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# COMPLETE GENERATORS AND MAXIMAL COMPLETIONS OF $MV$-ALGEBRAS

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$MV$-algebras are also called algebras of the infinite valued Łukasiewicz logic [2]. If $\mathcal{A}$ is an $MV$-algebra, then the notion of a set of generators of $\mathcal{A}$ has the usual meaning.

By considering complete $MV$-algebras (cf., e.g., [2], [8]) we can define the notion of the complete homomorphism and of the set of the complete generators. (For definitions, cf. Section 1 below.)

Analogous definitions have been applied for complete Boolean algebras, complete lattice ordered groups and complete vector lattices.

In [5] it was proved that if $\alpha$ is an infinite cardinal, then there exists no free complete Boolean algebra with $\alpha$ free complete generators. A similar result was proved in [10] for complete lattice ordered groups and in [11] for complete vector lattices.

In the present paper we show that an analogous result is true also in the case of complete $MV$-algebras.

It is well-known that each $MV$-algebra $\mathcal{A}$ can be constructed by means of an appropriately chosen abelian lattice ordered group $G$ with a strong unit $u$ (cf. [12]).

There exist exactly three nonisomorphic types of lattice ordered groups $G$ with one generator. In each of these cases $G$ is complete. If $X$ is a set of nonzero orthogonal elements of $G$, then $\operatorname{card} X \leqslant 2$. Analogous results hold for Boolean algebras with one generator.

In view of the above mentioned relation between $MV$-algebras and abelian lattice ordered groups we can ask whether analogous results are valid for $MV$-algebras with one generator. The answer is "No".

An $MV$-algebra with one generator need not be complete; moreover, it need not be archimedean. There exist infinitely many nonisomorphic complete $MV$-algebras with one complete generator. If $\mathcal{A}$ is an $MV$-algebra with one generator (or a complete $MV$-algebra with one complete generator) and if $X$ is an orthogonal subset of $\mathcal{A}$, then the set $X$ can be infinite.

Further we investigate maximal completions of $MV$-algebras. The analogous notion for abelian lattice ordered groups was studied in [3] and [6]. It will be shown that each $MV$-algebra possesses a unique maximal completion.

## 1. Preliminaries

We apply the terminology and notation from [7]. Thus an $MV$-algebra is a system $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$, where $A$ is a nonempty set, $\oplus, *$ are binary operations, $\neg$ is a unary operation and $0, 1$ are nulary operations on $A$ such that the identities $(\mathrm{m}_1)$–$(\mathrm{m}_9)$ from [7] are satisfied.

We quote the following results which will be needed in the sequel.

**1.1. Theorem.** (Cf. [4].) *Let $\mathcal{A}$ be an $MV$-algebra. For each $x, y \in A$ put $x \vee y = (x * \neg y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$. Then $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge)$ is a distributive lattice with the least element* $0$ *and the greatest element* $1$.

**1.2. Theorem.** (Cf. [12].) *Let $G$ be an abelian lattice ordered group with a strong unit $u$. Let $A$ be the interval $[0, u]$ of $G$. For each $a$ and $b$ in $A$ we put*

$$a \oplus b = (a + b) \wedge u, \ \neg a = u - a, \ 1 = u, \ a * b = \neg(\neg a \oplus \neg b).$$

*Then $\mathcal{A} = (A; \oplus, *, \neg, 0, 1)$ is an $MV$-algebra.*

The $MV$-algebra $\mathcal{A}$ from 1.2 will be denoted by $\mathcal{A}_0(G, u)$.

**1.3. Theorem.** (Cf. [12].) *Let $\mathcal{A}$ be an $MV$-algebra. Then there exists an abelian lattice ordered group $G$ with a strong unit $u$ such that $\mathcal{A} = \mathcal{A}_0(G, u)$.*

Let us remark that if $\mathcal{A}$ and $G$ are as in 1.2, then the partial order on $A$ inherited from $G$ is the same as the partial order on $A$ defined by means of 1.1.

An $MV$-algebra $\mathcal{A}$ is called complete if the lattice $\mathcal{L}(\mathcal{A})$ is complete.

For $a \in A$ we put $1 \cdot a = a$ and for $n > 1$ we define by induction $n \cdot a = a \oplus (n-1) \cdot a$. The $MV$-algebra $\mathcal{A}$ is called archimedean, if there exists no $a \in A$ such that $n \cdot a < (n+1) \cdot a < u$ for each positive integer $n$. (A formally different but equivalent definition was introduced in [9].)

Let $A_1$ be a nonempty subset of $A$ which is closed with respect to the operations $\oplus$, $*$, $\neg$, 0, 1. Then $\mathcal{A}_1 = (A_1, \oplus, *, \neg, 0, 1)$ is a *subalgebra* of $\mathcal{A}$. If no misunderstanding can occur, then we do not distinguish between $\mathcal{A}_1$ and $A_1$.

Suppose that $\mathcal{B}$ is a subalgebra of $\mathcal{A}$. If $\mathcal{L}(\mathcal{B})$ is a closed sublattice of $\mathcal{L}(\mathcal{A})$, then $\mathcal{B}$ is said to be a closed subalgebra of $\mathcal{A}$.

The notion of homomorphism of $MV$-algebras has the usual meaning. A homomorphism $\varphi$ of an $MV$-algebra $\mathcal{A}$ into an $MV$-algebra $\mathcal{B}$ is said to be complete if it satisfies the following condition (c) and the condition (c') dual to (c).

(c) Whenever $\{a_i\}_{i\in I} \subseteq A, a \in A$ and $a = \bigvee_{i\in I} a_i$ is valid in $\mathcal{L}(\mathcal{A})$, then $\varphi(a) = \bigvee_{i\in I} \varphi(a_i)$ is valid in $\mathcal{L}(\mathcal{B})$.

We apply the following standard definitions:

Let $X$ be a subset of $A$. If each subalgebra $\mathcal{B}_1$ of $\mathcal{A}$ with $X \subseteq \mathcal{B}_1$ coincides with $\mathcal{A}$, then $X$ is called *a system of generators* of $\mathcal{A}$. If, moreover, for each $MV$-algebra $\mathcal{C}$, each mapping $\psi$ of $X$ into the underlying set $C$ of $\mathcal{C}$ can be extended to a homomorphism $\varphi$ of $\mathcal{A}$ into $\mathcal{C}$, then $X$ is a set of *free generators* of $\mathcal{A}$.

For the case of complete $MV$-algebras we modify the above definition as follows.

Let $\mathcal{A}$ be a complete $MV$-algebra and let $X \subseteq A$. If for each closed subalgebra $\mathcal{B}_1$ of $\mathcal{A}$ with $X \subseteq \mathcal{B}$, the relation $\mathcal{B}_1 = \mathcal{A}$ is valid, then $X$ is said to be *a system of complete generators* of $\mathcal{A}$.

If, moreover, for each complete $MV$-algebra $\mathcal{C}$, each mapping $\psi\colon X \to C$ can be extended to a complete homomorphism of $\mathcal{A}$ into $\mathcal{C}$, then $X$ is a system of *free complete generators* of $\mathcal{A}$. In such a case we also say that $\mathcal{A}$ is a free complete $MV$-algebra with $\alpha$ free complete generators, where $\alpha = \operatorname{card} X$.

We will use analogous notions for complete lattice ordered groups and for complete Boolean algebras.

## 2. Generators and complete generators

We need the following result (cf. [3]).

**2.1. Proposition.** *Let $\alpha$ be an infinite cardinal. There exists a complete Boolean algebra $B_\alpha$ which satisfies the following conditions:*

(i) $\operatorname{card} B_\alpha \geqslant \alpha$.

(ii) *There exists a denumerable system $X$ of complete generators of $B_\alpha$.*

In fact, this is also a consequence of Theorem K in [13], p. 157. (It suffices to consider the Dedekind completion of the Boolean algebra from Theorem K; in [13] it is remarked that the method of constructing this Boolean algebra is due to Hales [5].)

**2.2. Proposition.** *Let $\alpha$ be an infinite cardinal. There exists a complete $MV$-algebra $\mathcal{A}$ such that*

(i) $\operatorname{card} A \geqslant \alpha$,

(ii) *there exists a denumerable system $X$ of complete generators of $\mathcal{A}$.*

P r o o f. Let $B = B_\alpha$ be as in 2.1. We consider the vector lattice $E$ of all elementary Carathéodory functions on $B$ and then we construct the lattice ordered group $G$ as in the concluding part of [7]. Put $\mathcal{A} = \mathcal{A}_0(G;u)$. Hence the lattice $\mathcal{L}(\mathcal{A})$ coincides with the Boolean algebra $B$. Thus $\operatorname{card} A \geqslant \alpha$. Let $X$ be as in 2.1. Then $X$ is a system of complete generators of $\mathcal{A}$. □

**2.3. Theorem.** *Let $\beta$ be an infinite cardinal. There exists no free complete $MV$-algebra with $\beta$ free complete generators.*

P r o o f. By way of contradiction, assume that $\mathcal{A}_\beta$ is a free complete $MV$-algebra with a system $X_\beta$ of free complete generators such that $\operatorname{card} X_\beta = \beta$. Let $A_\beta$ be the underlying set of $\mathcal{A}_\beta$. There exists a cardinal $\alpha$ with $\alpha > \operatorname{card} A_\beta$. Let $\mathcal{A}$ be as in 2.2.

There exists a denumerable subset $X_1$ of $X_\beta$. Hence there is an injective mapping $\psi$ of $X_1$ onto $X$. For each $x_\beta \in X_\beta \backslash X_1$ we put $\psi_1(x_\beta) = 0$; for $x_\beta \in X_1$ we set $\psi_1(x_\beta) = \psi(x_\beta)$. According to the assumption there exists a complete homomorphism $\varphi$ of $\mathcal{A}_\beta$ into $\mathcal{A}$ such that $\varphi$ is an extension of $\psi_1$. Thus $\varphi(X_1) = X$ and hence from 2.2 (ii) we obtain that $\varphi(\mathcal{A}_\beta) = \mathcal{A}$. Therefore $\operatorname{card} A_\beta \geqslant \operatorname{card} A \geqslant \alpha$, which is a contradiction. □

The additive group of all integers with the natural linear order will be denoted by $\mathbb{Z}$. The free lattice ordered group with one free generator is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. (Cf. [1], Chap. XIII, §4.) As a free generator we can take either the element $(1,-1)$ or the element $(-1,1)$. From this we immediately obtain:

**2.4. Lemma.** *Let $G_1$ be a lattice ordered group with one generator. Then $G_1$ is isomorphic to some of the following lattice ordered groups: $\{0\}$, $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$. In each of these three cases $G_1$ is complete.*

**2.5. Example.** Consider the lattice ordered group $\mathbb{Z}$ and a positive integer $n$. Put $u = n$ and let us construct the $MV$-algebra $\mathcal{A} = \mathcal{A}_n$ as in Theorem 1.2. Then $\mathcal{A}_n$ is a complete $MV$-algebra; the element 1 of $\mathcal{A}_n$ is a generator of $\mathcal{A}_n$. If $n(1)$ and $n(2)$ are distinct positive integers, then $\mathcal{A}_{n(1)}$ fails to be isomorphic to $\mathcal{A}_{n(2)}$. Hence the situation concerning $MV$-algebras with one generator essentially differs from that concerning lattice ordered groups with one generator.

**2.6. Example.** Let $G = \mathbb{Z} \circ \mathbb{Z}$ (where $\circ$ denotes the operation of lexicographic product). Put $u = (1,0)$. Then $u$ is a strong unit of $G$ and hence we can construct the $MV$-algebra $\mathcal{A}$ according to 1.2. It is obvious that $\mathcal{A}$ is not complete; moreover, it is not archimedean. The underlying set $A$ of $\mathcal{A}$ consists of all elements $(m, n) \in G$ such that either (i) $m = 0$, or (ii) $m = 1$ and $n \leqslant 0$.

Put $x = (0, 1)$. If $\mathcal{A}_1$ is a subalgebra of $\mathcal{A}$ with the underlying set $A_1$ and if $x \in A_1$, then clearly $(0, n) \in A_1$ for each $n \geqslant 0$; hence $(1, -n) = u - (0, n)$ also belongs to $A_1$. Thus $A_1 = A$ and hence $x$ is a generator of $\mathcal{A}$.

**2.7. Example.** For each positive integer $n$ let $G_n = \mathbb{Z}$ and $G = \prod_{n=1}^{\infty} G_n$. If $x \in G$, then we denote by $x_n$ the component of $x$ in $G_n$. Let $u \in G$ be such that $u_n = n$ for each positive integer $n$. The convex $\ell$-subgroup of $G$ which is generated by $u$ will be denoted by $G'$. Hence $u$ is a strong unit of $G'$. We can construct the $MV$-algebra $\mathcal{A} = \mathcal{A}_0(G', u)$ with the underlying set $A$. It is obvious that $G'$ is complete, hence in view of [8], 1.1, $\mathcal{A}$ is complete as well. There exists $x \in A$ such that $x_n = 1$ for each $n \in \mathbb{N}$.

Let $\mathcal{A}_1$ be a closed subalgebra of $\mathcal{A}$. Suppose that $A_1$ is the underlying set of $\mathcal{A}_1$ and $x \in A_1$. Hence $u - x \in A_1$. Put $(u - x) \oplus (u - x) = y$. Then in view of 1.2,

$$y = ((u - x) + (u - x)) \wedge u = (2u - 2x) \wedge u,$$

whence

$$y_n = (2n - 2) \wedge n$$

for each $n \in \mathbb{N}$. Therefore

$$y_n = \begin{cases} 0 & \text{if } n = 1, \\ n & \text{if } n > 1. \end{cases}$$

Put $x^1 = u - y$. We have $y \in A_1$, hence $x^1 \in A_1$. Clearly $x^1_1 = 1$ and $x^1_n = 0$ for $n > 1$.

Further we have

$$(u - x)_1 = 0 = (x - x^1)_1,$$

and

$$(u - x)_n = u_{n-1}, \quad (x - x^1)_n = x_{n-1}$$

for $n > 1$. Thus if the elements $u$ and $x$ in the above calculation are replaced by $u - x$ and $x - x^1$, respectively, then we obtain that there exists $x^2 \in A_1$ such that $x^2_2 = 1$ and $x^2_n = 0$ for each $n \in \mathbb{N} \setminus \{2\}$.

By applying the obvious induction we conclude that for each $m \in \mathbb{N}$ there is $x^m$ in $A_1$ such that $x^m_m = 1$ and $x^m_n = 0$ whenever $n \neq m$.

Let $m \in \mathbb{N}$. We put $z^{m1} = x^m$. If $1 < k \in \mathbb{N}$, $k \leqslant m$, then we define by induction

$$z^{m,k} = z^{m,k-1} \oplus z^{m1}.$$

Thus $z^{m,k} \in A_1$ for $k = 1, 2, \ldots, m$. In view of 1.2 we easily verify that $z_n^{m,k} = 0$ if $n \neq m$, and $z_m^{m,k} = k$.

Let $t \in A$. Hence $t_m \leqslant m$ for each $m \in \mathbb{N}$. Then

$$t = \bigvee_{m \in \mathbb{N}} z^{m,t_m} \tag{1}$$

is valid in $\mathcal{A}$. Since $\mathcal{A}_1$ is a closed subalgebra of $\mathcal{A}$, we obtain that $t$ belongs to $A_1$. Thus $\mathcal{A} = \mathcal{A}_1$.

Therefore $\mathcal{A}$ is a complete $MV$-algebra having one complete generator $x$. The subset $\{x^n\}_{n \in \mathbb{N}}$ of $A$ is orthogonal and infinite. Let us also remark that the cardinality of $A$ equals the power of the continuum.

**2.8. Example.** Let us apply the same notation as in 2.7. Further let $\mathcal{A}_0$ be the subalgebra of $\mathcal{A}$ generated by the element $x$ and let $A_0$ be the underlying set of $A$. Thus $\operatorname{card} A_0 \leqslant \aleph_0$. Moreover, all elements $z^{m,k}$ $(m \in \mathbb{N}, 1 \leqslant k \leqslant m)$ belong to $A_0$. The $MV$-algebra $\mathcal{A}_0$ fails to be complete.

This can be verified as follows. By way of contradiction, suppose that $\mathcal{A}_0$ is complete. Since $\operatorname{card} A_0 < \operatorname{card} A$, there exists $t \in A \setminus A_0$. Consider the system $\{z^{m,t_m}\}_{m \in \mathbb{N}} = S$. Thus there exists $t' \in A_0$ such that $t' = \sup S$ is valid in $\mathcal{A}_0$. Then $t'_m \geqslant z_m^{m,t_m} = t_m$ for each $m \in \mathbb{N}$, hence $t' \geqslant t$. Since $t \notin A_0$, we conclude that $t' > t$. Therefore there exists $m \in \mathbb{N}$ with $t'_m \geqslant t_m + 1$.

Further there exists $t''$ in $A_0$ such that

$$t'' \oplus x^m = t'.$$

Then $t'' < t'$ and $t''$ is an upper bound of the system $S$, which is a contradiction.

## 3. Maximal completions

For a subset $X$ of a lattice $L$ we denote by $X^u$ and $X^\ell$, respectively, the set of all upper bounds and the set of all lower bounds of $X$ in $L$. Let $d(L)$ be the system of all sets $(X^u)^\ell$, where $X$ runs over the system of all nonempty upper bounded subsets of $L$. The system $d(L)$ is partially ordered by the set-theoretical inclusion. Then $d(L)$ is a conditionally complete lattice.

The mapping $\varphi\colon L \to d(L)$ defined by

$$\varphi(x) = (\{x\}^u)^\ell \quad \text{for each} \quad x \in L$$

is an isomorphism of $L$ into $d(L)$. When no misunderstanding can occur we will identify $x$ with $\varphi(x)$ for each $x \in L$. Then $L$ turns out to be a sublattice of $d(L)$.

Moreover, if $X_1$ is a subset of $L$ and if $x_1$ is the supremum of $X_1$ in $L$, then $x_1$ is also the supremum of $X_1$ in $d(L)$. The corresponding dual assertion is valid as well. If $X$ is a nonempty upper bounded subset of $L$, then the relation

$$(X^u)^\ell = \bigvee x_i \quad (x_i \in X) \tag{1}$$

holds in $d(L)$.

Let $a, b \in L,\ a < b$, and let $L_1$ be the interval $[a, b]$ in $L$. For $\emptyset \neq X \subseteq L_1$ we denote

$$X^{u(1)} = X^u \cap L_1, \quad X^{\ell(1)} = X^\ell \cap L_1.$$

Hence $d(L_1)$ is the system of all sets $(X^{u(1)})^{\ell(1)}$, where $X$ runs over the system of all nonempty subsets of $L_1$.

Let $L_1^*$ be the interval with the endpoints $a$ and $b$ in $d(L)$. For $Z \in L_1^*$ and $T \in d(L_1)$ we put

$$\varphi_1(Z) = Z \cap [a, b], \quad \varphi_2(T) = (T^u)^\ell.$$

By applying (1) we obtain

**3.1. Lemma.** *$\varphi_1$ is an isomorphism of $L_1^*$ onto $d(L_1)$ and $\varphi_2 = \varphi_1^{-1}$. Moreover, $\varphi_1(x) = x$ for each $x \in L_1$.*

Now let $\mathcal{A}$ be an $MV$-algebra and let $G$ be a lattice ordered group with a strong unit $u$ such that $\mathcal{A} = \mathcal{A}_0(G, u)$.

Denote $d(\mathcal{L}(\mathcal{A})) = d(A)$. Let $A^*$ be the interval with the endpoints 0 and $u$ in $d(G)$. For each $P \in A^*$ we put $\varphi_1(P) = P \cap A$. From 3.1 we obtain

**3.2. Corollary.** *$\varphi_1$ is an isomorphism of $A^*$ onto $d(A)$. Moreover, $\varphi_1(x) = x$ for each $x \in A$.*

For $Y_1, Y_2 \in d(G)$ we put

$$Y_1 + Y_2 = (\{y_1 + y_2 \colon\ y_1 \in Y_1 \text{ and } y_2 \in Y_2\}^u)^\ell.$$

In view of (1) we have

$$Y_1 + Y_2 = \sup\{y_1 + y_2\} \quad (y_1 \in Y_1, y_2 \in Y_2), \tag{2}$$

where the supremum is taken with respect to $d(G)$.

The following results 3.3 and 3.4 have been proved in [3]; cf. also [6], 1.1 and 1.2.

**3.3. Proposition.** *The set $d(G)$ with the operation $+$ is a semigroup. The element $0$ is a neutral element of $(d(G);+)$. If $a,b,c\in d(G)$, $a\leqslant b$, then $a+c\leqslant b+c$. Next, $G$ is a subsemigroup of $d(G)$.*

**3.4. Theorem.** *Let $M(G)$ be the set of all elements of $d(G)$ which have an inverse in the semigroup $(d(G),+)$. Then*

(a) *$(M(G);+,\leqslant)$ is a lattice ordered group;*
(b) *$(M(G);\leqslant)$ is a sublattice of $d(G)$.*

In what follows we will write $M(G)$ instead of $(M(G);+,\leqslant)$. In [3], $M(G)$ is called the Dedekind completion of $G$; in [6], $M(G)$ was called the maximal completion of $G$.

We define a binary operation $\oplus$ on $d(A)$ as follows. For $T_1,T_2\in d(A)$ we put

$$T_1\oplus T_2=(\{t_1\oplus t_2\colon\ t_1\in T_1\quad\text{and}\quad t_2\in T_2\}^{u(1)})^{\ell(1)}$$

where $u(1)$ and $\ell(1)$ have analogous meanings as in the case of $L_1$ above.

Then according to (1) we have

$$T_1\oplus T_2=\sup\{t_1\oplus t_2\}\quad(t_1\in T_1\text{ and }t_2\in T_2)\tag{3}$$

where the supremum is taken with respect to $d(A)$.

It is easy to verify that the just defined operation $\oplus$ on $d(A)$ is an extension of the original operation $\oplus$ on $A$.

From the fact that the operation $\oplus$ on $A$ is commutative and associative we infer (by applying (3))

**3.5. Lemma.** *The set $d(A)$ with the operation $\oplus$ is an abelian semigroup.*

**3.6. Definition.** *Let $\mathcal{A}$ be as above and let $\mathcal{B}$ be an $MV$-algebra such that the following conditions are satisfied:*

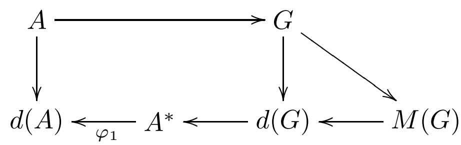
(a) *$\mathcal{A}$ is a subalgebra of $\mathcal{B}$.*
(b) *$\mathcal{L}(\mathcal{B})$ is a sublattice of $d(A)$.*
(c) *$(B;\oplus)$ is a subsemigroup of the semigroup $(d(A);\oplus)$.*

*Then $\mathcal{B}$ is called a c-extension of $\mathcal{A}$.*

**3.7. Definition.** *Let $\mathcal{B}_1$ be a c-extension of $\mathcal{A}$. If for each c-extension $\mathcal{B}$ of $\mathcal{A}$ the $MV$-algebra $\mathcal{B}$ is a subalgebra of $\mathcal{B}_1$, then $\mathcal{B}_1$ is called a maximal completion of $\mathcal{A}$.*

The above definition yields that if a maximal completion of $\mathcal{A}$ does exist, then it is uniquely determined.

For structures dealt with in the present section we can apply the following diagram:




Here the mapping $\varphi_1$ is as in 3.2; all the remaining mappings are embeddings.

We define a binary operation $\oplus$ on $A^*$ as follows. For $Z_1, Z_2 \in A^*$ we put

$$Z_1 \oplus Z_2 = \sup\{(z_1 + z_2) \wedge u \colon\ z_1 \in Z_1 \text{ and } z_2 \in Z_2\}, \tag{4}$$

where the supremum is taken with respect to the complete lattice $A^*$.

**3.8. Lemma.** *$\varphi_1$ is an isomorphism of $(A^*, \oplus)$ onto $(d(A), \oplus)$.*

Proof. Let $Z_1, Z_2 \in A^*$. Put $\varphi_1(Z_i) = T_i$ $(i = 1, 2)$. In view of (3),

$$\begin{aligned} T_1 \oplus T_2 = \ & \sup\{(t_1 + t_2) \wedge u \colon\ t_1 \in T_1 \text{ and } t_2 \in T_2\} = \\ = \ & \sup\{(\varphi(z_1) + \varphi(z_2)) \wedge \varphi(u) \colon\ z_1 \in Z_1 \text{ and } z_2 \in Z_2\} = \\ = \ & \sup\{\varphi((z_1 + z_2) \wedge u) \colon\ z_1 \in Z_1 \text{ and } z_2 \in Z_2\}, \end{aligned}$$

where sup is taken with respect to $d(A)$. Hence in view of (4) and 1.2, $T_1 \oplus T_2 = \varphi(Z_1 \oplus Z_2)$. □

From the construction of $M(G)$ we infer that $u$ is a strong unit of $M(G)$. Let $M_0$ be the interval of $M(G)$ with the endpoints 0 and $u$. Hence we can construct (by means of 1.2) the $MV$-algebra $\mathcal{A}_0(M(G), u)$; we will denote this $MV$-algebra by the symbol $M_0$ of its underlying set.

**3.9. Lemma.** *The $MV$-algebra $\mathcal{A}$ is a subalgebra of $M_0$.*

Proof. Since $G$ is an $\ell$-subgroup of $M(G)$, from the relations

$$\mathcal{A} = \mathcal{A}_0(G, u), \quad M_0 = \mathcal{A}_o(M(G), u)$$

we obtain that $\mathcal{A}$ is a subalgebra of $M_0$. □

**3.10. Lemma.** *$\varphi_1(M_0)$ is a sublattice of $d(A)$.*

Proof. In view of 3.4 (b), $(M(G); \leqslant)$ is a sublattice of $d(G)$. This yields that $M_0$ is a sublattice of $A^*$. Hence according to 3.2, $\varphi_1(M_0)$ is a sublattice of $d(A)$. □

The following result is well-known.

($*$) Let $a, b, c \in G^+$, $c \leqslant a+b$. Then there are $a_1, b_1 \in G$ such that $a_1 \in [0, a]$, $b_1 \in [0, b]$ and $c = a_1 + b_1$.

**3.11. Lemma.** *Let $a, b \in A$. Then there are $a_1 \in [0, a]$ and $b_1 \in [0, b]$ such that $a \oplus b = a_1 + b_1$.*

Proof. We have $a \oplus b = (a + b) \wedge u$. Put $c = a \oplus b$. Now it suffices to apply ($*$). □

**3.12. Corollary.** *Let $T_1, T_2 \in d(A)$. Then*

$$T_1 \oplus T_2 = \sup\{t_1 + t_2\colon\ t_1 \in T_1, t_2 \in T_2 \text{ and } t_1 + t_2 \leqslant u\},$$

*where the supremum is taken with respect to $d(A)$.*

**3.13. Lemma.** *$(\varphi_1(M_0); \oplus)$ is a subsemigroup of the semigroup $(d(A), \oplus)$.*

Proof. This is a consequence of 3.10 and 3.12. □

**3.14. Lemma.** *$\varphi_1(M_0)$ is a c-extension of $\mathcal{A}$.*

Proof. This follows from 3.10, 3.9 and 3.13. □

**3.15. Lemma.** *Let $\mathcal{B}$ be a c-extension of $\mathcal{A}$. Let $Z \in B$, $Z \neq u$. Then there are $a \in A$ and $Z_1 \in d(A)$ such that $a < u$ and $Z \oplus Z_1 = a$.*

Proof. There exists a lattice ordered group $G'$ such that $u$ is a strong unit in $G'$ and $\mathcal{B} = \mathcal{A}_0(G', u)$. Since $Z \neq u$, the set $Z$ must be upper bounded in $A \setminus \{u\}$. Hence there is $a \in A$ such that $Z \leqslant a < u$. This implies that there is $b \in B$ such that $Z + b = a$ holds in $G'$. Then $0 \leqslant b < u$ and therefore $a = Z \oplus b$ is valid in $\mathcal{B}$. Now it suffices to put $Z_1 = b$. □

**3.16. Lemma.** *Let $Z$ be as in 3.15. Then $\varphi_1^{-1}(Z) \in M_0$.*

Proof. We have $Z \oplus Z_1 = a$. Hence in view of 3.8 the relation

$$\varphi_1^{-1}(Z) \oplus \varphi_1^{-1}(Z_1) = a$$

is valid in $A^*$. Since $a < u$, we get

$$a = \varphi_1^{-1}(Z) + \varphi_1^{-1}(Z_1).$$

From the relation $a \in M(G)$ we infer that there is $-a \in M(G)$; hence

$$0 = \varphi_1^{-1}(Z) + (\varphi_1^{-1}(Z_1) - a).$$

Now according to 3.4, $\varphi_1^{-1}(Z)$ belongs to $M(G)$. Next, $\varphi_1^{-1}(Z) \in A^*$ and thus $\varphi_1^{-1}(Z) \in M_0$. □

**3.17. Corollary.** *Let $\mathcal{B}$ be a c-extension of $\mathcal{A}$. Then $B \subseteq \varphi_1(M_0)$.*

Therefore we obtain

**3.18. Theorem.** *Let $\mathcal{A}$ be an $MV$-algebra. Let $\varphi_1$ and $M_0$ be as above. Then $\varphi_1(M_0)$ is the maximal completion of $\mathcal{A}$.*

**3.19. Proposition.** *Let $\mathcal{A}$ be an $MV$-algebra. Then the maximal completion of $\mathcal{A}$ is the set of all $T \in d(A)$ which satisfy the following condition:*

(c) *either $T = u$, or there are $a \in A$ and $T_1 \in d(A)$ such that $a < u$ and $T \oplus T_1 = a$.*

P r o o f. a) Let $T$ belong to the maximal completion of $\mathcal{A}$. Then clearly $T \in d(A)$. According to 3.15, the condition (c) is satisfied.

b) Let $T \in d(A)$ and suppose that the condition (c) is valid. If $T = u$, then in view of 3.18, $u$ belongs to the maximal completion $\varphi_1(M_0)$ of $\mathcal{A}$. Let $T < u$. In view of 3.16, $T \in \varphi_1(M_0)$. □

An $MV$-algebra will be called $m$-complete if it coincides with its maximal completion. We conclude by remarking without proof that the class of all $m$-complete $MV$-algebras is closed with respect to direct products, but it fails to be closed with respect to homomorphic images.

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