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THE MAXIMUM GENUS, MATCHINGS AND THE CYCLE SPACE OF A GRAPH

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Abstract. In this paper we determine the maximum genus of a graph by using the matching number of the intersection graph of a basis of its cycle space. Our result is a common generalization of a theorem of Glukhov [5] and a theorem of Nebeský [15].

Keywords: Maximum genus, matching, cycle space

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1. Introduction

The maximum genus $\gamma_M(G)$ of a connected graph G is the largest integer k such that G has a cellular embedding in the orientable surface of genus k. Formally the maximum genus was introduced by Nordhaus, Stewart and White [16] in 1971, but already in 1970 Khomenko and Yavorsky [11, Section 5] derived a criterion for a graph to have a 2-cell embedding with a single face and by using it calculated the maximum genus of the complete bipartite graphs $K_{n,n}$ and the n-cubes Q_n.

One of the most remarkable facts about the maximum genus is that this topological invariant can be characterized in a purely combinatorial manner. The first combinatorial characterization of the maximum genus is due to Khomenko, Ostroverkhy and Kuzmenko [10]. Basically the same result was later independently proved by Xuong [21], and an essential part of it by Jungerman [8]. It is convenient to state these results in terms of an equivalent quantity called the *Betti deficiency* of G. It is

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defined by the equation $\xi(G) = \beta(G) - 2\gamma_M(G)$, where $\beta(G) = |E(G)| - |V(G)| + 1$ is the Betti number, i.e., the cycle rank of G.

For a graph H, let $c(H)$, $d(H)$ and $b(H)$ denote the number of components, the number of components with odd number of edges, and the number of components with odd Betti number, respectively.

Theorem 1. (Xuong [21]) *Let G be a connected graph. Then*

$$\xi(G) = \min\{d(G - E(T));\ T \text{ a spanning tree of } G\}.$$

A characterization which is in a sense complementary was subsequently found by Khomenko and Glukhov [9] and independently by Nebeský [13] in a slightly different form. In contrast to Theorem 1, these results express the Betti deficiency as the maximum of a certain combinatorial function.

Let A be a subset of $E(G)$. Set $\nu(G, A) = c(G - A) + b(G - A) - |A| - 1$.

Theorem 2. (Nebeský [13]) *Let G be a connected graph. Then*

$$\xi(G) = \max\{\nu(G, A);\ A \subseteq E(G)\}.$$

Extensions, generalizations and variations on these two theorems were established by various authors, see e.g. [2, 14, 15, 18]. In this paper we pursue the connection between the maximum genus and matchings and prove a characterization of the maximum genus of a graph similar to Theorem 2 by employing the classical theorems due to Hall [7], Tutte [20] and its generalization due to Berge [1].

The relationship between maximum genus and matchings is implicit in the proofs of Theorem 1. By Theorem 1, a connected graph G has zero Betti deficiency (equivalently, it admits a single-face 2-cell embedding in some orientable surface) if and only if it has a spanning tree whose cotree consists of components with even number of edges. Such a component can be decomposed into pairs of adjacent edges, and each of the pairs contributes to the maximum genus by one. The adjacent pairs form what is sometimes called an adjacency matching (see, e.g., [6]) and in turn corresponds to a usual matching in a suitably defined graph. In general, the Betti deficiency of a graph equals the minimum number of edges not covered by an adjacency matching.

As regards Theorem 2 and similar characterizations, the role of matchings is less obvious. The first step in revealing this connection was taken by Glukhov [5]. To state this result, let B be a basis of the cycle space $\mathscr{C}(G)$ of a graph G. Define the graph $J(G, B)$ to have B as its vertex set, with two elements $C_1 \neq C_2$ in B adjacent if they have a vertex in common. We call $J(G, B)$ the *intersection graph* of the basis B. Note that $J(G, B)$ is connected whenever G is 2-edge-connected.

Theorem 3. (Glukhov [5, Theorem 1]) *A connected graph G has a single-face orientable 2-cell embedding (i.e., $\xi(G) = 0$) if and only if for each basis B of the cycle space of G the intersection graph $J(G, B)$ of B has a perfect matching.*

As a further step in this direction, Nebeský [15] proved the following. For a tree T of a graph G let $G\#T$ be the graph with $V(G\#T) = E(G) - E(T)$ and with the property that ef, where $e, f \in E(G) - E(T)$, forms an edge if $T + e + f$ has only one non-trivial leaf. (Recall that a *leaf* of G is a 2-edge-connected subgraph maximal with respect to inclusion.) Let $\omega(H)$ denote the number of unsaturated vertices of a maximum matching in a graph H.

Theorem 4. (Nebeský [15, Theorem 1]) *Let G be a connected graph different from a tree. Then*

$$\xi(G) = \max\{\omega(G\#T);\ T \text{ a spanning tree of } G\}.$$

Moreover, there is a spanning tree Y of G such that

$$d(G - E(Y)) = \xi(G) = \omega(G\#Y).$$

Note, however, that every spanning tree T of a graph G determines the standard basis B_T of $\mathscr{C}(G)$ where each element of B_T uses only one cotree edge. Moreover, one easily derives from the definitions that $G\#T$ coincides with $J(G, B_T)$. These observations suggest that there should be a common generalization of Glukhov's Theorem 3 and Nebeský's Theorem 4. The aim of this paper is to prove such a theorem.

Theorem 5. *Let G be a connected graph. Then*

$$\xi(G) = \max\{\omega(J(G, B));\ B \text{ a basis of } \mathscr{C}(G)\}.$$

Moreover, there is a spanning tree Y of G such that

$$d(G - E(Y)) = \xi(G) = \omega(J(G, B_Y)).$$

It is easy to find a basis for $\mathscr{C}(G)$ that is not of the form B_T for some spanning tree T of G. It follows that the maximum in Theorem 5 is taken over a larger set than in Theorem 4, and so our main result indeed improves Nebeský's Theorem 4.

2. Definitions and auxiliary results

All graphs considered in this paper are finite and may have loops or multiple edges.

A *circuit* in a graph G is a connected regular subgraph of valency 2, whereas a *cycle* is a subgraph of G in which every vertex has even valency greater than or equal to 2. The *cycle space* $\mathscr{C}(G)$ of a graph G is the vector space over the 2-element field spanned by the cycles of G; the sum of two vectors is obtained by taking the symmetric difference of the corresponding sets of edges and omitting all resulting isolated vertices. It follows that the non-zero elements of $\mathscr{C}(G)$ are cycles. The dimension of $\mathscr{C}(G)$ is $\beta(G) = |E(G)| - |V(G)| + k$, where k denotes the number of components of G. It is called the *Betti number* of G.

Let G be a connected graph and let T be a spanning tree of G. For a cotree edge $e \in E(G) - E(T)$ let $T(e)$ denote the unique cycle in $T + e$. Then $\{T(e);\ e \in E(G) - E(T)\}$ is a basis for the cycle space of G. As noted above, not every basis of the cycle space can be obtained in this way.

Our results rely on the use of matchings in graphs. We therefore recall some pertinent definitions and theorems.

Let H be a connected loopless graph. A subset $M \subseteq E(H)$ will be called a *matching* if no two edges in M have a vertex in common. A matching with maximum cardinality is called a *maximum matching* of H and a matching which covers every vertex of H is said to be *perfect*. The size of a maximum matching in the graph H is its *matching number* and is denoted by $\mu(H)$. The number of vertices that are not covered by a maximum matching in H is denoted by $\omega(H)$. If n is the order of H, then $\omega(H) = n - 2\mu(H)$.

For a graph H let $o(H)$ denote the number of components of H with odd order. The following generalization of Tutte's celebrated 1-factor theorem [20] is due to Berge [1].

Theorem 2.1. (Berge) *Let H be a simple graph of order n. Then*

$$\omega(H) = \max\{o(H - X) - |X|;\ X \subseteq V(H)\}.$$

Another result which we need is the König-Hall theorem [7, 12] about matchings in bipartite graphs. For a vertex x of a graph H let $N_H(x)$ denote the neighbourhood of x, the set of vertices adjacent to x. If $X \subseteq V(H)$, let $N_H(X) = \bigcup_{x \in X} N_H(x)$.

Theorem 2.2. (König-Hall) *Let H be a bipartite graph with bipartition (U, W). Then H contains a matching that covers all vertices in U if and only if $N_H(X) \geqslant |X|$ for every subset $X \subseteq U$.*

The following lemma is a bridge between the theorems of Berge and Xuong. Recall that the *line graph* $L(H)$ of a graph H is the graph whose vertices correspond to the edges of H, and where two vertices are joined by an edge if and only if the corresponding edges have an end-vertex in common.

Lemma 2.3. *Let G be a connected graph and let T be a spanning tree of G. Then $d(G-E(T)) = \omega(L(G-E(T)))$.*

Proof. Consider a component K of the cotree $G-E(T)$ and let K have m edges. For each subset $X \subseteq E(K)$ we obviously have $d(K-X) \leqslant |X|$ if m is even, and $d(K-X) \leqslant |X|+1$ if m is odd. Since $\omega(L(K)) = \max\{d(K-X)-|X|;\ X \subseteq E(K)\}$ by Theorem 2.1, we have $\omega(L(K)) = 0$ if m is even, and $\omega(L(K)) = 1$ if m is odd. (In other words, a connected line graph has either a perfect matching or a matching that misses only one vertex, cf. [3, 19].) Hence,

$$\omega(L(G-E(T))) = \sum\{\omega(L(K));\ K \text{ a component of } G-E(T)\} = d(G-E(T)).$$

□

We conclude this section with developing a useful technical machinery to handle maximum-genus problems. It is based on the concept of a frame decomposition which was extensively used by Širáň and Škoviera in [18] within the context of signed graphs. Here, however, we only use the unsigned restriction of this concept.

A pair (F, A) is called a *frame decomposition* of a connected graph G if F, a *frame*, is a connected spanning subgraph of G and $A = E(G) - E(F)$. Denote by $\mathrm{ol}(F)$ the number of leaves of F with odd Betti number.

Lemma 2.4. [9, 18] *Let (F, A) be a frame decomposition of a connected graph G. Then $\xi(G) \geqslant \mathrm{ol}(F) - |A|$.*

A frame decomposition (F, A) is said to be *strong* if it satisfies the following properties:

(1) every non-trivial leaf R of F is *critical*, i.e., $\xi(R) = 1$ and $\xi(R-e) = 0$ for every edge e of R;
(2) (F, A) admits a *pairing*, i.e., there is an injective mapping which to every edge $e \in A$ assigns a non-trivial leaf R_e of F such that R_e is incident with e.

Lemma 2.5. [18] *If (F, A) is a strong frame decomposition of a connected graph G, then $\xi(G) = \mathrm{ol}(F) - |A|$.*

The following theorem is one of the main results of [18].

Theorem 2.6. [18] *Every connected graph admits a strong frame decomposition.*

We note in passing that Theorem 2.6 easily implies Theorem 2. Indeed, if (F, A) is a strong frame decomposition of G and $A' = A \cup I$ where I is the set of all bridges of F, then $\nu(G, A') = \xi(G)$.

3. Proof of Theorem 5

First we show that $\xi(G) \geqslant \omega(J(G, B))$ for every basis B of the cycle space of G. Fix a basis B of $\mathscr{C}(G)$ and choose an optimal spanning tree T in G, i.e., one with $d(G - E(T)) = \xi(G)$. Define a bipartite graph $H = H(B, T)$ with bipartition (U, W) by setting $U = B$, $W = E(G) - E(T)$ and by joining $C \in U$ to $e \in W$ if the edge e belongs to the cycle C. Clearly, $|U| = |W|$. We claim that H has a perfect matching. Suppose not. Theorem 2.2 then implies that there exists a subset $B' \subseteq B$ such that

$$|N_H(B')| < |B'|. \tag{1}$$

On the other hand, the set $D = \{T(e);\ e \in E(G) - E(T)\}$ is also a basis for $\mathscr{C}(G)$. By elementary linear algebra, D contains a subset D' with $|D'| \geqslant |B'|$ such that every cycle $C \in B'$ is a linear combination of elements of D'. Take D' to have the minimum number of elements. Then any cycle $C \in B'$ can be written in the form $C = \sum T(e)$, where all cycles $T(e)$ appearing in this expression belong to D' and $e \in E(C) - E(T)$. By the minimality of D',

$$|D'| = \left| \bigcup_{C \in B'} (E(C) - E(T)) \right|. \tag{2}$$

At the same time, the definition of $H(B, T)$ implies that

$$\bigcup_{C \in B'} (E(C) - E(T)) = \bigcup_{C \in B'} N_H(C) = N_H(B'). \tag{3}$$

However, from (1)–(3) we infer that

$$|B'| \leqslant |D'| = \left| \bigcup_{C \in B'} (E(C) - E(T)) \right| = |N_H(B')| < |B'|,$$

which is absurd. Thus $H(B, T)$ has a perfect matching. We may therefore denote by C_e the cycle from B matched with the cotree edge e. Obviously, if the cotree edges e and f are adjacent, then C_e and C_f intersect. It follows that the bijection $e \mapsto C_e$ provides an isomorphism φ of the line graph $L(G - E(T))$ with a spanning subgraph of $J(G,B)$. By Lemma 2.3, $L(G-E(T))$ contains a matching N with $\xi(G)$ unsaturated vertices. Hence $\varphi(N)$ is a matching in $J(G,B)$ and has $\xi(G)$ unsaturated vertices, too.

Summing up, for every basis B of $\mathscr{C}(G)$ we have obtained

$$\xi(G) \geqslant \omega(J(G,B)),$$

and so

$$\xi(G) \geqslant \max_B \omega(J(G,B)),$$

as well.

To prove the reverse inequality it is sufficient to exhibit a basis B of $\mathscr{C}(G)$ for which $\omega(J(G,B)) \geqslant \xi(G)$. By Theorem 2.6, there is a strong frame decomposition (F,A) of G. We claim that $\omega(J(G,B_T)) \geqslant \xi(G)$ for any spanning tree T of the frame F. (Note that T is at the same time a spanning tree of G.)

Fix a spanning tree T of F and set $Z = \{T(e);\ e \in A\}$. We estimate $\omega(J(G,B_T))$. Theorem 2.1 implies that

$$\begin{aligned}\omega(J(G,B_T)) &= \max\{o(J(G,B_T)-X)-|X|;\ X \subseteq B_T\}\\ &\geqslant o(J(G,B_T)-Z)-|Z|.\end{aligned}$$

However, $J(G,B_T)-Z = J(G-A,B_T)$ is just the disjoint union $\bigcup\limits_R J(R,B_{T\cap R})$ which is taken over all leaves R of F. Recall that $\mathrm{ol}(F)$ is the number of leaves of F with odd Betti number. It follows that

$$o(J(G,B_T)-Z) = \mathrm{ol}(F).$$

Since $|Z| = |A|$, Lemma 2.5 yields

$$\omega(J(G,B_T)) \geqslant \mathrm{ol}(F) - |A| = \xi(G),$$

as claimed. This establishes the first part of our theorem.

In the second part, we again utilize a strong frame decomposition (F,A) of G. So far we have proved that for any spanning tree T of the frame F we have $\omega(J(G,B_T)) = \xi(G)$. Thus to complete the proof it is sufficient to show that among the spanning trees of F there is one, denoted by Y, that is optimal for G.

By the definition of a strong frame decomposition, (F,A) admits a pairing which to every edge $e \in A$ assigns a non-trivial leaf R_e such that e is incident with R_e. Besides paired leaves R_e, $e \in A$, F may contain some unpaired leaves as well. For each $e \in A$ choose in R_e an edge e' adjacent to e and form the set $A' = \{e';\ e \in A\}$. Now, take Y to be any optimal spanning tree of $F - A'$.

To show that Y is optimal also for G, we construct a matching in $L(G - E(Y))$ that has at most $\mathrm{ol}(F) - |A|$ unsaturated vertices. Obviously, the tree $Y_R = Y \cap R$ is an optimal spanning tree for any leaf R of $F - A'$. Every leaf of $F - A'$ is either a leaf S of F that does not contain an edge of A' or a leaf of $R_e - e'$ for some paired leaf R_e of F. In the former case, S is critical; hence $d(S - E(Y_S)) = 1$ and by Lemma 2.3 there is a matching P_S of $L(S - E(Y_S))$ with a single unsaturated vertex. In the latter case we have $d(R_e - e' - E(Y \cap R_e)) = 0$ by the criticality of R_e. The same lemma then implies that the line graph of the corresponding cotree has a perfect matching Q_e. Taking the union of all the perfect matchings Q_e ($e \in A$) with the matchings P_S (S an unpaired leaf of F) and with the additional matching $\{ee';\ e \in A\}$ we obtain a matching M of $L(G - E(Y))$. It is easy to see that M has (at most) one unsaturated vertex per each unpaired leaf S of F and that there are no other unsaturated vertices. Consequently, $\omega(L(G - E(Y)))$ does not exceed the number of unpaired leaves of F, i.e., $\mathrm{ol}(F) - |A|$. Using Lemma 2.5 and Lemma 2.3 again we finally get

$$\xi(G) \leqslant d(G - E(Y)) \leqslant \omega(L(G - E(Y))) \leqslant \mathrm{ol}(F) - |A| = \xi(G)$$

implying that $d(G - E(Y)) = \xi(G)$. Thus Y is an optimal spanning tree of G. The first part of this proof now yields $\omega(J(G, B_Y)) = \xi(G) = d(G - E(Y))$, and the theorem follows.

4. Corollaries

Nebeský's Theorem 3 and Glukhov's Theorem 4 stated in Introduction are obvious corollaries of our Theorem 5. Here we give some more corollaries of this result. We first restate Theorem 5 in a different form.

Theorem 4.1. *Let G be a connected graph. Then the maximum genus of G is*

$$\gamma_M(G) = \min_B \mu(J(G, B)),$$

where the minimum is taken over all bases B of the cycle space of G. Moreover, there is a spanning tree T such that $\gamma_M(G) = \mu(J(G, B_T))$.

Here are some corollaries to Theorem 4.1.

Corollary 4.2. [17] *Let G be a connected graph. Then $\gamma_M(G) = 0$ if and only if no two circuits of G have a vertex in common.*

Before the next corollary, we need two definitions. A *necklace* is a graph with vertex set $V = \{v_1, v_2, \ldots, v_{2r}\}$, such that the vertex v_{2s-1} is connected by a single edge to the vertex v_{2s}, $s = 1, \ldots, r$, and the vertex v_{2s-2} is connected by a pair of parallel edges to the vertex v_{2s-1} (where $v_0 = v_{2r}$), and some loops are added at distinct interior points of those non-multiple edges. Next, a graph G is called a *cluster of three cycles* if it contains a pair of intersecting circuits C_1 and C_2 such that $V(C_1) \cap V(C_2)$ induces a path (possibly a single vertex) and $G - (E(C_1) \cup E(C_2))$ is a path that joins a vertex of C_1 to a vertex of C_2 and is internally disjoint from $C_1 \cup C_2$.

Corollary 4.3. [4] *Let G be a 2-edge-connected graph. Then $\gamma_{M}(G) = 1$ if and only if it is homeomorphic to a necklace or a cluster of three cycles.*

Proof. Using Theorem 4.1 it is easy to check that any graph which is homeomorphic to a necklace or to a cluster of three cycles has maximum genus 1. Conversely, if $\gamma_{M}(G) = 1$ and G is neither homeomorphic to a necklace nor to a cluster of three cycles, then there are two pairs of intersecting circuits, say $\{C_1, C_2\}$ and $\{C_3, C_4\}$, such that C_1, C_2, C_3 and C_4 are linearly independent. Let $e_i \in E(C_i) - \bigcup\limits_{j \neq i} E(C_j)$, where the pairs e_1, e_2 and e_3, e_4 are adjacent. Then for any basis B of the cycle space of G we can find four members of B, say D_1, D_2, D_3 and D_4 such that $e_i \in E(D_i)$ for each $i = 1, \ldots, 4$. It follows that $\{D_1, D_2\}$ and $\{D_3, D_4\}$ are intersecting pairs of cycles in B, whence $\mu(J(G, B)) \geqslant 2$. This contradicts Theorem 4.1. □

One can go on and ask for a description of graphs having maximum genus 2. However, with Theorem 4.1 and Corollary 4.3 this is easier done than said. Roughly speaking, graphs of maximum genus two are certain combinations of two graphs of the kind described in Corollary 4.3, including a cluster of four or five cycles.

5. Concluding remarks

1. The proof of Theorem 5 given above does not depend on Theorem 3 and Theorem 4 which Theorem 5 generalizes. Nonetheless, the inequality $\xi(G) \leqslant \max\ \omega(J(G, B))$ and the existence of a spanning tree Y with $d(G - E(Y)) = \xi(G) = \omega(J(G, B_Y))$ can be derived from Theorem 4 since $J(G, B_Y) = G \# Y$.

2. It seems that the results of this paper might be extended to the maximum Euler genus of a signed graph [18]. A signed graph is a graph whose edges are labelled with signs + and −. An embedding of a signed graph (G, σ) in a closed surface, orientable or non-orientable, is an embedding of its underlying graph G where cycles

of G that preserve or reverse orientation of the surface are specified in advance by the signature σ. This is done as follows. A cycle (or a circuit) of (G, σ) is said to be *balanced* if the product of signs on its edges is $+$. In an embedding of (G, σ), a circuit of G must be embedded in the surface so as to preserve orientation precisely when it is balanced. This kind of embeddings was studied, e.g., in [18] and [22].

It turns out that signed graph embeddings provide a very natural generalization of embeddings in orientable surfaces where the orientable case is obtained simply by only allowing all-positive signatures. In [18], Širáň and Škoviera introduced the maximum Euler genus of a signed graph and proved characterization theorems similar to Theorem 1 (Xuong [21]) and Theorem 2 (Nebeský [13]). Since Theorem 1 and Theorem 2 (or, more precisely, Theorem 2.6 which implies Theorem 2) are crucial to our proofs, there is hope that our present results may be extended to the signed case. However, such an extension is far from immediate. It must include an appropriate definition of the intersection graph $J(G, B)$ to reflect the balance in the signed graph G and may involve extensions of the König-Hall theorem and the Tutte-Berge theorem.

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