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Ram U. Verma

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GENERALIZED VARIATIONAL INEQUALITIES AND ASSOCIATED NONLINEAR EQUATIONS

Ram U. Verma, Orlando

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Abstract. Here we consider the solvability based on iterative algorithms of the generalized variational inequalities and associated nonlinear equations.

1. Introduction

The theory of nonlinear variational inequalities [1] has turned out to be a powerful tool in providing us a unified framework in dealing with a wide class of problems in physics, economics, and engineering sciences. The study of associated nonlinear equations is equally important in the sense that a class of variational inequalities are equivalent to some associated equations involving strongly monotone operators and other combinations leading to the strongly monotone operators. The strongly monotone operators and their variant forms are widely applied in variational inequalities as well as in hemivariational inequalities [2]. For a selected detail on nonlinear equations, we refer to [3–6].

We consider the solvability of a generalized variational inequality involving strongly monotone and relaxed Lipschitz operators. Among the special cases of the obtained result is the variational inequality problem of Yao [7].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let K be a nonempty closed convex subset of H and P_K be the projection of H onto K.

We consider for given operators $f, T\colon H \to H$, the generalized variational inequality (GVI) problem: Find an element x in H such that $f(x)$ is in K and

$$\langle f(x) - T(x), v - f(x)\rangle \geqslant 0 \quad \text{for all } v \text{ in } K. \tag{1}$$

Next, we need recall some necessary definitions for our problem at hand.

Definition 2.1. An operator $T\colon H \to H$ is said to be a relaxed Lipschitz operator if for a given constant $k \geqslant 0$,

$$\langle T(u) - T(v), u - v\rangle \leqslant -k\|u - v\|^2 \quad \text{for all } u,\ v \text{ in } H. \tag{2}$$

The operator T is called Lipschitz continuous if for a constant $m > 0$,

$$\|T(u) - T(v)\| \leqslant m\|u - v\| \quad \text{for all } u,\ v \text{ in } H. \tag{3}$$

When $m = 1$ in (3), the operator T is said to be nonexpansive, that is,

$$\|T(u) - T(v)\| \leqslant \|u - v\| \quad \text{for all } u,\ v \text{ in } H. \tag{4}$$

Definition 2.2. An operator $f\colon H \to H$ is said to be strongly monotone if for all u, v in H and for a constant $r > 0$,

$$\langle f(u) - f(v), u - v\rangle \geqslant r\|u - v\|^2. \tag{5}$$

Inequality (5) implies that

$$\|f(u) - f(v)\| \geqslant r\|u - v\|. \tag{6}$$

The operator f satisfying (6) is called r-expanding, and when $r = 1$, f is called just expanding.

3. Nonlinear equations and variational inequalities

In this section we first give some lemmas on the equivalence of variational inequalities to some sort of nonlinear Wiener-Hopf equations. Then we consider the main result on the solvability of the GVI problems.

Lemma 3.1 [1]. *For a given element z in H, an element x in K satisfies*

$$\langle x - z, v - x\rangle \geqslant 0 \quad \text{for all } v \text{ in } K, \tag{7}$$

iff $x = P_K z$.

Lemma 3.2. *An element x in H such that $f(x)$ is in K is a solution of the GVI (1) iff x in H with $f(x)$ in K satisfies the equation*

$$f(x) = P_K[(1-\lambda)f(x) + \lambda T(x)], \tag{8}$$

where $\lambda \geqslant 0$ is arbitrary.

Proof. The proof follows from an application of Lemma 3.1. □

Based on (8) we generate an iterative algorithm

Algorithm 3.1. For $n = 0, 1, 2, \ldots,$

$$f(x_{n+1}) = P_K[(1-\lambda)f(x_n) + \lambda T(x_n)]. \tag{9}$$

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space H and let $f\colon H \to H$ be strongly monotone and Lipschitz continuous with corresponding constants 1 and $s \geqslant 1$. Let $T\colon H \to H$ be relaxed Lipschitz and Lipschitz continuous with corresponding constants $k \geqslant 0$ and $m \geqslant 1$. Then the sequences $\{x_n\}$ and $\{f(x_n)\}$, as generated by Algorithm 3.1 with x_0 in H, $f(x_0)$ in K, and*

$$\begin{aligned}&\Bigl|\lambda - \frac{1+k+p(1-p)}{1+2k+m^2-p^2}\Bigr|\\ &< \frac{\sqrt{[1+k+p(1-p)]^2 - (1+2k+m^2-p^2)[1-(1-p)^2]}}{1+2k+m^2-p^2},\end{aligned}$$

where $1+k > p(p-1) + \sqrt{(1+2k+m^2-p^2)[1-(1-p)^2]}$, $1+2k+m^2-p^2 > 0$, $k < m$, and $p = \sqrt{s^2-1} < 1$, converge to x and $f(x)$, respectively, the solution of the equation (8).

P r o o f. Since operator P_K is nonexpansive, we obtain

$$
\begin{aligned}
(10)\qquad &\|f(x_{n+1})-f(x_n)\|\\
&\quad\leqslant \|(1-\lambda)f(x_n)+\lambda T(x_n)-(1-\lambda)f(x_{n-1})-\lambda T(x_{n-1})\|\\
&\quad= \|(1-\lambda)[f(x_n)-f(x_{n-1})]+\lambda[T(x_n)-T(x_{n-1})]\|\\
&\quad\leqslant \|(1-\lambda)\{x_n-x_{n-1}-[f(x_n)-f(x_{n-1})]\}\|\\
&\qquad+\|(1-\lambda)(x_n-x_{n-1})+\lambda[T(x_n)-T(x_{n-1})]\|.
\end{aligned}
$$

Since

$$
\begin{aligned}
(11)\qquad &(1-\lambda)^2\|x_n-x_{n-1}-[f(x_n)-f(x_{n-1})]\|^2\\
&= (1-\lambda)^2\{\|x_n-x_{n-1}\|^2-2\langle f(x_n)-f(x_{n-1}),x_n-x_{n-1}\rangle\\
&\quad+\|f(x_n-f(x_{n-1})\|^2\}\\
&\leqslant (1-\lambda)^2(s^2-1)\|x_n-x_{n-1}\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
(12)\qquad &\|(1-\lambda)(x_n-x_{n-1})+\lambda[T(x_n)-T(x_{n-1})]\|^2\\
&= (1-\lambda)^2\|x_n-x_{n-1}\|^2+2\lambda(1-\lambda)\langle T(x_n)-T(x_{n-1}),x_n-x_{n-1}\rangle\\
&\quad+\lambda^2\|T(x_n)-T(x_{n-1})\|^2\\
&\leqslant \{(1-\lambda)^2-2\lambda(1-\lambda)k+\lambda^2m^2\}\|x_n-x_{n-1}\|^2,
\end{aligned}
$$

this implies that

$$
\begin{aligned}
(13)\qquad \|f(x_{n+1})-f(x_n)\| &\leqslant \big((1-\lambda)\sqrt{s^2-1}\\
&\quad+\sqrt{(1-\lambda)^2-2\lambda(1-\lambda)k+\lambda^2m^2}\big)\|x_n-x_{n-1}\|.
\end{aligned}
$$

Since f is strongly monotone with constant 1 (and hence expanding), it follows that

$$
\begin{aligned}
(14)\qquad &\|x_{n+1}-x_n\|\leqslant\|f(x_{n+1})-f(x_n)\|\\
&\leqslant\{(1-\lambda)p+\sqrt{(1-\lambda)^2-2\lambda(1-\lambda)k+\lambda^2m^2}\}\|x_n-x_{n-1}\|,
\end{aligned}
$$

where $p=\sqrt{s^2-1}$. Therefore,

$$
(15)\qquad \|x_{n+1}-x_n\|\leqslant\theta\|x_n-x_{n-1}\|,
$$

where $\theta=(1-\lambda)p+\sqrt{(1-\lambda)^2-2\lambda(1-\lambda)k+\lambda^2m^2}$. Now, it follows that $0<\theta<1$ for all λ such that

$$
\begin{aligned}
&\Big|\lambda-\frac{1+k+p(1-p)}{1+2k+m^2-p^2}\Big|\\
&\quad<\frac{\sqrt{[1+k+p(1-p)]^2-(1+2k+m^2-p^2)[1-(1-p)^2]}}{1+2k+m^2-p^2},
\end{aligned}
$$

where $1+k > p(p-1)+\sqrt{(1+2k+m^2-p^2)[1-(1-p)^2]}$, $1+2k+m^2-p^2>0$, $k<m$, and $p=\sqrt{s^2-1}<1$.

Consequently, for all q in N,

$$\|x_{n+q}-x_n\| \leqslant \frac{\theta^n}{1-\theta}\|x_1-x_0\|. \tag{16}$$

This implies that $\{x_n\}$ is a Cauchy sequence, and since H is complete, there exists an element x in H such that $x_n \to x$. Now the Lipschitz continuity of the operators f and T implies the solvability of (8). Therefore, it leads to the solvability of the GVI (1). □

Corollary 3.1. *Let $T\colon H \to H$ be relaxed Lipschitz and Lipschitz continuous with respective constants $k \geqslant 0$ and $m \geqslant 1$, and let $f\colon H \to H$ be strongly monotone (with constant 1) and nonexpansive. Then the sequence $\{x_n\}$ and $\{f(x_n)\}$, as generated by Algorithm 3.1 for x_0 in H, and $f(x_0)$ in K and, $0<\lambda<2(1+k)/(1+2k+m^2)$, converge, respectively, to x and $f(x)$, the solution of (8).*

Corollary 3.2. *When f is the identity, Theorem 3.1 reduces to [7, Theorem 3.6].*

P r o o f o f C o r o l l a r y 3.1. Since under the assumptions using (11) and (12),

$$\begin{aligned}&(1-\lambda)^2\|x_n-x_{n-1}-[f(x_n)-f(x_{n-1})]\|^2\\ &\qquad\leqslant (1-\lambda)^2(1-1)\|x_n-x_{n-1}\|^2=0,\end{aligned}$$

and

$$\begin{aligned}&\|(1-\lambda)(x_n-x_{n-1})+\lambda[T(x_n)-T(x_{n-1})]\|^2\\ &\qquad\leqslant \{(1-\lambda)^2-2\lambda(1-\lambda)k+\lambda^2m^2\}\|x_n-x_{n-1}\|^2,\end{aligned}$$

it follows that

$$\|x_{n+1}-x_n\| \leqslant \theta\|x_n-x_{n-1}\|, \tag{17}$$

where $\theta=\sqrt{(1-\lambda)^2-2\lambda(1-\lambda)k+\lambda^2m^2}$. Now the rest of the proof is similar to that of Theorem 3.1. □

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Author's addresses: International Publications, 12046 Coed Drive, Orlando, Florida 32826, U.S.A.; Instituto per la Ricerca di Base, Division of Mathematics, I-86075 Monteroduni (IS), Molise, Italy.