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VARIATIONAL EQUATIONS ALONG INTEGRAL CURVES OF A PROJECTABLE SYSTEM OF VECTOR FIELDS

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There exists a rich literature on systems of connections and systems of vector fields, stimulated by their importance in geometry and physis. In the previous papers [T1], [T2] we examined a simple type of systems of vector fields, called parameter dependent vector fields, and established their varionational equation.

In this paper we generalize the above equation to the projectable system of vector fields. The material is organized as follows: in the first section the geometry of the product bundle is presented. In the second we introduce the notion of derivative along a direction and prove Theorem 1. The third section is devoted to Theorem 2, which represents the main result of the paper. Some examples are presented in the last section. In a further paper we will apply the results in order to investigate some special systems as strong systems, "nice" systems and systems of connections generated by systems of vector fields.

1. The geometry of the product bundle

(1.a) The vertical splitting. Let $\chi\colon\ H \to B$, $\pi\colon\ E \to B$ be two fibred manifolds over the same base space B. We denote

$$H \oplus E = H \times_B E = \big\{(h,e); \chi(h) = \pi(e) = x \in B\big\}. \tag{1.1}$$

One gets the following fibred manifolds:

$$\begin{aligned} p_E\colon\ H \oplus E \to E, &\quad (h,e) \to e,\\ p_H\colon\ H \oplus H \to H, &\quad (h,e) \to h,\\ p\colon\ H \oplus E \to B, &\quad (h,e) \to x. \end{aligned}$$

They are related by the relation

$$p = \chi \circ p_H = \pi \circ p_E.$$

Proposition 1. *We have the following vertical splittings:*

$$\begin{aligned} Vp &= Vp_E \oplus Vp_H, \\ Vp_E &= H \oplus E \times_H V\chi, \\ Vp_H &= H \oplus E \times_E V\pi. \end{aligned}$$

They are consequences of $p = \chi \circ p_H = \pi \circ p_E$. It is easy to verify that the last two splittings are described by the followings isomorphisms:

$$\begin{aligned} \mu_E \colon\ Vp_E &\to H \oplus E \times_H V\chi, \\ Vp_E \ni \Xi_{,(h,e)} &\to \big((h,e), (p_H)_* \cdot \Xi_{,(h,e)}\big), \\ \mu_H \colon\ Vp_H &\to H \oplus E \times_E V\pi, \\ Vp_H \ni \Xi_{,(h,e)} &\to \big((h,e), (p_H)_* \cdot \Xi_{,(h,e)}\big). \end{aligned} \tag{1.2}$$

These splittings allow us to prolong a vertical vector field of χ to a vertical vector field of p_E.

Let $h \in H$ be a fixed point and let $\Xi \in V\chi_{,h}$ be a vertical vector; then Ξ induces a vector field along the set

$$\{h\} \times E_x = p_H^{-1}(h) \text{ where } x = \chi(h),$$

which is vertical along p_E at every point $(h,e) \in p_H^{-1}(h)$.

Indeed, if we set

$$\Xi^{\uparrow}(h,e) = \mu_E^{-1}\big((h,e), \Xi_{,h}\big) \tag{1.3}$$

we obtain a field of vertical vectors along p_E, called *the vertical prolongation of* Ξ. Let us write the above consideration in coordinate expressions: let (U, x^i), $i = 1, \ldots, n$, be a coordinate chart on B; let $\big(\chi^{-1}(U), x^i, z^a\big)$ be the adapted coordinate chart on H and $\big(\pi^{-1}(U), x^i, u^\alpha\big)$ the adapted coordinate chart on E. Then, on the open set $\chi^{-1}(U) \oplus \pi^{-1}(U)$, we get the coordinate expression $(\overline{x}^i, \overline{z}^a, \overline{u}^\alpha)$ with the property

$$\begin{aligned} \overline{x}^i(h,e) &= x^i(x), \quad x = \chi(h) = \pi(e), \\ \overline{z}^a(h,e) &= z^a(h), \\ \overline{u}^\alpha(h,e) &= u^\alpha(e). \end{aligned}$$

If $\Xi \in V\chi_{,h}$, $\Xi = \Xi^a \cdot \partial z^a_{,h}$, the one gets

$$\Xi^{\uparrow}(h,e) = \Xi^a \cdot \partial z^a_{,(h,e)}, \quad (h,e) \in p_H^{-1}(h). \tag{1.4}$$

(1.b) Connections of the fibred manifold $p_E\colon\ H \oplus E \to E$.

Proposition 2. *Let $\sigma\colon\ H \times_B TB \to TH$ be a connection of the fibred manifold $\chi\colon\ H \to B$; we denote the vertical projector of σ by*

$$P_\sigma\colon\ TH \to V\chi.$$

Then the formula

$$P_{\overline{\sigma}} = \mu_E^{-1} \circ (\underset{H\oplus E}{\mathrm{id}} \times_H P_\sigma) \circ p_H \tag{1.5}$$

represents a splitting of the exact sequence

$$0 \to Vp_E \to T(H \oplus E) \to H \oplus E \times_E TE \to 0.$$

Consequently, (1.5) represents the vertical projector of a connection

$$\overline{\sigma}\colon\ H \oplus E \times_E TE \to T(H \oplus E).$$

The coordinate expression of σ described as follows: if

$$\sigma(h, \partial x^i) = \partial x^i + \sigma_i^a(x,h) \cdot \partial z^a = \delta x^i$$

then

$$\begin{aligned} \overline{\sigma}\big((h,e), \partial x^i\big) &= \delta \overline{x}^i = \partial \overline{x}^i + \sigma_i^a(x,h) \cdot \partial \overline{z}^a, \\ \overline{\sigma}\big((h,e), \partial u^\alpha\big) &= \partial \overline{u}^\alpha. \end{aligned} \tag{1.6}$$

The connection $\overline{\sigma}$ splits the tangent space $T(H \oplus E)$ into the decomposition

$$T(H \oplus E) = Vp_E \oplus Hp_E. \tag{1.7}$$

2. Systems of vector fields

(2.a) The horizontal prolongation of a (p.s.v.f). We use the abbreviation (s.v.f) for systems of vector fields and (p.s.v.f) for projectable (s.v.f). Let $\eta\colon\ H\oplus E\to TE$ be a (s.v.f) and let σ be a connection of the fibred manifold $\chi\colon\ H\to B$.

We set

$$\begin{aligned} &\overline{\eta}\colon\ H\oplus E\to T(H\oplus E),\\ &\overline{\eta}(h,e)=\overline{\sigma}\big((h,e),\eta(h,e)\big), \end{aligned}\tag{2.1}$$

where $\overline{\sigma}$ is the induced connection described by Proposition 2. The coordinate description of (2.1) is: if

$$\eta(h,e)=\eta^i(h,e)\cdot\partial x^i+\eta^\alpha(h,e)\cdot\partial u^\alpha$$

then

$$\overline{\eta}(h,e)=\eta^i(h,e)\cdot\delta\overline{x}^i+\eta^\alpha(h,e)\cdot\partial\overline{u}^\alpha\tag{2.2}$$

where $\delta\overline{x}^i$, $\partial\overline{u}^\alpha$ are described by (1.6).

We denote by $\eta\colon\ S(\chi)\to\chi(E)$ the sheaf morphism

$$\begin{aligned} &S(\chi)\ni s\to\mathbf{s}\in\chi(E),\\ &\mathbf{s}(e)=\eta\big(s(\pi(e)),e\big) \end{aligned}\tag{2.3}$$

(see [M-M] for details). The (s.v.f) is projectable if there exists a morphism

$$\eta\colon\ H\longrightarrow TB$$

such that the diagram

$$\begin{array}{ccc} \eta\colon H\oplus E & \longrightarrow & TE\\ \downarrow & & \downarrow\\ \eta\colon H & \longrightarrow & TB \end{array}$$

commutes. According to [M-M], η is projectable if and only if $\eta^i\in F(H)$. Let us suppose η is a (p.s.v.f). Then its horizontal prolongation $\overline{\eta}$ has the property that it projects onto the horizontal vector field η', $\eta'\colon\ H\to TH$, described by the following diagram:

$$\begin{array}{ccc} H & \longrightarrow & H\times_B TB\\ & \searrow^{\eta'} & \downarrow{\scriptstyle\sigma}\\ & & TH \end{array}$$

One gets locally the expression

$$\eta'(h) = \eta^i(h) \cdot \delta x^i = \eta^i(h) \cdot \big(\partial x^i + \sigma_i^a(x,h) \cdot \partial z^a\big). \tag{2.4}$$

Let $\{\overline{\varphi}_t\colon\ H \oplus E \to H \oplus E,\ t \in \mathbb{R}\}$ be the one-parametric group of $\overline{\eta}$ and let $\{\varphi'_t\colon\ H \to H,\ t \in \mathbb{R}\}$ be the one-parametric group of η'. Because $\overline{\eta}$ projects onto η', one obtains that the diagram

$$\begin{array}{ccc} \overline{\varphi}_t\colon H \oplus E & \longrightarrow & H \oplus E \\ \big\downarrow & & \big\downarrow \\ \varphi'_t\colon H & \longrightarrow & H \end{array} \tag{2.5}$$

commutes.

We may use this diagram in order to describe how an integral curve of η' determines a fields of integral curves of $\overline{\eta}$.

Let $h_0 \in H_{,x_0} = \chi^{-1}(x_0)$ be a fixed point and let

$$t \to \varphi'_t(h_0) = c'(t)$$

be the integral curve of η' passing through h_0. Let $e_0 \in E_{,x_0} = \pi^{-1}(x_0)$ be any point and let

$$t \to \overline{\varphi}_t(h_0, e_0) = \overline{c}(t) \tag{2.6}$$

be the integral curve of $\overline{\eta}$ passing through (h_0, e_0). The formula (2.6) determines the family of integral curves of $\overline{\eta}$ starting from every point of $E_{,x_0}$.

Let $s \in S(\chi)$ be an integral section of η' which contains $c'(t)$, and let

$$\{\mathbf{s}_t\colon\ E \to E,\ t \in \mathbb{R}\}$$

be the one-parametric group of the vector field $\mathbf{s} = \eta(s)$. Let us denote the integral curves of $\mathbf{s}$ which passes through e_0 by

$$t \to \mathbf{s}_t(e_0) = \mathbf{c}(t). \tag{2.7}$$

Then we obtain the following decomposition of $\overline{c}(t)$:

$$\overline{c}(t) = \big(c'(t), \mathbf{c}(t)\big). \tag{2.8}$$

One can verify that (2.8) is independent of the chosen section satisfying the above properties.

(2.b) The derivative along a direction. Let η be a (s.v.f) and let $\Xi \in \chi(H)$ be a vertical vector field of the fibred manifold χ. We call such a vector field a *direction.*

Let

$$\{\zeta_\lambda\colon\ H \to H,\ \lambda \in \mathbb{R}\}$$

be the one-parametric group of Ξ.

Definition 1. Let $(h,e) \in H \oplus E$ be a fixed point. By the derivative of η along the direction Ξ at the point (h,e) we mean the limit

$$(\partial\eta/\partial\Xi)(h,e) = \lim_{\lambda\to 0}(1/\lambda)\cdot\bigl(\eta(h,e) - \eta(\zeta_\lambda(h),e)\bigr). \tag{2.9}$$

One notices that $\eta(h,e)$, $\eta\bigl(\zeta_\lambda(h),e\bigr)$ belong to T_eE and so $\partial\eta/\partial\Xi$ does. If the above limit exists at every point $(h,e) \in H \oplus E$, then a new (s.v.f)

$$\partial\eta/\partial\Xi\colon\ H \oplus E \to TE$$

called *the derivative of* η *along the direction* Ξ*,* is well defined.

Theorem 1. *Let* η *be a (s.v.f) and let* $\Xi \in \chi(H)$ *be a direction. Then the relation*

$$(p_E)_{*,(h,e)}\cdot[\Xi^\uparrow,\eta] = (\partial\eta/\partial\Xi)(h,e) \tag{2.10}$$

holds.

Proof. Let $\Gamma\colon\ H \oplus E \times_E TE \to T(H \oplus E)$ be a connection of the fibred manifold $p_E\colon\ H \oplus E \to E$. Locally, Γ can be described by the formulas

$$\Gamma(\partial x^i) = \delta\overline{x}^i = \partial x^i + \Gamma_i^a(h,e)\cdot\partial z^a,$$
$$\Gamma(\partial u^\alpha) = \delta\overline{u}^\alpha = \partial u^\alpha + \Gamma_\alpha^a(h,e)\cdot\partial u^\alpha.$$

As in (2.1), we can prolong η to the horizontal vector field

$$\eta^\Gamma\colon\ H \oplus E \to T(H \oplus E)$$

with the coordinate expression

$$\eta^\Gamma(h,e) = \eta^i(h,e)\cdot\delta\overline{x}^i + \eta^\alpha(h,e)\cdot\delta\overline{u}^\alpha.$$

Let $\{\zeta_\lambda^\uparrow\colon\ H \oplus E \to H \oplus E,\ \lambda \in \mathbb{R}\}$ be the one-parametric group of the vertical prolongation $\Xi^\uparrow$. Then it acts as

$$\zeta_\lambda^\uparrow(h,e) = \bigl(\zeta_\lambda(h),e\bigr),$$

where $\{\zeta_\lambda\colon\ H \to H,\ \lambda \in \mathbb{R}\}$ represents the one-parametric group of $\Xi \in \chi(H)$. Consequently, the differential of $\zeta_\lambda^\uparrow$ satisfies

$$(\zeta_{-\lambda}^\uparrow)_{*,(\zeta_\lambda(h),e)} \cdot \partial x^i = \partial x^i_{,(h,e)},$$
$$(\zeta_{-\lambda}^\uparrow)_{*,(\zeta_\lambda(h),e)} \cdot \partial u^\alpha = \partial u^\alpha_{,(h,e)}$$

and

$$(\zeta_{-\lambda}^\uparrow)_{*,(\zeta_\lambda(h),e)} \cdot \partial z^a = Y^a(h,e) \in Vp_{E,(h,e)}.$$

One obtains

$$\begin{aligned}
[\Xi^\uparrow, \eta^\Gamma](h,e) &= \lim_{\lambda\to 0}(1/\lambda)\cdot\big(\eta^\Gamma(h,e) - (\zeta_{-\lambda}^\uparrow)_* \cdot \eta^\Gamma(\zeta_\lambda(h),e)\big) \\
&= \lim_{\lambda\to 0}(1/\lambda)\cdot\Big(\Gamma((h,e),\eta(h,e)) - \Gamma\big((h,e),\eta(\zeta_\lambda(h),e)\big)\Big) \\
&\quad + \lim_{\lambda\to 0}(1/\lambda)\cdot\Big(\Gamma((h,e),\eta(\zeta_\lambda(h),e)) - (\zeta_{-1}^\uparrow)_* \cdot \eta^\Gamma\big(\zeta_\lambda(h),e\big)\Big).
\end{aligned}$$

The first limit is equal to

$$\Gamma\big((h,e),(\partial\eta/\partial\Xi)(h,e)\big)$$

as we can see from Proposition 1. We shall prove that the second limit represents a vertical vector along $p_E\colon\ H \oplus E \to E$.

One has

$$\begin{aligned}
(\zeta_{-\lambda}^\uparrow)_* \cdot \eta^\Gamma\big(\zeta_\lambda(h),e\big) &= (\zeta_{-\lambda}^\uparrow)_* \cdot (\eta^i \cdot \partial x^i + \eta^\alpha \cdot \partial u^\alpha + \Gamma_i^a \cdot \partial z^a + \Gamma_i^a \cdot \partial z^a)_{,\big(\zeta_\lambda(h),e\big)} \\
&= \eta^i\big(\zeta_\lambda(h),e\big)\cdot\partial x^i_{,(h,e)} + \eta^\alpha\big(\zeta_\lambda(h),e\big)\cdot\partial u^\alpha_{,(h,e)} \\
&\quad + \Big(\Gamma_i^a\big(\zeta_\lambda(h),e\big) + \Gamma_\alpha^a\big(\zeta_\lambda(h),e\big)\Big)\cdot Y^a(h,e) \\
&= \eta^i\big(\zeta_\lambda(h),e\big)\cdot\partial x^i_{,(h,e)} + \eta^\alpha\big(\zeta_\lambda(h),e\big)\cdot\partial u^\alpha_{,(h,e)} \\
&\quad + \text{ a vertical vector} \\
&= \Gamma\Big((h,e),\eta\big(\zeta_\lambda(h),e\big)\Big) + \text{ a vertical vector}
\end{aligned}$$

This implies that the vector

$$(1/\lambda)\cdot\Big(\Gamma\big((h,e),\eta(\zeta_\lambda(h),e)\big) - (\zeta_{-\lambda}^\uparrow)_* \cdot \eta^\Gamma\big(\zeta_\lambda(h),e\big)\Big)$$

belongs to the vector space $Vp_{E,(h,e)}$. Because $Vp_{E,(h,e)}$ is a closed set, we conclude that

$$\lim_{\lambda\to 0}(1/\lambda)\cdot\Big(\Gamma\big((h,e),\eta(\zeta_\lambda(h),e)\big) - (\zeta_{-\lambda}^\uparrow)_* \cdot \eta^\Gamma\big(\zeta_\lambda(h),e\big)\Big)$$

belongs to $Vp_{E,(h,e)}$.

Hence, because of the property of the connection Γ, one obtains

$$(p_E)_{*,(h,e)} \cdot [\Xi^{\uparrow}, \eta^{\Gamma}] = (p_E)_{*,(h,e)} \cdot \Gamma\big((h,e), \partial\eta/\partial\Xi(h,e)\big) = (\partial\eta/\partial\Xi)(h,e).$$

This concludes the proof.

Theorem 1 allows us to compute the local expression of the derivative along a direction. Let $\Xi \in \chi(H)$ be a direction with the coordinate expression

$$\Xi(h) = \Xi^a(x,h) \cdot \partial z^a.$$

Then, according to (1.4), one has

$$(\partial\eta/\partial\Xi)(h,e) = \Xi^a(h) \cdot \big((\partial\eta^i/\partial z^a)(h,e) \cdot \partial x^i_{,e} + (\partial\eta^\alpha/\partial z^a)(h,e) \cdot \partial u^\alpha_{,e}\big). \tag{2.11}$$

The above relation leads to the following conclusion: the derivative along a direction depends only on the value of the direction at the point. □

3. The variational equation of a (p.s.v.f)

(3.a) The evolution map. Let $r \in S(\pi)$ be a section; we set

$$r^{\uparrow}\colon\ H \to E,$$
$$r^{\uparrow}(h) = r\big(\chi(h)\big);$$

then $r^{\uparrow}$ becomes a morphism between the fibred manifold H and E.

Proposition 3. *Let $h_0 \in H$ be a fixed point. Then there exist*
- *a real number $\varepsilon > 0$,*
- *a fibred neighborhood U_0 of h_0*
- *a map*

$$\overline{\gamma}\colon\ (-\varepsilon,\varepsilon) \times U_0 \to H \oplus E$$

with the property

$$\overline{\gamma}_{*,(t,h)} \cdot \partial t = \eta\big(\overline{\gamma}(t,h)\big), \qquad \overline{\gamma}(0,h) = \big(h, r^{\uparrow}(h)\big). \tag{3.1}$$

Proof. By a continuity argument, we can see that the map

$$\overline{\gamma}(t,h) = \overline{\varphi}_t\big(h, r^{\uparrow}(h)\big) \tag{3.2}$$

is well defined on a neighborhood of $(0,h_0) \in \mathbb{R} \times H$ and satisfies (3.1). We call the map $\overline{\gamma}$ the *evolution map.* □

Corollary 1. *Let $\Xi \in V\chi_{,h_0}$ be a vertical vector and let us denote*

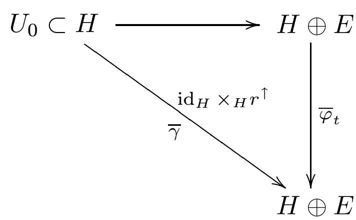
$$e_0 = r^{\uparrow}(h_0) = r(x_0).$$

Then the relation

$$\overline{\gamma}_{*,(t,h_0)} \cdot \Xi = (\overline{\varphi}_t)_{*,(h_0,e_0)} \cdot \Xi^{\uparrow}(h_0, e_0) \tag{3.3}$$

holds for every t belonging to the set $(-\varepsilon, \varepsilon) \subset \mathbb{R}$.

Proof. We can rewrite the relation (3.2) as the diagram



for every t belonging to $(-\varepsilon, \varepsilon)$. It is easy to check that

$$(\underset{H}{\mathrm{id}} \times_H r^{\uparrow})_{*,h_0} \cdot \Xi = \Xi^{\uparrow}(h_0, e_0).$$

This concludes the proof. □

(3.b) The variational vector along the integral curve of s. One notices that the curve

$$t \to \overline{\gamma}(t, h_0)$$

is nothing else but the integral curve of $\overline{\eta}$ starting from (h_0, e_0). Let us suppose η is a (p.s.v.f). According to (2.8), one gets

$$t \to \overline{\gamma}(t, h_0) = \overline{c}(t) = \big(c'(t), \mathbf{c}(t)\big).$$

Then, because of (3.3), one obtains a vector field along $\overline{c}(t)$, denoted by

$$\overline{J}(\Xi)\colon \ \big[\overline{c}(t)\big] \to T(H \oplus E)$$

and defined by the relation

$$\overline{J}(\Xi)\big(\overline{c}(t)\big) = (\overline{\gamma}_*)_{,(t,h_0)} \cdot \Xi = (\overline{\varphi}_t)_{*,(h_0,e_0)} \cdot \Xi^{\uparrow}(h_0, e_0). \tag{3.4}$$

According to the decomposition (1.7), one gets

$$\overline{J}(\Xi)\big(\overline{c}(t)\big) = \overline{J}^{v}(\Xi)\big(\overline{c}(t)\big) + \overline{J}^{h}(\Xi)\big(\overline{c}(t)\big) \tag{3.5}$$

where $\overline{J}^{v}(\Xi) \in VP_{E,\overline{c}(t)}$ and $\overline{J}^{h}(\Xi)\big(\overline{c}(t)\big) \in Hp_{E,\overline{c}(t)}$.

Definition 2. Let σ be a connection of the fibred manifold $\chi\colon H \to B$ and let $\Xi \in V\chi_{,h_0}$ be a vertical vector. Then the variational vector field induced by σ and Ξ along the curve $\mathbf{c}(t) = \mathbf{s}_t(h_0)$ is the field

$$
\begin{aligned}
&J(\Xi)\colon\ \big[\mathbf{c}(t)\big] \to TE,\\
&J(\Xi) = (p_E)_{*,\overline{c}(t)} \cdot \overline{J}^h(\Xi)\big(\overline{c}(t)\big).
\end{aligned}
\tag{3.6}
$$

(3.c) The variational equation of a (p.s.v.f). Let

$$j(\Xi)\colon\ \big[c'(t)\big] \to TH$$

be the vector field defined by the relation

$$j(\Xi)\big(c'(t)\big) = (\varphi'_t)_{*,h_0} \cdot \Xi \tag{3.7}$$

Then $j(\Xi)$ splits into the decomposition

$$j(\Xi)\big(c'(t)\big) = j^h(\Xi)\big(c'(t)\big) + f^v(\Xi)\big(c'(t)\big),$$

where $j^h(\Xi)\big(c'(t)\big) \in H\chi_{,c'(t)}$, $j^v(\Xi)\big(c'(t)\big) \in V\chi_{,c'(t)}$.

We set

$$\widetilde{\Xi}\big(c'(t)\big) = P_\sigma \cdot j(\Xi)\big(c'(t)\big) = j^v(\Xi)\big(c'(t)\big). \tag{3.8}$$

Theorem 2. *The formula*

$$\big[\mathbf{s}, J(\Xi)\big]\big(\mathbf{c}(t)\big) = (\partial\eta/\partial\widetilde{\Xi})\big(c(t)\big) \tag{3.9}$$

holds.

Proof. First, we prove that the relation

$$P_{\overline{\sigma}} \cdot \overline{J}(\overline{\Xi})\big(\overline{c}(t)\big) = \overline{J}^v(\Xi)\big(\overline{c}(t)\big) = \Big(\widetilde{\Xi}\big(c'(t)\big)\Big)^{\uparrow} \tag{$*$}$$

holds. Taking into account (2.5) and the expression of $p_H!$, one gets

$$
\begin{aligned}
P_{\overline{\sigma}} \cdot \overline{J}(\Xi)\big(\overline{c}(t)\big) &= \mu_E^{-1} \circ \big(\underset{H\oplus E}{\mathrm{id}} \times_H P_\sigma\big) \circ p_H! \cdot \big\{(\overline{\varphi}_t)_{*,(h_0,e_0)} \cdot \Xi^{\uparrow}(h_0,e_0)\big\}\\
&= \mu_E^{-1} \circ \Big(\big(c'(t), \mathbf{c}(t)\big), P_\sigma \cdot j(\Xi)_{,c'(t)}\Big)\\
&= \mu_E^{-1} \circ \Big\{\big(c'(t), \mathbf{c}(t)\big), \widetilde{\Xi}\big(c'(t)\big)\Big\}\\
&= \Big(\widetilde{\Xi}\big(c'(t)\big)\Big)^{\uparrow}\big(\overline{c}(t)\big).
\end{aligned}
$$

Along the set $(-\varepsilon, \varepsilon) \times \{h_0\}$, one has

$$[\partial t, \Xi] = 0.$$

Thus, one obtains

$$(\overline{\gamma}_*)_{,(t,h)} \cdot [\partial t, \Xi] = \big[\overline{\eta}, \overline{J}(\Xi)\big]\big(\overline{c}(t)\big) = 0.$$

Consequently, one obtains

$$\begin{aligned} 0 &= \big[\overline{\eta}, \overline{J}(\Xi)\big]\big(\overline{c}(t)\big) = \big[\overline{\eta}, \overline{J}^v(\Xi) + \overline{J}^h(\Xi)\big]\big(\overline{c}(t)\big) \\ &= \big[\overline{\eta}, \overline{J}^v(\Xi)\big]\big(\overline{c}(t)\big) + \big[\overline{\eta}, \overline{J}^h(\Xi)\big]\big(\overline{c}(t)\big), \end{aligned}$$

and so

$$\big[\overline{J}^v(\Xi), \overline{\eta}\big]\big(\overline{c}(t)\big) = \big[\overline{\eta}, \overline{J}^h(\Xi)\big]\big(\overline{c}(t)\big).$$

Applying $(p_H)_*$ on both sides of the equality and taking into account Theorem 1 and $(*)$, one obtains

$$\big[\mathbf{s}, J(\Xi)\big]\big(\mathbf{c}(t)\big) = (\partial\eta/\partial\widetilde{\Xi})\big(\mathbf{c}(t)\big).$$

This concludes the proof. □

4. Examples

(4.a) Parameter dependent vector fields. Let M be a manifold; we set

$$\begin{aligned} H = \mathbb{R}^m \times M \longrightarrow M, &\qquad (\lambda, x) \longrightarrow x, \\ E = M \longrightarrow M, &\qquad x \longrightarrow x. \end{aligned}$$

Then a (s.v.f) is a map

$$X\colon\ \mathbb{R}^m \times M \to M, \quad (\lambda, x) \to X(\lambda, x) \in T_xM.$$

In the paper [T1] we called such a (s.v.f) a parameter dependent vector field. It models the classical theory of the differential equations depending on parameters.

We notice that the integral sections are the constants.

Consequently, if $\lambda \in \mathbb{R}^m$, then it induces the vector field

$$\lambda\colon\ M \to TM, \quad \lambda(x) = X(\lambda, x).$$

The reader can verify that the formula (3.9) has the form

$$\big[\lambda, J(\Xi)\big]\big(\lambda_t(x)\big) = (\partial X/\partial\Xi)\big(\lambda_t(x)\big). \tag{4.1}$$

The above relation is nothing else but the variational equation of the differential equations depending on parameters.

(4.b) The geodesic flow. An example of a parameter dependent vector is the geodesic flow of the principal frame bundle $B(M)$ of the manifold M. Let θ be the canonical form and let ω be a connection form on $B(M)$. Then, if $\xi \in \mathbb{R}^m$ is a vector, there exists only one standard horizontal vector field

$$B(\xi)\colon\ B(M) \to TB(M)$$

defined by the relations

$$\theta\big(B(\xi)\big) = \xi,$$
$$\omega\big(B(\xi)\big) = 0.$$

We define the parameter depending vector field

$$\begin{aligned} &B\colon\ \mathbb{R}^m \times B(M) \to TB(M),\\ &B(\xi, u) = B(\xi)(u) = \sum_{i=1}^{m} \xi^i \cdot B(e_i)(u), \end{aligned} \tag{4.3}$$

where we denote by $e_1, \ldots, e_m$ the canonical basis on $\mathbb{R}^m$ and set

$$\xi = (\xi^1, \ldots, \xi^m).$$

A direction (vertical vector) $\Xi \in \chi\big(\mathbb{R}^m \times B(M)\big)$ can be written as

$$\Xi = \Xi^i(\xi, u) \cdot \partial e_i.$$

Let $\xi_0 \in \mathbb{R}^m$ be a fixed point; then the relation (4.1) combined with (4.3) leads to the formula

$$\big[B(\xi_0), J(\Xi)\big] = \big(\xi_t(u)\big) = \sum_{i=1}^{m} \Xi^i\big(\xi_t(u), u\big) \cdot B(e_i)\big(\xi_t(u)\big), \tag{4.4}$$

where $\big[t \to \xi_t(u)\big]$ represents the integral curve of $B(\xi_0)$ starting from u.

We have proved in [T2] that $J(\Xi)$ projects onto a Jacobi vector field along the geodesic determined by $B(\xi_0)$. This is the reason why we call (4.4) the lift of the Jacobi equation to the frame principal bundle.

(4.c) The system of horizontal projectable vector fields. Let $\pi\colon E \to B$ be a fibred manifold endowed with the connection

$$\gamma\colon\ E \times_B TB \to TE.$$

The lift of a vector $X_{,x} \in T_x B$ is defined as follows:

$$X_e^h = \gamma\big((e,x), X_{,x}\big), \quad \pi(e) = x,$$

as is well known.

Consequently, we obtain a (p.s.v.f), devoted

$$\gamma\colon\ TB \times_B E \to TE$$

and described by the relation

$$\gamma(X_{,x}, e) = X_e^h \in T_e E. \tag{4.5}$$

Let σ be a general connection on TB; then $\overline{\sigma}$ is a connection of the fibred manifold $p_E\colon\ TB \oplus E \to E$ and so one obtains

$$\begin{aligned}&\overline{\gamma}\colon\ TB \oplus E \to T(TB \oplus E),\\ &\overline{\gamma}(X_x, e) = \overline{\sigma}\big((X_{,x}, e), X_e^h\big).\end{aligned}$$

The projection of $\overline{\gamma}$ along p_H is nothing else but the lift of X with respect to the connection σ. So one obtains

$$\begin{aligned}&\gamma'\colon\ TB \to T(TB),\\ &\gamma'(X_{,x}) = \sigma(x, X_{,x}).\end{aligned}$$

Let $X_{,x} \in TB$ be fixed and let $\Xi \in V\chi_{,x}$ be a vertical vector in $X_{,x}$. Then, according to (3.9), one obtains the equation

$$\big[X^h, J(\Xi)\big]\big(\mathbf{c}(t)\big) = (\partial\gamma/\partial\widetilde{\Xi})\big(\mathbf{c}(t)\big).$$

The above equation has a simple form. Let (x^i), $i = 1, \ldots, m$ be a local coordinate chart of B and let (x^i, u^α), $\alpha = 1, \ldots, n$ be an adapted chart of E. The adapted chart of $H = TB$ becomes $(x^i, \dot{x}^j)$, $j = 1, \ldots, m$. The connection γ is described by the relations

$$\gamma(e, \partial x^i_{,x}) = \partial x^i_{,e} + \gamma^\alpha_i(e) \cdot \partial u^\alpha_{,e} = \delta x^i_{,e}.$$

Then, if $\Xi = \Xi^j \cdot \partial \dot{x}^j$, one obtains

$$\big[X^h, J(\Xi)\big]\big(\mathbf{c}(t)\big) = \widetilde{\Xi}^j \cdot \delta x^j$$

where we set

$$\widetilde{\Xi}\big(c'(t)\big) = \widetilde{\Xi}^j\big(c'(t)\big) \cdot \partial \dot{x}^j_{,c'(t)}.$$

Let us suppose we have defined an adapted covariant derivative on the fibred manifold E, denoted ∇.

Proposition 4. *The vector field $J(\Xi)$ is a Jacobi field along the geodesic $\mathbf{c}(t)$ if and only if the following relation is fulfilled:*

$$\nabla_{\partial\gamma/\partial\widetilde{\Xi}} X^h + \nabla_{X^h} \partial\gamma/\partial\widetilde{\Xi} = 0. \tag{4.6}$$

Proof. One has

$$\nabla_{X^h} J(\Xi) - \nabla_{J(\Xi)} X^h - T\big(X^h, J(\Xi)\big) = \partial\gamma/\partial\widetilde{\Xi}.$$

Deriving this relation with respect to ∇ and taking into account that X^h is parallel along $\mathbf{c}(t)$, one obtains

$$\nabla_{X^h}\nabla_{X^h} J(\Xi) + \Omega\big(J(\Xi), X^h\big)X^h + \nabla_{X^h} T\big(J(\Xi), X^h\big) = \nabla_{\partial\gamma/\partial\widetilde{\Xi}} X^h + \nabla_{X^h} \partial\gamma/\partial\widetilde{\Xi}.$$

We have denoted by Ω and T the curvature and the torsion tensors with respect to ∇ (see [K-N] for details). □

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