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ON THE EXTENSION OF D-POSET VALUED MEASURES

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Abstract. A variant of Alexandrov theorem is proved stating that a compact, subadditive D-poset valued mapping is continuous. Then the measure extension theorem is proved for MV-algebra valued measures.

Keywords: D-posets, extension of measures, observables in quantum mechanics

MSC 2000: 28E10

0. Introduction

The notion of a D-poset was introduced in a connection with quantum mechanical models ([5], [6], [7], [9], [12], [13]). If H is a D-poset, then a state is a morphism from H to the unit interval, an observable is a morphism from the Borel σ-algebra $\mathscr{B}(\mathbb{R})$ to H. An observable $x\colon \mathscr{B}(\mathbb{R}) \to H$ can be regarded as an H-valued measure.

For constructions of some observables it is useful to construct them first on intervals and then extend them to the family of all Borel sets. Therefore we first prove a variant of the Alexandrov theorem (with values in D-posets) and then a measure extension theorem (with values in MV-algebras).

Of course, the results mentioned are formulated and proved only for some special cases of D-posets.

Our variant of the Alexandrov theorem works in the so-called regular D-posets ([15]). Recall that every l-group as well as any MV-algebra is regular in the mentioned sense ([18] and Section 3). Our extension theorem works in the so-called weakly σ-distributive MV algebras. Of course in the case of Riesz spaces H the condition is necessary and sufficient for extendability of every H-valued measure from a ring to the generated σ-ring ([21]).

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1. Preliminaries

A D-poset is a partially ordered set with the greatest element 1 and the least element 0 and a partial binary operation—such that $a - b$ is defined if $b \leqslant a$, which satisfies the following conditions:

(i) If $b \leqslant a$, then $a - b \leqslant a$ and $a - (a - b) = b$.

(ii) If $a \leqslant b \leqslant c$, then $c - b \leqslant c - a$ and $(c - a) - (c - b) = b - a$.

A D-poset is called a D-σ-poset, if $b_i \leqslant a_i$, $a_i \nearrow a$, $b_i \searrow b$ implies $a_i - b_i \nearrow a - b$.

Let $\mathscr{R}$ be an algebra of subsets of a set Ω, let H be a D-poset. By an H-valued measure on $\mathscr{R}$ we shall understand a mapping $\mu\colon \mathscr{R} \to H$ satisfying the following conditions:

(i) $\mu(\emptyset) = 0$.

(ii) If $A, B \in \mathscr{R}$ and $A \subset B$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

(iii) If $a_n \in \mathscr{R}$, $A_n \subset A_{n+1} \subset A$ $(kn = 1, 2, \ldots)$, $A \in \mathscr{R}$ and $A = \bigcup\limits_{n=1}^{\infty} A_n$, then $\mu(A_n) \nearrow \mu(A)$.

An observable is a measure $x\colon \mathscr{B}(\mathbb{R}) \to H$ defined on the σ-algebra of all Borel subsets of $\mathbb{R}$ such that $x(\mathbb{R}) = 1$.

Since $a \leqslant 1$ for all $a \in H$, $1 - a$ exists. We shall denote $a^{\perp} = 1 - a$. Two elements a_1, a_2 are said to be orthogonal, if $a_1 \leqslant a_2^{\perp}$ (or equivalently $a_2 \leqslant a_1^{\perp}$). For orthogonal elements the sum $a_1 + a_2$ can be defined by the formula

$$a_1 + a_2 = 1 - ((1 - a_1) - a_2)\,,$$

or equivalently by the formula

$$a_1 + a_2 = 1 - ((1 - a_2) - a_1)\,.$$

The notion of orthogonality can be extended to an arbitrary finite collection of elements. A family P of $n+1$ elements is orthogonal, if every its subfamily containing at most n elements is orthogonal and every element of P is orthogonal to the sum of the remaining elements. If $a_1, \ldots, a_n$ are orthogonal, then we define again by induction

$$a_1 + \ldots + a_n = (a_1 + \ldots + a_{n-1}) + a_n.$$

We shall prove the following simple assertion.

Proposition 1. *If $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant a_{n+1}$, then the sum $(a_1 - a_2) + (a_2 - a_3) + \ldots + (a_n - a_{n+1}) = a_1 - a_{n+1}$ exists.*

P r o o f. Assume first that $c \leqslant b \leqslant a$. Then we shall prove that $a - b$ and $b - c$ are orthogonal and

$$(a - b) + (b - c) = a - c.$$

Indeed,

$$b - c = (a - c) - (a - b) \leqslant 1 - (a - b)$$

so that $b - c$ and $a - b$ are orthogonal. Moreover,

$$\begin{aligned}(a - b) + (b - c) &= 1 - ((1 - (a - b)) - (b - c)) \\ &= 1 - ((1 - (a - b)) - ((a - c) - (a - b))) \\ &= 1 - (1 - (a - c)) = a - c.\end{aligned}$$

Namely, $f \leqslant e \leqslant d$ implies $1 - d \leqslant 1 - e \leqslant 1 - f$, hence

$$\begin{aligned}(d - f) - (e - f) &= ((1 - f) - (1 - d)) - ((1 - f) - (1 - e)) \\ &= (1 - e) - (1 - d) = d - e.\end{aligned}$$

The assertion can be now proved by induction. □

If we now consider an H-valued measure $\mu \colon \mathscr{R} \to H$ and a finite chain $A_1 \supset A_2 \supset A_n \supset A_{n+1}$ of sets of $\mathscr{R}$, then $\mu(A_1) \geqslant \mu(A_2) \geqslant \ldots \geqslant \mu(A_n) \geqslant \mu(A_{n+1})$, the sum

$$\begin{aligned}(\mu(A_1) - \mu(A_2)) &+ (\mu(A_1) - \mu(A_3)) + \ldots + (\mu(A_n) - \mu(A_{n+1})) \\ &= \mu(A_1 \setminus A_2) + \mu(A_2 \setminus A_3) + \ldots + \mu(A_n \setminus A_{n+1})\end{aligned}$$

exists and equals $\mu(A_1) - \mu(A_{n+1}) = \mu(A_1 \setminus A_{n+1})$. Similarly, if $B_1, \ldots, B_n$ are pairwise disjoint sets of $\mathscr{R}$ and we put

$$A_{n+1} = \emptyset,\ A_n = B_n,\ A_{n-1} = B_n \cup B_{n-1}, \ldots, A_1 = B_1 \cup B_2 \cup \ldots \cup B_n,$$

then $A_1 \supset A_2 \supset \ldots \supset A_{n+1}$, $B_i = A_i \setminus A_{i+1}$ $(i = 1, \ldots, n)$, $A_1 \setminus A_{n+1} = \bigcup\limits_{i=1}^{n} B_i$, hence

$$\mu(B_1) + \mu(B_2) + \ldots + \mu(B_n) = \mu\Bigl(\bigcup_{i=1}^{n} B_i\Bigr).$$

2. Alexandrov theorem

We will say that a D-poset H is weakly regular ([15]), if for any bounded double sequence $(a_{ij})_{i,j}$ such that $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$) and every $a > 0$ there are $i_1 < i_2 < i_3 < \ldots$ such that

$$a \leqslant b_1 + \ldots + b_n$$

holds for no n and no $b_1 \leqslant a_{1i_1}$, $b_2 \leqslant a_{2i_2}$, …, $b_n \leqslant a_{ni_n}$ with existing sum $b_1 + \ldots + b_n$.

It is interesting that every commutative lattice ordered group is weakly regular ([18]).

We will say that a mapping $\lambda\colon \mathscr{R} \to H$ is subadditive, if

$$\lambda(A) \leqslant \lambda(A_1) + \lambda(A_2) + \ldots + \lambda(A_n)$$

whenever $A \subset A_1 \cup A_2 \cup \ldots \cup A_n$ and $A_1, \ldots, A_n$ are pairwise disjoint.

A family $\mathscr{C}$ of subsets of Ω is called compact, if for every sequence $(C_n)_n \subset \mathscr{C}$

$$\left(\forall n\colon \bigcap_{i=1}^{n} C_i \neq \emptyset\right) \Longrightarrow \bigcap_{i=1}^{\infty} C_i \neq \emptyset.$$

A mapping $\lambda\colon \mathscr{R} \to H$ is called compact, if there exists a compact family $\mathscr{C}$ such that for every $A \in \mathscr{R}$ there are $B_j \in \mathscr{R}$, $C_j \in \mathscr{C}$, $B_j \subset C_j \subset A$ ($j = 1, 2, \ldots$) and $\lambda(A \setminus B_j) \searrow 0$ ($j \to \infty$).

A mapping $\lambda\colon \mathscr{R} \to H$ is upper continuous in $\emptyset$, if $\lambda(A_n) \searrow 0$ whenever $A_n \searrow \emptyset$.

A D-poset H is called monotonously σ-complete, if every decreasing sequence $(a_n)_n$ in H has the greatest lower bound $\bigwedge_{n=1}^{\infty} a_n$ and every increasing sequence $(b_n)_n$ in H has the least upper bound $\bigvee_{n=1}^{\infty} b_n$.

Theorem 1 (Alexandrov). *Let H be a weakly regular, monotonously σ-complete D-poset. Let $\lambda\colon \mathscr{R} \to H$ be subadditive and compact. Then λ is upper continuous in $\emptyset$.*

Proof. Let $A_n \searrow \emptyset$. We want to prove $\lambda(A_n) \searrow 0$. Since $(\lambda(A_n))_n$ is decreasing, there exists $a = \bigwedge_{n=1}^{\infty} \lambda(A_n)$. Assuming $a > 0$, we obtain a contradiction.

Since λ is compact, for every i there are $C_j^i \in \mathscr{C}$ and $B_j^i \in \mathscr{R}$ such that $B_j^i \subset C_j^i \subset A_i$ and

$$a_{ij} = \lambda(A_i \setminus B_j^i) \searrow 0 \quad (j \to \infty, i = 1, 2, \ldots).$$

Since H is weakly regular, there are $i_1 < i_2 < \ldots$ such that

$$a \leqslant b_1 + \ldots + b_n$$

for no n and no $b_1, \ldots, b_n$ with the corresponding properties. Put

$$B_n = B^1_{i_1} \cap B^2_{i_2} \cap \ldots \cap B^n_{i_n}, \ C_n = C^1_{i_1} \cap C^2_{i_2} \cap \ldots \cap C^n_{i_n}.$$

We want to prove that $C_n \neq \emptyset$ $(n = 1, 2, \ldots)$. Indirectly, let $C_n = \emptyset$ for some n. Then $B_n = \emptyset$ and

$$\begin{aligned} a &\leqslant \lambda(A_n) = \lambda(A_n \setminus B_n) \\ &= \lambda\left((A_1 \setminus B_n) \cup (A_2 \setminus A_1 \setminus B_n) \cup \ldots \cup (A_n \setminus A_{n-1} \setminus B_n)\right) \\ &\leqslant \lambda\left((A_1 \setminus B^1_{i_1}) \cup (A_2 \setminus A_1 \setminus B^2_{i_2}) \cup \ldots \cup (A_n \setminus A_{n-1} \setminus B^n_{i_n})\right) \\ &\leqslant \lambda(A_1 \setminus B^1_{i_1}) + \lambda(A_2 \setminus A_1 \setminus B^2_{i_2}) + \ldots + \lambda(A_n \setminus A_{n-1} \setminus B^n_{i_n}). \end{aligned}$$

Put $b_1 = \lambda(A_1 \setminus B^1_{i_1})$, $b_2 = \lambda(A_2 \setminus A_1 \setminus B^2_{i_2})$, …, $b_n = \lambda(A_n \setminus A_{n-1} \setminus B^n_{i_n})$.

Then $b_1 = a_{1i_1}$, $b_2 \leqslant a_{2i_2}$, …, $b_n \leqslant a_{ni_n}$ and $b_1 + \ldots + b_n$ exists, hence

$$a \leqslant b_1 + \ldots + b_n,$$

which is impossible. We have obtained a contradiction, hence $C_n \neq \emptyset$ for all n.

Since $C_n \in \mathscr{C}$, $C_n \neq \emptyset$, $C_n \supset C_{n+1}$ $(n = 1, 2, \ldots)$ and $\mathscr{C}$ is a compact family, we obtain $\bigcap\limits_{n=1}^{\infty} C_n \neq \emptyset$. But $\bigcap\limits_{n-1}^{\infty} C_n \subset \bigcap\limits_{n=1}^{\infty} A_n = \emptyset$, which contradicts the assumption $a > 0$. Therefore $0 = a = \bigwedge\limits_{n=1}^{\infty} \lambda(A_n)$. □

As an example of using the above theorem we present the following result concerning the Lebesgue-Stieltjes measure.

Theorem 2. *Let H be a weakly regular, monotonously σ-complete D_σ-poset. Let $F\colon \mathbb{R} \to H$ be a mapping satisfying the following conditions:*

(i) *F is increasing,*

(ii) $\bigwedge\limits_{n=1}^{\infty} F(-n) = 0$,

(iii) *$\bigvee\limits_{n=1}^{\infty} F(t_n) = F(t)$ for every increasing sequence $(t_n)_n$ such that $t_n \nearrow t$.*

Let $\mathscr{R}$ be the ring generated by the family of all intervals $\langle a, b)$, $a, b \in \mathbb{R}$, $a < b$. Then there is exactly one measure $\lambda_F\colon \mathscr{R} \to H$ such that

$$\lambda_F\left(\langle a, b)\right) = F(b) - F(a)$$

whenever $a, b \in \mathbb{R}$, $a < b$.

P r o o f. If $\langle a_1, b_1), \ldots, \langle a_n, b_n)$ are pairwise disjoint, then we denote

$$\lambda_F\Bigl(\bigcup_{i=1}^{n}\langle a_i, b_i)\Bigr) = (F(b_1) - F(a_1)) + (F(b_2) - F(a_2)) + \ldots + (F(b_n) - F(a_n))\,.$$

By Proposition 1 the above sum exists and does not depend on the choice of $\langle a_i, b_i)$. By the definition of a D-poset the additivity of λ_F follows.

Let $\mathscr{C}$ be the family of all compact subsets of the set $\mathbb{R}$ of real numbers. Evidently $\mathscr{C}$ is a compact family. Moreover, if $A \in \mathbb{R}$, $A = \bigcup\limits_{i=1}^{n} \langle a_i, b_i)$, $\langle a_i, b_i)$ pairwise disjoint, put

$$B_j = \bigcup_{i=1}^{n}\Bigl\langle a_i, b_i - \frac{2}{j}\Bigr), \quad C_j = \bigcup_{i=1}^{n}\Bigl\langle a_i, b_i - \frac{1}{j}\Bigr\rangle.$$

Then $C_j \in \mathscr{C}$, $B_j \in \mathscr{R}$, $B_j \subset C_j \subset A$ and

$$A \setminus B_j = \bigcup_{i=1}^{n}\Bigl\langle b_i - \frac{1}{j}, b_i\Bigr),$$

$$\lambda_F(A \setminus B_j) = \biggl(F(b_1) - F\Bigl(b_1 - \frac{2}{j}\Bigr)\biggr) + \ldots + \biggl(F(b_n) - F\Bigl(b_n - \frac{2}{j}\Bigr)\biggr)$$

for sufficiently large j. Since H is a D_σ-poset and F satisfies (iii), we obtain $F(b_i) - F(b_i - \frac{2}{j}) \searrow 0$ $(j \to \infty,\ i = 1, \ldots, n)$ and

$$\lambda_F(A \setminus B_j) \searrow 0.$$

By Theorem 1, λ_F is upper continuous in $\emptyset$. Finally, the additivity and the upper continuity in $\emptyset$ imply the continuity of λ_F. □

3. Measure extension theorem

In this section we will assume that H is an MV σ-algebra ([2], [3], [10], [11], [13], [14]). It is known that H is a D_σ-lattice ([3]). An MV σ-algebra H is called weakly σ-distributive, if for any double sequence $(a_{ij})_{i,j}$ such that $a_{ij} \searrow 0$ $(j \to \infty,\ i = 1, 2, \ldots)$, we have

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0.$$

Theorem 3. *Let $\mathscr{R}$ be a ring of subsets of a set Ω, $\mu\colon \mathscr{R} \to H$ a measure. Let H be a weakly σ-distributive MV algebra. Then there is exactly one measure $\overline{\mu}\colon \sigma(\mathscr{R}) \to H$ extending μ.*

P r o o f. The main tools are the measure extension theorem for group-valued measures from [16] and the Mundici representation theorem from [14]. For every MV-algebra H there is an abelian lattice ordered group G with a strong unit u such that $H = \langle 0, u \rangle$, where

$$a \oplus b = (a+b) \wedge u,\ a^* = u - a,\ 1 = u,\ a \odot b = (a^* \oplus b^*)^*.$$

The measure $\mu\colon \mathscr{R} \to H \subset G$ can be regarded as a measure with values in G. We shall prove that G satisfies the following assumptions of the group valued measure extension theorem:

(i) G is σ-complete,

(ii) G is weakly σ-distributive.

The σ-completeness of G follows by the σ-completness of $\langle 0, u \rangle$. It can be proved by the same method as it was done in [10], Lemma 1.1.

We shall prove the weak σ-distributivity of G. Let $(a_{ij})_{i,j}$ be a bounded sequence of G, $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$). Put

$$b_{ij} = u \wedge a_{ij}.$$

Then

$$b_{ij} \searrow 0\ (j \to \infty,\ i = 1, 2, \ldots).$$

By the weak σ-distributivity of H we obtain

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} b_{i\varphi(i)} = 0.$$

Since G is a distributive lattice,

$$0 = \bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} b_{i\varphi(i)} = u \wedge \Bigl(\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \Bigr).$$

Put

$$v = \bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)},$$

hence

$$u \wedge v = 0.$$

Since every strong unit in G is a weak unit ([1], Chapter XIII, §11, Lemma 4), the relation $u \wedge v = 0$ implies $v = 0$, hence

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0.$$

By [16] there exists a measure $\overline{\mu}\colon \sigma(\mathscr{R}) \to G$ extending μ. Since $\mu(A) \leqslant u$ for all $A \in \mathscr{R}$, $\overline{\mu}(B) \leqslant u$ for all $B \in \sigma(\mathbb{R})$. Indeed, for every $B \in \sigma(\mathbb{R})$ there exists a sequence $(A_n) \subset \mathscr{R}$ such that $A_n \subset A_{n+1}$ $(n = 1, 2, \ldots)$ and $B \subset \bigcup\limits_{n=1}^{\infty} A_n$. Therefore $\overline{\mu}(B) \leqslant \overline{\mu}\Big(\bigcup\limits_{n=1}^{\infty} A_n\Big) = \bigvee\limits_{n=1}^{\infty} \overline{\mu}(A_n) = \bigvee\limits_{n=1}^{\infty} \mu(A_n) \leqslant u$.

Since $\overline{\mu}(B) \leqslant u$ for every $B \in \sigma(\mathbb{R})$, the mapping $\overline{\mu}$ can be regarded as a mapping $\overline{\mu}\colon \sigma(\mathscr{R}) \to \langle 0, u\rangle = H$. The mapping $\overline{\mu}$ is additive and continuous, hence $\overline{\mu}\colon (\mathscr{R}) \to H$ is a measure extending μ.

If $\nu\colon \sigma(\mathscr{R}) \to H$ is a measure extending μ, then $\mathscr{K} = \big\{A \in \sigma(\mathscr{R});\ \overline{\mu}(A) = \nu(A)\big\} \supset \mathscr{R}$ and $\mathscr{K}$ is a monotone family. Therefore $\mathscr{K} \supset \mathscr{M}(\mathscr{R}) = \sigma(\mathscr{R})$, hence $\overline{\mu}(A) = \nu(A)$ for every $A \in \sigma(\mathscr{R})$. □

Theorem 4. *Let H be a weakly σ-distributive MV σ-algebra. Let $F\colon \mathbb{R} \to H$ be a mapping satisfying the following conditions:*

(i) *F is increasing,*

(ii) $\bigwedge\limits_{n=1}^{\infty} F(-n) = 0$,

(iii) $\bigvee\limits_{n=1}^{\infty} F(n) = 1$,

(iv) $\bigvee\limits_{n=1}^{\infty} F(t_n) = F(t)$ *for every increasing sequence $(t_n)_n$ such that $t_n \nearrow t$.*

Then there exists exactly one observable $x\colon \mathscr{B}(\mathbb{R}) \to H$ such that $x((-\infty, t)) = F(t)$ for every $t \in \mathbb{R}$.

Proof. First we prove that every MV algebra considered as a D-poset is weakly regular. Let $a_{ij} \searrow 0$ $(j \to \infty,\ i = 1, 2, \ldots)$, $a_{ij} \in H$, $a \in H$, $a > 0$. Let G be a group and u a strong unit of G such that $H = \langle 0, u\rangle$. Since every abelian l-group G is weakly regular ([18]), there are $i_1 < i_2 < i_3 < \ldots$ such that

$$a \leqslant a_{1i_1} + \ldots + a_{ni_n}$$

does not hold for any n. Hence, if $b_1, \ldots, b_n \in H$ are such that $b_1 + \ldots + b_n$ exists and $b_1 \leqslant a_{1i_1}, \ldots, b_n \leqslant a_{ni_n}$, then $a \leqslant b_1 + \ldots + b_n$ implies $a \leqslant a_{1i_1} + \ldots + a_{ni_n}$, which is impossible.

Now by Theorem 2 there is exactly one measure

$$\begin{aligned} &\lambda_F\colon \mathscr{R} \to H \quad \text{such that} \\ &\lambda_F\left(\langle a, b)\right) = F(b) - F(a) \end{aligned}$$

whenever $a, b \in \mathbb{R}$, $a < b$. Then also

$$\begin{aligned} \lambda_F\left((-\infty, t)\right) &= \lambda_F\Bigl(\bigcup_{n=1}^{\infty}\langle t-n, t)\Bigr) \\ &= \bigvee_{n=1}^{\infty}\left(F(t) - F(t-n)\right) \\ &= F(t) - \bigwedge_{n=1}^{\infty} F(t-n) = F(t) \end{aligned}$$

for every $t \in \mathbb{R}$. By Theorem 3 there exists exactly one measure (denote it by x) extending λ_F. Therefore

$$x\left((-\infty, t)\right) = \lambda_F\left((-\infty t)\right) = F(t)$$

for every $t \in \mathbb{R}$. Moreover, $x\colon \mathscr{B}(\mathbb{R}) \to H$ is additive, continuous and $x(\mathbb{R}) = x\Bigl(\bigcup\limits_{n=1}^{\infty}\langle -n, n)\Bigr) = \bigvee\limits_{n=1}^{\infty}\left(F(n) - F(-n)\right) = \bigvee\limits_{n=1}^{\infty} F(n) - \bigwedge\limits_{n=1}^{\infty} F(-n) = 1 - 0 = 1$, hence x is an observable. $\square$

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