Czechoslovak Mathematical Journal

Yusuf Karakuş

Best simultaneous L_p approximations

Czechoslovak Mathematical Journal, Vol. 48 (1998), No. 3, 457–463

Persistent URL: http://dml.cz/dmlcz/127432

Czechoslovak Mathematical Journal, 48 (123) (1998), 457–463

BEST SIMULTANEOUS L_p APPROXIMATIONS

Yusuf Karakuş, Adana

(Received November 22, 1995)

Abstract. In this paper we study simultaneous approximation of n real-valued functions in $L_p[a,b]$ and give a generalization of some related results.

1. Best simultaneous L_p approximations

The problem of simultaneous approximation to two or more real-valued functions belonging to $L_p\,[a,b]$ by elements of a subset S of $L_p\,[a,b]$ has been studied by several authors. Phillips and Sahney [3] gave results for the L_1 and L_2 norms.The problem of the best simultaneous approximation to an arbitrary number of functions discussed by Holland and Sahney [1], who generalized the results in [3] for the L_2 norms. Ling [2] considered simultaneous Chebyshev approximation in the Sum norm. Holland, McCabe, Phillips and Sahney [4] considered the best simultaneous L_1 approximations and studied the relation between the best simultaneous approximations and the best L_1 approximations to the arithmetics mean of n functions. The problem of simultaneous L_p approximation to two real valued functions f_1 and f_2 when p is an odd natural number was discussed by Karakuş in [5] and when p is non-integer real number by Karakuş-Atacik in [6].

In this paper we study the best simultaneous L_p approximation to n functions.

Definition 1. Let $p \geqslant 1$ be real number and $S \subset L_p[a,b]$ a non-empty set of real-valued functions. Let us assume that real-valued functions $f_1, f_2, \ldots, f_n$ and all $s \in S$ are L_p integrable. If there exists an element $s^* \in S$ such that

$$\inf_{s\in S} \sum_{i=1}^{n} \|f_i - s\|_p^p = \sum_{i=1}^{n} \|f_i - s^*\|_p^p, \tag{1}$$

then s^* is said to be a *best simultaneous approximation to the functions* $f_1, f_2, \ldots, f_n$ *in the* L_p *norm.*

Theorem 1. *Let* f_i, $i = 1, 2, \ldots, n$ $(n \geqslant 2)$ *and* s *be as defined above.*
a) *If* p *is an even natural number, then*

$$
\tag{2}
\begin{aligned}
&\inf_{s\in S} \sum_{i=1}^{n} \|f_i - s\|_p^p \\
&\quad = \inf_{s\in S} \left\{ \frac{2}{n-1} \sum_{k=0}^{p/2} \binom{p}{2k} \sum_{i<j} \int_a^b \left[\frac{f_i(x)+f_j(x)}{2} - s(x) \right]^{p-2k} \right. \\
&\qquad \left. \times \left[\frac{f_i(x)-f_j(x)}{2} \right]^{2k} \mathrm{d}x \right\}
\end{aligned}
$$

b) *If* p *is an odd natural number, the*

$$
\tag{3}
\begin{aligned}
&\inf_{s\in S} \sum_{i=1}^{n} \|f_i - s\|_p^p = \inf_{s\in S} \left\{ \frac{2}{n-1} \sum_{k=0}^{(p-1)/2} \binom{p}{2k} \right. \\
&\times \sum_{i<j} \int_a^b \left[\max\left\{ \left| \frac{f_i(x)+f_j(x)}{2} - s(x) \right|, \left| \frac{f_i(x)-f_j(x)}{2} \right| \right\} \right]^{p-2k} \\
&\times \left. \left[\min\left\{ \left| \frac{f_i(x)+f_j(x)}{2} - s(x) \right|, \left| \frac{f_i(x)-f_j(x)}{2} \right| \right\} \right]^{2k} \mathrm{d}x \right\}.
\end{aligned}
$$

We first prove the following lemma.

Lemma 1. *Let* $n \geqslant 2$ *be a natural number and let* $1 \leqslant i < j \leqslant n$. *For arbitrary real numbers* a_i, a_j *and* $p \geqslant 1$ *let*

$$
\tag{4}
A_{ij} = \binom{p}{2k} \left(\frac{a_i + a_j}{2} \right)^{p-2k} \left(\frac{a_i - a_j}{2} \right)^{2k}.
$$

a) *If* p *is an even natural number, then*

$$
\tag{5}
\sum_{i=1}^{n} a_i^p = \frac{2}{n-1} \sum_{k=0}^{p/2} \sum_{i<j} A_{ij}.
$$

b) *If* p *is an odd natural number, then*

$$
\tag{6}
\sum_{i=1}^{n} a_i^p = \frac{2}{n-1} \sum_{k=0}^{(p-1)/2} \sum_{i<j} A_{ij}.
$$

c) *If p is a non-integer real number, $a_i + a_j = 1$ and $-1 \leqslant a_i - a_j \leqslant 1$, then*

$$\sum_{i=1}^{n} a_i^p = \frac{2}{n-1} \sum_{k=0}^{\infty} \sum_{i<j} A_{ij}. \tag{7}$$

Proof. a) To prove Lemma 1(a), we use the identity

$$(a+b)^p + (a-b)^p = 2 \sum_{k=0}^{p/2} \binom{p}{2k} a^{p-2k} b^{2k} \tag{8}$$

where p is an even natural number and a, b are arbitrary real numbers. Let us choose $a + b = a_i$, $a - b = a_j$. Then we have

$$a_i^p + a_j^p = 2 \sum_{k=0}^{p/2} A_{ij}. \tag{9}$$

By using (9), we obtain

$$\begin{aligned} \frac{2}{n-1} \sum_{k=0}^{p/2} \sum_{i<j} A_{ij} &= \frac{2}{n-1} \sum_{k=0}^{p/2} \{(A_{12} + A_{13} + \ldots + A_{1n}) \\ &\quad + (A_{23} + \ldots + A_{2n}) + \ldots + (A_{(n-1)n})\} \\ &= \frac{2}{n-1} \Big\{ \Big(\frac{a_1^p + a_2^p}{2} + \frac{a_1^p + a_3^p}{2} + \ldots + \frac{a_1^p + a_n^p}{2} \Big) \\ &\quad + \Big(\frac{a_2^p + a_3^p}{2} + \ldots + \frac{a_2^p + a_n^p}{2} \Big) + \ldots + \Big(\frac{a_{n-1}^p + a_n^p}{2} \Big) \Big\} \\ &= a_1^p + a_2^p + \ldots + a_n^p. \end{aligned} \tag{10}$$

b) In this case, we have

$$a_i^p + a_j^p = 2 \sum_{k=0}^{(p-1)/2} A_{ij}. \tag{11}$$

The proof of part (b) is similar to part (a).

c) By using the series

$$(1+y)^p + (1-y)^p = 2 \sum_{k=0}^{\infty} \binom{p}{2k} y^{2k}, \quad -1 \leqslant y \leqslant 1 \tag{12}$$

and writing $a_i + a_j = 1$, $a_i - a_j = y$ we have

$$a_i^p + a_j^p = 2 \sum_{k=0}^{\infty} A_{ij}. \tag{13}$$

Using this result, we obtain (7). □

P r o o f o f T h e o r e m 1. We first show the existence of the right hand side of (2) in the sense of the L_p norm. From the Hölder inequality

$$\int_a^b |g(x)h(x)|\,\mathrm{d}x \leqslant \left[\int_a^b |g(x)|^r\,\mathrm{d}x\right]^{1/r}\left[\int_a^b |h(x)|^t\,\mathrm{d}x\right]^{1/t}$$

where $1/r+1/t=1$, $g\in L_r$ and $h\in L_t$. If any $s\in S$, $1/r=(p-2k)/p$ and $1/t=2k/p$ we have

$$\begin{aligned}
&\frac{2}{n-1}\sum_{k=0}^{p/2}\binom{p}{2k}\sum_{i<j}\int_a^b\left[\frac{f_i(x)+f_j(x)}{2}-s(x)\right]^{p-2k}\left[\frac{f_i(x)-f_j(x)}{2}\right]^{2k}\mathrm{d}x\\
&\quad\leqslant\frac{2}{n-1}\sum_{k=0}^{p/2}\binom{p}{2k}\sum_{i<j}\left\{\int_a^b\left[\frac{f_i(x)+f_j(x)}{2}-s(x)\right]^{p}\mathrm{d}x\right\}^{(p-2k)/p}\\
&\quad\quad\times\left\{\int_a^b\left[\frac{f_i(x)-f_j(x)}{2}\right]^{p}\mathrm{d}x\right\}^{2k/p}\\
&\quad=\frac{2}{n-1}\sum_{k=0}^{p/2}\binom{p}{2k}\sum_{i<j}\left\|\frac{f_i+f_j}{2}-s\right\|_p^{p-2k}\left\|\frac{f_i-f_j}{2}\right\|_p^{2k},
\end{aligned}$$

which implies the existence of the right hand side of (2). On the other hand, for any $s\in S$ and a pair (i,j) we define

$$\begin{aligned}
g_s(x)&=\max\left\{\left|\frac{f_i(x)+f_j(x)}{2}-s(x)\right|,\left|\frac{f_i(x)-f_j(x)}{2}\right|\right\},\\
h_s(x)&=\min\left\{\left|\frac{f_i(x)+f_j(x)}{2}-s(x)\right|,\left|\frac{f_i(x)-f_j(x)}{2}\right|\right\}.
\end{aligned}$$

We have g_s, $h_s\in L_p$. The existence of the right hand side of (3) in the sense of the L_p norm is shown as in the proof of (2).

a) Let $s\in S$ and $a_i=f_i-s$. Then by Lemma 1(a)

$$\begin{aligned}
(14)\qquad \sum_{i=1}^{n}[f_i(x)-s(x)]^p&=\frac{2}{n-1}\sum_{k=0}^{p/2}\binom{p}{2k}\\
&\quad\times\sum_{i<j}\left[\frac{f_i(x)+f_j(x)}{2}-s(x)\right]^{p-2k}\left[\frac{f_i(x)-f_j(x)}{2}\right]^{2k}.
\end{aligned}$$

Integrating each side from a to b and then taking the infimum over all $s\in S$, we obtain (2).

b) Let $s \in S$ and $a_i = |f_i - s|$. Then by Lemma 1(b)

$$\sum_{i=1}^{n} |f_i(x) - s(x)|^p \tag{15}$$
$$= \frac{2}{n-1} \sum_{k=0}^{(p-1)/2} \binom{p}{2k} \sum_{i<j} \left[\frac{|f_i(x) - s(x)| + |f_j(x) - s(x)|}{2}\right]^{p-2k} \left[\frac{|f_i(x) - s(x)| - |f_j(x) - s(x)|}{2}\right]^{2k}.$$

For arbitrary real numbers m and n we have identities

$$\begin{aligned} |m+n| + |m-n| &= 2\max\{|m|, |n|\}, \\ \big||m+n| - |m-n|\big| &= 2\min\{|m|, |n|\}. \end{aligned} \tag{16}$$

If we replace $m+n$ and $m-n$ in (16) by $f_i(x) - s(x)$ and $f_j(x) - s(x)$, respectively, and use (15), we obtain

$$\sum_{i=1}^{n} |f_i(x) - s(x)|^p$$
$$= \frac{2}{n-1} \sum_{k=0}^{(p-1)/2} \binom{p}{2k} \sum_{i<j} \left[\max\left\{\Big|\frac{f_i(x) + f_j(x)}{2} - s(x)\Big|, \Big|\frac{f_i(x) - f_j(x)}{2}\Big|\right\}\right]^{p-2k}$$
$$\times \left[\min\left\{\Big|\frac{f_i(x) + f_j(x)}{2} - s(x)\Big|, \Big|\frac{f_i(x) - f_j(x)}{2}\Big|\right\}\right]^{2k}.$$

Integrating both sides of this equality from a to b and taking the infimum over all $s \in S$, we have the result of Theorem 1(b).

Remark 1. In Theorem 1(a):

a) If we take $p = 2$ we see that Theorem 1 in [1] is a special case of Theorem 1(a). On the other hand, if we take $n = 2$, we obtain Theorem 3 in [2].

b) If p is an even natural number, then for $n = 2$ Theorem II in [1] is a special case of Theorem 1(a).

Remark 2. In Theorem 1(b):

a) If we replace $[a, b]$ by $[0, 1]$ put $p = 1$ and $n = 2$, then ve obtain Theorem 2 in [3].

b) If we take $p = 1$, we see that Theorem 5 in [4] is a special case of Theorem 1(b). Really, $\operatorname{sgn}(f_i(x) - s(x))$ is always positive or negative according to the hypothesis of Theorem 5 in [4]. Hence

$$\max\left\{\Big|\frac{f_i(x) + f_j(x)}{2} - s(x)\Big|, \Big|\frac{f_i(x) - f_j(x)}{2}\Big|\right\} = \Big|\frac{f_i(x) + f_j(x)}{2} - s(x)\Big|.$$

Then (3) implies

$$\inf_{s\in S}\sum_{i=1}^{n}\|f_i - s\|_1 = \inf_{s\in S}\left\{\frac{2}{n-1}\int_a^b\sum_{i<j}\Big(\frac{f_i(x)+f_j(x)}{2} - s(x)\Big)\,\mathrm{d}x\right\}$$
$$= \inf_{s\in S} n\left\|\frac{1}{n}\sum_{i=1}^{n} f_i - s\right\|_1.$$

c) If p is an odd natural number, then for $n = 2$ Theorem 1 in [5] is a special case of Theorem 1(b).

Theorem 2. *Let $f_1, f_2, \ldots, f_n$ and s be as in Definition 1 and let $p > 1$ be a non-integer real number. If $f_i(x) - s(x) \neq 0$, then*

$$\inf_{s\in S}\sum_{i=1}^{n}\|f_i - s\|_p^p$$
$$= \inf_{s\in S}\left\{\frac{2}{n-1}\sum_{k=0}^{\infty}\binom{p}{2k}\sum_{i<j}\int_a^b\left[\max\left\{\left|\frac{f_i(x)+f_j(x)}{2} - s(x)\right|, \left|\frac{f_i(x)-f_j(x)}{2}\right|\right\}\right]^{p-2k}\right.$$
$$\left.\times\left[\min\left\{\left|\frac{f_i(x)+f_j(x)}{2} - s(x)\right|, \left|\frac{f_i(x)-f_j(x)}{2}\right|\right\}\right]^{2k}\,\mathrm{d}x\right\}.$$

P r o o f. The existence of the right hand side of the equality in the sense of the L_p-norm is shown as in the proof of Theorem 1(b) and using the absolute convergence of the series on the right hand side under the given hypothesis. To prove Theorem 2 it is sufficient to take $a_i = |f_i(x) - s(x)|$ in Lemma 1(c). □

Remark 3. In Theorem 2, if we take $n = 2$, we see that Theorem in [6] is a special case of Theorem 2.

References

[1] *A.S.B. Holland and B.N. Sahney*: Some remarks on best simultaneous approximation. Theory of Approximation with Application (A.G. Law and B.N. Sahney, eds.). Academic Press, New York, 1976, pp. 332–337.

[2] *W.H. Ling*: On simultaneous Chebyshev approximation in the sum norm. Proc. Amer. Mat. Soc. *48* (1975), 185–188.

[3] *G.M. Phillips and B.N. Sahney*: Best simultaneous approximation in the L_1 and L_2 norms. Theory of Approximation with Applications (A.G. Law and B.N. Sahney, eds.). Academic press, New York, 1976, pp. 213–219.

[4] *A.S.B. Holland, J.H. McCabe, G.M. Phillips and B.N. Sahney*: Best simultaneous L_1-approximations. Journal of Approximation Theory *24* (1978), 361–365.

[5] *Y. Karakuş*: Simultaneous approximation in L_p norm. Doga-Turkish Journal of Mathematics, Tübitak-Ankara *15* (1991), 25–28.

[6] *Y. Karakuş and S. Atacik*: Simultaneous Approximation in $L_p[a, b]$ when p is non-integer real number. Doga-Turkish Journal of Mathematics, Tübitak-Ankara *15* (1991), 165–168.

Author's address: University of Çukurova, Faculty of Arts and Sciences, Department of Mathematics, 01330 Adana, Turkey.