fuzzy ideals in semirings. The theory of semirings has been studied by many authors [1, 5]. This paper is a continuation of [2].

By a *semiring* we will mean a set R endowed with two associative binary operations called *addition* and *multiplication* (denoted by $+$ and $\cdot$, respectively) satisfying the following conditions:

(i) addition is a commutative operation,
(ii) there exists $0 \in R$ such that $x + 0 = x$ and $x0 = 0x = 0$ for each $x \in R$, and
(iii) multiplication distributes over addition both from the left and from the right.

From now on we write R and S for semirings. A subset A of R is a *left* (*right*) *ideal* if $x, y \in A$ and $r \in R$ imply that $x + y \in A$ and $rx \in A$ ($xr \in A$). If A is both a left and a right ideal of R, we say that A is a two sided ideal, or simply, *ideal* of R. A mapping $f\colon R \to S$ is called a *homomorphism* if $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in R$.

We note that if $f\colon R \to S$ is an onto homomorphism and if A is a left (right) ideal of R, then $f(A)$ is a left (right) ideal of S.

Throughout this paper $L = (L, \leqslant, \wedge, \vee)$ will be a completely distributive lattice, which has least and greatest elements, say $\mathbf{0}$ and $\mathbf{1}$, respectively. Let X be a non-empty (usual) set. An *L-fuzzy set* in X is a map $\mu\colon X \to L$, and $\mathcal{F}(X)$ will denote the set of all L-fuzzy sets in X. If $\mu, \nu \in \mathcal{F}(X)$, then $\mu \subseteq \nu$ if and only if $\mu(x) \leqslant \nu(x)$ for all $x \in X$, and $\mu \subset \nu$ if and only if $\mu \subseteq \nu$ and $\mu \neq \nu$. It is easily seen that $\mathcal{F}(X) = (\mathcal{F}(X), \subseteq, \wedge, \vee)$ is a completely distributive lattice, which has least and greatest elements, say $\mathbf{0}$ and $\mathbf{1}$, respectively, in a natural manner, where $\mathbf{0}(x) = 0$, $\mathbf{1}(x) = 1$ for all $x \in X$.

Definition 1.1. Let $\mu \in \mathcal{F}(R)$. Then μ is called an *L-fuzzy left* (*right*) *ideal* of R if for all $x, y \in R$,

(i) μ is an L-fuzzy subsemigroup of $(R, +)$; i.e., $\mu(x+y) \geqslant \min\{\mu(x), \mu(y)\}$,

(ii) $\mu(xy) \geqslant \mu(y)$ $(\mu(xy) \geqslant \mu(x))$.

An L-fuzzy set μ is called an *L-fuzzy ideal* of R if and only if it is both an L-fuzzy left and right ideal of R. It follows from Definition 1.1-(ii) that if μ is an L-fuzzy left (right) ideal of R, then $\mu(0) \geqslant \mu(x)$ for all $x \in X$. As the idea of a semiring is a generalization of the idea of a ring, the notion of L-fuzzy left (right) ideal of a semiring is also a generalization of the notion of L-fuzzy left (right) ideal in rings. Hence, every L-fuzzy left (right) ideal of a ring is L-fuzzy left (right) ideal of a semiring. But the converse need not be true at all.

Proposition 1.2. *Let $\mu \in \mathcal{F}(R)$. Then μ is an L-fuzzy left (right) ideal of R if and only if, for any $t \in L$ such that $\mu_t \neq \emptyset$, μ_t is a left (right) ideal of R, where $\mu_t = \{x \in R \mid \mu(x) \geqslant t\}$, which is called a level subset of μ.*

If μ is an L-fuzzy left (right) ideal of R, we call μ_t $(\neq \emptyset)$ a *level left* (*right*) *ideal* of μ.

Corollary 1.3 ([2]). *If $\mu \in \mathcal{F}(R)$ is an L-fuzzy left (right) ideal of R, then the set*

$$R_\mu = \{x \in R \mid \mu(x) = \mu(0)\}$$

is a left (right) ideal of R.

Theorem 1.4 ([2]). *Let A be any left (right) ideal of R. Then there exists an L-fuzzy left (right) ideal μ of R such that $\mu_t = A$, where $t \in L$.*

Theorem 1.5 ([2]). *Let $\mu \in \mathcal{F}(R)$ be an L-fuzzy left (right) ideal of R. Then two level left (right) ideals μ_s, μ_t (with $s < t$ in L) of μ are equal if and only if there is no $x \in R$ such that $s \leqslant \mu(x) < t$.*

For any $\mu \in \mathcal{F}(R)$ we denote by $\mathrm{Im}(\mu)$ the image set of μ, i.e., $\mathrm{Im}(\mu) = \mu(R)$.

Theorem 1.6 ([2]). *Let $\mu \in \mathcal{F}(R)$ be an L-fuzzy left (right) ideal of R. If $\mathrm{Im}(\mu) = \{t_1, t_2, \ldots, t_n\}$, where $t_1 < t_2 < \ldots < t_n$, then the family of left (right) ideals μ_{t_i} $(i = 1, \ldots, n)$ constitutes the collection of all level left (right) ideals of μ.*

2. Level subsets

We next discuss some properties of L-fuzzy ideals in semirings R. Using the above results we obtain the following

Lemma 2.1. *Let R be a semiring and μ an L-fuzzy left (right) ideal of R. If $\mathrm{Im}(\mu)$ is finite, say $\{t_1, \ldots, t_n\}$, then for any $t_i, t_j \in \mathrm{Im}(\mu)$, $\mu_{t_i} = \mu_{t_j}$ implies $t_i = t_j$.*

Proof. Assume that $t_i \neq t_j$, say $t_i < t_j$. Since $t_i \in \mathrm{Im}(\mu)$, there is an $x \in R$ such that $\mu(x) = t_i$ and hence $t_i \leqslant \mu(x) < t_j$. By virtue of Theorem 1.5, $\mu_{t_i} = \mu_{t_j}$ does not hold. This contradicts our assumption. □

Theorem 2.2. *Let μ and ν be two L-fuzzy left (right) ideals of a semiring R with a single family of level left (right) ideals. If $\mathrm{Im}(\mu) = \{t_0, t_1 \ldots, t_r\}$ and $\mathrm{Im}(\nu) = \{s_0, s_1, \ldots, s_k\}$ where $t_0 > t_1 > \ldots > t_r$ and $s_0 > s_1 > \ldots > s_k$, then*
(a) *$r = k$,*
(b) *$\mu_{t_i} = \nu_{s_i}$ $(0 \leqslant i \leqslant k)$,*
(c) *if $x \in R$ such that $\mu(x) = t_i$ then $\nu(x) = s_i$ $(0 \leqslant i \leqslant k)$.*

Proof. By Theorem 1.6 the only level subalgebras of μ and ν are the two families μ_{t_i} and ν_{s_i}. Since μ and ν have the same family of level left (right) ideals, it follows that $r = k$. Using Theorem 1.5 we have two chains of level left (right) ideals

$$\mu_{t_0} \subset \mu_{t_1} \subset \ldots \subset \mu_{t_k} = R$$

and

$$\nu_{s_0} \subset \nu_{s_1} \subset \ldots \subset \nu_{s_k} = R.$$

It follows that if $t_i, t_j \in \mathrm{Im}(\mu)$ with $t_i > t_j$ then

$$(*1) \qquad \mu_{t_i} \subset \mu_{t_j},$$

and if $s_i, s_j \in \mathrm{Im}(\nu)$ with $s_i > s_j$ then

$$(*2) \qquad \nu_{s_i} \subset \nu_{s_j}.$$

Since the two families of level left (right) ideals are identical, it is clear that $\mu_{t_0} = \nu_{s_0}$. By hypothesis $\mu_{t_1} = \nu_{t_j}$ since $j > 0$. Assume that $\mu_{t_1} \neq \nu_{t_1}$. Then $\mu_{t_1} = \nu_{s_j}$ for some $j > 1$, and $\nu_{s_1} = \mu_{t_i}$ for some $t_i < t_1$. Thus by $(*1)$ and $(*2)$ we obtain

$$\nu_{s_j} = \mu_{t_1} \subset \mu_{t_i} \quad \text{and} \quad \mu_{t_i} = \nu_{s_1} \subset \nu_{s_j},$$

which leads to a contradiction. Hence $\mu_{t_1} = \nu_{s_1}$. By induction on i, we obtain that $\mu_{t_i} = \nu_{s_i}$ $(0 \leqslant i \leqslant k)$. Let $x \in R$ with $\mu(x) = t_i$ and $\nu(x) = s_j$ for some $i, j \in \{0, 1, \ldots, k\}$. We claim that $s_i = s_j$. If $\mu(x) = t_i$, then by (b) $x \in \mu_{t_i} = \nu_{s_i}$. It follows that $s_j = \nu(x) > s_i$ and from $(*2)$ we have $\nu_{s_j} \subseteq \nu_{s_i}$. Now, $\nu(x) = s_j$ implies $x \in \nu_{s_j} = \mu_{t_j}$ by (b). It follows from $(*1)$ that $t_j \leqslant \mu(x) = t_i$ and from $(*1)$ that $\mu_{t_1} \subseteq \mu_{t_j}$. Hence we have $\nu_{s_i} = \mu_{t_i} \subseteq \mu_{t_j} = \nu_{s_j}$. Thus $\nu_{s_i} = \nu_{s_j}$, and by Lemma 2.1 we have $s_i = s_j$. This completes the proof. □

Theorem 2.3. *Let μ and ν be two L-fuzzy left (right) ideals of a semiring R having the same family of level left (right) ideals. Then $\mu = \nu$ if and only if* $\mathrm{Im}(\mu) = \mathrm{Im}(\nu)$.

Proof. Assume that $\mathrm{Im}(\mu) = \mathrm{Im}(\nu)$. For convenience, let $\mathrm{Im}(\mu) = \{t_0, t_1, \ldots, t_r\}$ and $\mathrm{Im}(\nu) = \{s_0, s_1, \ldots, s_r\}$, where $t_0 > t_1 > \ldots > t_r$ and $s_0 > s_1 > \ldots > s_r$. Then $s_0 \in \mathrm{Im}(\nu) = \mathrm{Im}(\mu)$ and thus $s_0 = t_{k_0}$ for some k_0. Assume that $t_{k_0} \neq t_0$. So $t_{k_0} < t_0$. Now, $s_1 \in \mathrm{Im}(\nu) = \mathrm{Im}(\mu)$, and hence $s_1 = t_{k_1}$ for some k_1. Since $s_0 > s_1$, we have $t_{k_0} > t_{k_1}$. Continuing in this way, we have $t_0 > t_{k_0} > t_{k_1} > \ldots > t_{k_r}$. This means that $|\mathrm{Im}(\mu)| = r + 1$. Hence we have $s_0 = t_0$. Proceeding in this manner, we obtain that $s_i = t_i$, $(0 \leqslant i \leqslant r)$. Let $x \in X$ with $\mu(x) = t_i$ for some $i \in \{0, 1, \ldots, r\}$. Then by Theorem 2.2, $\nu(x) = s_i$ $(0 \leqslant i \leqslant r)$. Since $s_i = t_i$, it follows that $\mu(x) = \nu(x)$ for each $x \in X$. Hence $\mu = \nu$. This proves the theorem. □

3. Semiring order

Suppose that R is a semiring. Define a relation $<$ on R as follows:

$$x < y \quad \text{provided} \quad x + y = y \quad \text{and} \quad xy = x,\ x \neq y.$$

Thus, since $0 + y = y$ and $0y = 0$, it follows that if $y \neq 0$, then always $0 < y$, i.e., 0 is a unique miminal element.

L1. *$x < y$ and $y < x$ is impossible.*

Proof. Suppose not. Then $x + y = y$ and $y + x = x$. Since $x + y = y + x$ in a semiring, hence $x = y$, a contradiction. □

L2. *$x < y$ and $y < z$ implies $x < z$.*

P r o o f. Since $x+y = y, y+z = z$, it follows that $x+z = x+(y+z) = (x+y)+z = y + z = z$. Similarly, $xy = x, yz = y$ yields $x = xy = x(yz) = (xy)z = xz$. If $x = z$, then $x < y$ and $y < x$ or $x = y$. But the assumption $x < y$ makes $x = y$ impossible. By L1 this requires also that $x \neq z$. □

The set $(R, <)$ is a poset with a unique miminal element 0. We will refer to it as the *semiring order* of R. A non-empty subset I of a semiring order $(R, <)$ is called an *order ideal* if $x \in I, y < x$ imply $y \in I$.

Example 3.1. Let $\mathbb{R}^+$ be the collection of non-negative real numbers with the usual operations "+" and "·". Then $(\mathbb{R}^+, +, \cdot)$ is a semiring. Also, if $x < y$ then $x + y = y$ means $x = 0$ and $y \neq 0$. In particular, if $x \neq y$ and $x \neq 0, y \neq 0$, then $x \circ y$, i.e., x and y are incomparable. Hence $(\mathbb{R}^+ - \{0\}, <)$ is an antichain. We will consider $\mathbb{R}^+$ to be an *antichain semiring.*

Example 3.2. Let $\mathbb{R}^+$ be the collection of non-negative real numbers. Define operations "$\oplus$" and "$\odot$" by $x \oplus y := \max\{x, y\}$, $x \odot y := \min\{x, y\}$. Then we are dealing with a semiring. Indeed, suppose that $x \leqslant y$. Then $x \oplus y = y$ and $x \odot y = x$. If $r \in \mathbb{R}^+$, then $x \leqslant y$ implies $r \odot x \leqslant r \odot y$ as well. Hence $(r \odot x) \oplus (r \odot y) = r \odot y = r \odot (x \oplus y)$. Thus $(\mathbb{R}^+, \oplus, \odot)$ is a semiring.

In this case, if $x < y$ in $\mathbb{R}^+$, then also $x < y$ in the semiring order, whence the two orders are the same since an order extension of a chain is precisely the chain itself. Thus, we will consider $(\mathbb{R}^+, \oplus, \odot)$ to be a *chain semiring.*

Example 3.3. If $(R, +, \cdot)$ is a ring and if $x < y$, then $x + y = y$ implies $x = 0$ and $xy = 0y = 0 = x$, i.e., $0 < y$ and $x \neq y$, $x \neq 0$, $y \neq 0$, implies $x \circ y$, i.e., $(R - \{0\}, <)$ is an antichain.

If a semiring $(R, +, \cdot)$ can be embedded into a ring $(S, +, \cdot)$, then $x < y$ implies $x+y = y$ and $x = 0$ and $(R, +, \cdot)$ is an antichain semiring as well. Antichain semirings are therefore generalizations of semirings which can be embedded in rings.

Lemma 3.4. *Any L-fuzzy left (right) ideal of R is order reversing, i.e, $x < y$ implies $\mu(x) \geqslant \mu(y)$.*

P r o o f. If $x < y$ then $\mu(xy) = \mu(x) \geqslant \mu(y)$. □

Theorem 3.5. *Suppose that $(R, +, \cdot)$ is a chain semiring such that*
(i) $x + 2x = 2x$, $2x^2 = x \Rightarrow x = 0$,
(ii) $x + x^2 = x^2$, $x(x^2) = x \Rightarrow x = 0$.

Then, if $\mu\colon R \to L$ is any order reversing mapping, it is also an L-fuzzy left ideal of the semiring.

P r o o f. If $x < y$ then $\mu(x+y) = \mu(y) \geqslant \min(\mu(x), \mu(y))$. If $x > y$ then $\mu(x+y) = \mu(x) \geqslant \min(\mu(x), \mu(y))$. If $x = y$ then $\mu(2x) < \mu(x)$ means $x \neq 2x$, and not $2x < x$. Thus $x < 2x$ and $x + 2x = 2x$, $x(2x) = x$, whence $x = 0$, a contradiction. Hence $\mu(2x) \geqslant \mu(x)$. If $x < y$ then $\mu(xy) = \mu(x) \geqslant \mu(y)$. If $x > y$ then $\mu(xy) = \mu(y) \geqslant \mu(y)$. If $x = y$ then $\mu(xx) = \mu(x)$ means $x < x^2$ and $x + x^2 = x^2$, $x(x^2) = x$, whence $x = 0$, so that $\mu(x^2) = \mu(x)$. Thus in any case $\mu(x+y) \geqslant \min(\mu(x), \mu(y))$, $\mu(xy) \geqslant \mu(y)$ and μ is also an L-fuzzy left ideal of the semiring. □

Proposition 3.6. *If $\mu\colon R \to L$ is an L-fuzzy left ideal of the semiring $(R, +, \cdot)$, then μ_t is an order ideal of $(R, <)$.*

P r o o f. Suppose that $x \leqslant y$ and $y \in \mu_t$. Then, since μ is order reversing by Lemma 3.4, it follows that $\mu(x) \geqslant \mu(y)$ and $\mu(x) \geqslant t$, i.e., $x \in \mu_t$ as well. □

Theorem 3.7. *If $\mu\colon R \to L$ is an L-fuzzy left ideal of the chain semiring $(R, +, \cdot)$, then any finite order ideal I is a level subset of μ.*

P r o o f. Let $t := \inf\{\mu(x) \mid x \in I\}$. Since I is finite, it follows that $t = \mu(x_0)$ for some $x_0 \in I$. Hence $I = \mu_t$ since $\mu(x) \geqslant \mu(x_0) = t$ implies $x \leqslant x_0$ and thus $x \in I$. □

Corollary 3.8. *If $\mu\colon R \to L$ is an L-fuzzy left ideal of a finite chain semiring $(R, +, \cdot)$ then the collection of order ideals of $(R, <)$ is the collection of level subsets of μ. Furthermore, this collection is linearly ordered by set inclusion.*

The examples discussed here show that the order properties of the semiring order are important in discussing L-fuzzy left ideals on semirings. A diagram for *ordinal sum* $P \oplus Q$ of two partially ordered sets P and Q is obtained by placing a diagram for P directly below a diagram for Q and then adding a line segment from *each* maximal element of P to *each* minimal element of Q. We also denote the n-element antichain simply by $\underline{n}$. On rings the semiring order is a poset $\underline{1} \oplus A$, where A is an antichain which need not be finite. For antichain semirings the class of order reversing L-fuzzy sets $\mu\colon R \to L$ is rather large. Chain semirings are close to the situation where order reversing L-fuzzy sets $\mu\colon R \to L$ are L-fuzzy left ideals. Conditions (i) and (ii) in Theorem 3.5 were designed to make certain that order reversing L-fuzzy sets $\mu\colon R \to L$ would indeed be L-fuzzy left ideals. It is a useful question to ask what other conditions besides (i) and (ii) would guarantee the same conclusion. Chain semirings are also those semirings for which order ideals come close to being level sets. Thus, if I is an order ideal, then $I \subseteq \mu_t$, $t = \inf\{\mu(x) \mid x \in I\}$. If $I \neq \mu_t$, then $u \in \mu_t - I$ has $\mu(u) = t$ and for some sequence $\{x_1, x_2, \ldots, \}$, $\mu(x_1) > \mu(x_2) > \ldots > \mu(x_n) > \lim_{n\to\infty} \mu(x_n) = t = \mu(u)$, where $x_i \in I$. Even if I is infinite then $I = \mu_t$ is quite possible under a variety of circumstances. Recently, Kim, Jun and Kim [4] discussed some connections between posets and BCK-algebras. With the notion of semiring order, one can then establish similar connections between semirings and BCK-algebras as well.

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