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RADICAL CLASSES OF GENERALIZED BOOLEAN ALGEBRAS

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The notion of the radical class of lattice ordered groups was introduced and studied in [4]; cf. also [2], [3], [5], [6], [7].

An analogous notion can be defined for generalized Boolean algebras. Namely, a nonempty subclass of the class $\mathcal{A}$ of all generalized Boolean algebras will be defined to be *a radical class* if it is closed with respect to isomorphisms, convex subalgebras and joins of convex subalgebras.

(For terminology and notation cf. Section 1 below.)

The collection of all radical classes of generalized Boolean algebras will be denoted by $\mathfrak{A}$. For $\mathcal{A}_1, \mathcal{A}_2 \in \mathfrak{A}$ we put $\mathcal{A}_1 \leqslant \mathcal{A}_2$ if $\mathcal{A}_1$ is a subcollection of $\mathcal{A}_2$. The notion of an atom of $\mathfrak{A}$ is defined in the usual way.

In the present paper we prove that there exists an injective mapping ψ of the class of infinite cardinals α into the collection of all atoms of $\mathfrak{A}$ such that, whenever $A \in \psi(\alpha)$, then each interval of A is complete.

Let us mention the following examples of radical classes of generalized Boolean algebras:

(a) The class of all $A \in \mathcal{A}$ such that each interval of A is complete.
(b) The class of all $A \in \mathcal{A}$ such that each interval of A is α-complete, where α is a fixed infinite cardinal.
(c) The class of all $A \in \mathcal{A}$ such that A is completely distributive.
(d) The class of all $A \in \mathcal{A}$ which are α-distributive, where α is a fixed infinite cardinal.
(e) The class of all $A \in \mathcal{A}$ such that each interval of A is finite.

We construct further types of radical classes by applying cardinal functions defined on the class of all Boolean algebras which were introduced in [8].

1. Preliminaries

A lattice L with the least element 0 such that each interval $[0,x]$ of L is a Boolean algebra is called *a generalized Boolean algebra.*

A convex sublattice L_1 of L with $0 \in L_1$ is called *a convex subalgebra* of L. Let us denote by $C(L)$ the system of all convex subalgebras of L. This system is partially ordered by the set-theoretical inclusion. It is clear that $C(L)$ is a complete lattice. The lattice operations in $C(L)$ will be denoted by $\wedge$ and $\vee$.

Let $\{L_i\}_{i\in I}$ be a nonempty subset of $C(L)$. Next, let L^1 be the set of all $x \in L$ such that there exists a finite subset $\{x_1, x_2, \dots, x_n\}$ of $\bigcup\limits_{i\in I} L_i$ with $x = x_1 \vee x_2 \vee \dots \vee x_n$.

1.1. Lemma. *Let $\{L_i\}_{i\in I}$ and L^1 be as above. Then*

(i) $\bigwedge\limits_{i\in I} L_i = \bigcap\limits_{i\in I} L_i$;

(ii) $\bigvee\limits_{i\in I} L_i = L^1$.

The proof is simple and will be omitted.

Let $\mathcal{A}$ be as in the introduction.

1.2. Definition. A nonempty subclass $\mathcal{A}_1$ of $\mathcal{A}$ is called *a radical class* if it satisfies the following conditions:

(i) $\mathcal{A}_1$ is closed with respect to isomorphisms;

(ii) if $A_1 \in \mathcal{A}_1$ and A_2 is a convex subalgebra of A_1, then $A_2 \in \mathcal{A}_1$;

(iii) if $A \in \mathcal{A}$ and A_i $(i \in I)$ are convex subalgebras of A such that $A_i \in \mathcal{A}_1$ for each $i \in I$, then $\bigvee\limits_{i\in I} A_i$ belongs to $\mathcal{A}_1$.

1.3. Lemma. *Let $\{L_i\}_{i\in I}$ be as in 1.1 and let $L^0 \in C(L)$. Then*

$$L^0 \wedge \left(\bigvee_{i\in I} L_i\right) = \bigvee_{i\in I} (L^0 \wedge L_i).$$

Proof. Put

$$P = L^0 \wedge \left(\bigvee_{i\in I} L_i\right), \quad Q = \bigvee_{i\in I} (L^0 \wedge L_i).$$

Clearly $Q \leqslant P$. Let $x \in P$. In view of 1.1 (i) we have $x \in L^0$ and $x \in \bigvee\limits_{i\in I} L_i$. Further, according to 1.1 (ii) we obtain that there are $x_1, x_2, \dots, x_n \in \bigcup\limits_{i\in I} L_i$ such that $x = x_1 \vee x_2 \vee \dots \vee x_n$. From $x \in L^0$ we infer that $x_j = x \wedge x_j \in L^0$ for $j = 1, 2, \dots, n$. Thus for each $j \in \{1, 2, \dots, n\}$ there is $i(j) \in I$ such that $x_j \in L^0 \wedge L_{i(j)}$. By applying 1.1 (ii) again we conclude that $x \in Q$, completing the proof. □

Let I be a nonempty set and for each $i \in I$ let $L_i \in \mathcal{A}$. The direct product $L = \prod_{i\in I} L_i$ is defined in the usual way. It is clear that L belongs to $\mathcal{A}$. If $I = \{1, 2, \ldots, n\}$, then we use also the notation $L = L_1 \times L_2 \times \ldots \times L_n$. The sublattice L^0 of L consisting of all $x \in L$ such that the set $\{i \in I \colon x(i) \neq 0\}$ is finite will be denoted by $\sum_{i\in I} L_i$; it is called the direct sum of generalized Boolean algebras L_i $(i \in I)$. If $I = \emptyset$, then we consider the direct sum to be equal to $\{0\}$. For $i(1) \in I$ we put $L^0_{i(1)} = \{y \in L \colon x(j) = 0 \text{ for each } j \in I \setminus i(1)\}$. Hence $L^0_{i(1)}$ is isomorphic to $L_{i(1)}$. When no ambiguity can occur we will identify $L_{i(1)}$ and $L^0_{i(1)}$.

2. Radical mappings

Let f be a mapping of $\mathcal{A}$ into $\mathcal{A}$ such that the following conditions are satisfied for each $A \in \mathcal{A}$:

(i) $f(A) \in C(A)$;

(ii) if $A_1 \in C(A)$, then $f(A_1) = A_1 \cap f(A)$;

(iii) if $A_1 \in \mathcal{A}$ and φ is an isomorphism of A onto A_1, then $\varphi(f(A)) = f(A_1)$.

Under these assumptions f will be called *a radical mapping*. The class of all radical mappings will be denoted by F. For $f_1, f_2 \in F$ we put $f_1 \leqslant f_2$ if $f_1(A) \subseteq f_2(A)$ for each $A \in \mathcal{A}$. Thus $\leqslant$ is a partial order on the class F.

Let $\mathfrak{A}$ be as in the introduction. Next, let $f \in F$ and $\mathcal{A}_1 \in \mathfrak{A}$. We put

a) $\mathcal{A}_f = \{A \in \mathcal{A} \colon f(A) = A\}$;

b) for each $A \in \mathcal{A}$ we set

$$f_1(A) = \bigvee_{i\in I} A_i,$$

where $\{A_i\}_{i\in I}$ is the set of all elements of $C(A)$ which belong to $\mathcal{A}_1$.

2.1. Proposition. *Let $f, \mathcal{A}_1, \mathcal{A}_f$ and f_1 be as above. Then*

(i) *$\mathcal{A}_1 \in \mathfrak{A}$ and $f_1 \in F$;*

(ii) *the mapping $f \longrightarrow \mathcal{A}_f$ is an isomorphism of the partially ordered class F onto the partially ordered collection $\mathfrak{A}$; moreover, under the notation as above, the corresponding inverse mapping is given by putting $\mathcal{A}_1 \longrightarrow f_1$ for each $\mathcal{A}_1 \in \mathfrak{A}$.*

Proof. The proof of (i) is analogous to that of 2.2 in [4]; it will be omitted. The assertion (ii) is an immediate consequence of the definitions of $\mathcal{A}_f$ and of f_1. □

Let $\mathcal{A}_0$ be the class of all one-element generalized Boolean algebras. It is obvious that $\mathcal{A}_0$ is the least element of $\mathfrak{A}$ and that $\mathcal{A}$ is the greatest element of $\mathfrak{A}$. We denote by f_0 and $\overline{f}$ the least element or the greatest element of F, respectively.

For a nonempty subclass F_1 of F we define mappings f_1 and f_2 of $\mathcal{A}$ into $\mathcal{A}$ as follows:

$$f_1(A) = \bigvee_{f \in F_1} f(A),$$
$$f_2(A) = \bigwedge_{f \in F_1} f(A)$$

for each $A \in \mathcal{A}$.

We obviously have

2.2. Lemma. *Let F_1, f_1 and f_2 be as above. Then f_1 and f_2 is the supremum or the infimum, respectively, of F_1 in F.*

It will be proved below that F is a proper class. Nevertheless, in view of 2.2 we shall apply to F the usual lattice-theoretic terminology and notation. Next, according to 2.1 (ii) we can do the same for the partially ordered collection $\mathfrak{A}$.

Let $\mathfrak{A}_1$ be a nonempty subcollection of $\mathfrak{A}$. Denote

$$F_1 = \{f \in F\colon \mathcal{A}_f \in \mathfrak{A}_1\}, \quad f_1 = \sup F_1.$$

There exists $\mathcal{A}_1 \in \mathfrak{A}$ such that $\mathcal{A}_1 = \mathcal{A}_{f_1}$. Then 2.1 and 2.2 yield

2.3. Lemma. *Let $\mathfrak{A}_1 = \{\mathcal{A}_i\}_{i \in I}$ be a nonempty subclass of $\mathfrak{A}$. Then under the above notation we have*

$$\bigwedge_{i \in I} \mathcal{A}_i = \bigcap_{i \in I} \mathcal{A}_i,$$

$$\bigvee_{i \in I} \mathcal{A}_i = \mathcal{A}_1.$$

We can describe $\bigvee_{i \in I} \mathcal{A}_i$ in a more constructive way (without applying the isomorphism from 2.1) as follows.

For a subclass X of $\mathfrak{A}$ we define $S_C X$ to be the class of all $A \in \mathcal{A}$ such that there exists $A_1 \in X$ with $A \in C(A_1)$. Next, let X^* be the class of all $A \in \mathcal{A}$ such that there are $A_i \in C(A)$, $A_i' \in X$ $(i \in I)$ with

$$\bigvee_{i \in I} A_i = A \quad \text{and} \quad A_i \cong A_i' \quad \text{for each} \quad i \in I,$$

where $A_i \cong A_i'$ expresses the fact that A_i and A_i' are isomorphic.

2.4. Lemma. *Let X be a nonempty subclass of $\mathcal{A}$. Then $(S_C X)^* \in \mathfrak{A}$.*

P r o o f. We consider the conditions (i), (ii) and (iii) from 1.2. It is obvious that $(S_C X)^*$ satisfies the conditions (i) and (iii). Let $A_1 \in (S_C X)^*$ and let A_2 be a convex subalgebra of A_1. There exist $A_i \in C(A_1)$ and $A_i' \in S_C X$ $(i \in I)$ such that $A_i \cong A_i'$ for each $i \in I$ and $\bigvee\limits_{i \in I} A_i = A_1$. In view of 1.3 we have

$$A_2 = A_2 \wedge A_1 = A_2 \wedge \left(\bigvee_{i \in I} A_i \right) = \bigvee_{i \in I} (A_2 \wedge A_i).$$

Let $i \in I$. There exists $A_i'' \in C(A_i')$ such that $A_i'' \cong A_2 \wedge A_i$. Hence A_i'' belongs to $S_C X$ for each $i \in I$ and so $A_2 \in (S_C X)^*$. Thus $(S_C X)^*$ satisfies the condition (ii).

□

2.5. Corollary. *Let X be a nonempty subclass of $\mathcal{A}$. Then $(S_C X)^*$ is the least radical class having X as a subclass.*

2.6. Corollary. *Let $\{\mathcal{A}_i\}_{i \in I}$ be a nonempty subcollection of $\mathfrak{A}$. Then*

$$\bigvee_{i \in I} \mathcal{A}_i = (S_C X)^*,$$

where $X = \{A \in \mathcal{A} :$ there is $i \in I$ with $A \in \mathcal{A}_i\}$.

2.7. Theorem. *Let $f \in F$ and let $\{f_i\}_{i \in I}$ be a nonempty subclass of F. Then*

$$f \wedge \left(\bigvee_{i \in I} f_i \right) = \bigvee_{i \in I} (f \wedge f_i).$$

P r o o f. Put

$$f_1 = f \wedge \left(\bigvee_{i \in I} f_i \right), \quad f_2 = \bigvee_{i \in I} (f \wedge f_i).$$

Let $A \in \mathcal{A}$. We have to verify that $f_1(A) = f_2(A)$. Since

$$f_1(A) = f_1(A) \wedge \left(\bigvee_{i \in I} f_i(A) \right),$$

in view of 1.3 we obtain

$$f_1(A) = \bigvee_{i \in I} (f(A) \wedge f_i(A)) = f_2(A).$$

□

From 2.7 and 2.1 we infer

2.8. Corollary. *Let* $\mathcal{A}_1 \in \mathfrak{A}$ *and let* $\{\mathcal{A}_i\}_{i\in I}$ *be a nonempty subcollection of* $\mathfrak{A}$. *Then*

$$\mathcal{A}_1 \wedge \Big(\bigvee_{i\in I} \mathcal{A}_i\Big) = \bigvee_{i\in I} (\mathcal{A}_1 \wedge \mathcal{A}_i).$$

3. On the classes (a)–(e)

The aim of the present section is to prove that the classes (a)–(e) mentioned in the introduction are radical classes. We need two lemmas.

3.1. Lemma. *Let* B *be a Boolean algebra,* $b \in B$, $y_i \in B$ $(i = 1, 2, \ldots, n)$, $b = y_1 \vee y_2 \vee \ldots \vee y_n$. *Then there exist elements* $y_1^1, y_2^1, \ldots, y_n^1$ *in* B *such that* $b = y_1^1 \vee y_2^1 \vee \ldots \vee y_n^1$, $y_i^1 \leqslant y_i$ *for* $i = 1, 2, \ldots, n$, *and* $y_{i(1)} \wedge y_{i(2)} = 0$ *whenever* $i(1)$ *and* $i(2)$ *are distinct elements of the set* $\{1, 2, \ldots, n\}$.

Proof. We proceed by induction on n. For $n = 1$ the assertion is valid; suppose that it holds for $n-1$. Hence there are $y_1^1, y_2^1, \ldots, y_{n-1}^1$ in B such that $y_1 \vee y_2 \vee \ldots \vee y_{n-1} = y_1^1 \vee y_2^1 \vee \ldots \vee y_{n-1}^1$, $y_i^1 \leqslant y_i$ for $i = 1, 2, \ldots, n-1$ and $y_{i(1)}^1 \wedge y_{i(2)}^1 = 0$ whenever $i(1), i(2)$ are distinct indices belonging to the set $\{1, 2, \ldots, n-1\}$. There exists $t \in B$ such that t is a relative complement of $y_1 \vee y_2 \vee \ldots \vee y_{n-1}$ in the interval $[0, b]$.Put $y_n^1 = y_n \wedge t$. Then $y_1^1, y_2^1, \ldots, y_n^1$ satisfy the required conditions. □

3.2. Lemma. *Let* B *be a Boolean algebra and let* b, $y_1, y_2, \ldots, y_n$ *be elements of* B *such that*

(i) $b = y_1 \vee y_2 \vee \ldots \vee y_n$,
(ii) $y_{i(1)} \wedge y_{i(2)} = 0$ *whenever* $i(1), i(2)$ *are distinct indices belonging to the set* $\{1, 2, \ldots, n\}$.

For each $x \in [0, b]$ *put* $\varphi(x) = (x \wedge y_i)_{i=1,2,\ldots,n}$. *Then* φ *is an isomorphism of the interval* $[0, b]$ *onto the direct product* $[0, y_1] \times [0, y_2] \times \ldots \times [0, y_n]$.

The proof is simple and will be omitted.

Let α be an infinite cardinal. A lattice is said to be *conditionally* α*-complete* if each of its intervals is α-complete. The notion of conditional completeness is defined analogously.

3.3. Lemma. *Let* α *be an infinite cardinal and let* $A \in \mathcal{A}$, $A_i \in C(A)$ $(i \in I)$. *Suppose that all* A_i *are conditionally* α*-complete and that* $A = \bigvee_{i\in I} A_i$. *Then* A *is conditionally* α*-complete.*

P r o o f. Let $[a,b]$ be an interval in A. For proving that it is α-complete it suffices to verify that the interval $[0,b]$ is α-complete.

There exists a subset $\{y_1, y_2, \ldots, y_n\}$ of the set $\bigcup_{i\in I} A_i$ such that $b = y_1 \vee y_2 \vee \ldots \vee y_n$. In view of 3.1 we can suppose, without loss of generality, that $y_{i(1)} \wedge y_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of $\{1, 2, \ldots, n\}$. Hence we can apply the isomorphism φ from 3.2. Since all intervals $[0, y_i]$ are α-complete, the interval $[0,b]$ must be α-complete as well. □

3.4. Corollary. *Let $A \in \mathcal{A}$, $A_i \in C(A)$ $(i \in I)$. If all A_i are conditionally complete and $\bigvee_{i\in I} A_i = A$, then A is conditionally complete.*

Let us remark that

(i) neither 3.3 nor 3.4 remain valid for general lattices with the least element;

(ii) the conditional α-completeness in 3.3 (or conditional completeness in 3.4) cannot be replaced by α-completeness (or completeness).

Let us denote by $\mathcal{A}_{b(\alpha)}$ the class defined in (b) above (cf. the introduction).

3.5. Proposition. *$\mathcal{A}_{b(\alpha)}$ is a radical class.*

P r o o f. The conditions (i) and (ii) from 1.2 are obviously satisfied. In view of 3.3, the condition (iii) from 1.2 is also valid. □

3.6. Corollary. *Let $\mathcal{A}_a$ be the class of all $A \in \mathcal{A}$ which are conditionally complete. Then $\mathcal{A}_a$ is a radical class.*

P r o o f. We have $\mathcal{A}_a = \inf \mathcal{A}_{b(\alpha)}$, where α runs over the class of all infinite cardinals. □

We will apply the following definition.

3.7. Definition. Let α be an infinite cardinal and let $A \in \mathcal{A}$. We say that A is *α-distributive* if, whenever $u, v \in A$, $\{x_{ij}\}_{i\in I, j\in J} \subseteq A$ such that

$$\operatorname{card} I \leqslant \alpha, \quad \operatorname{card} J \leqslant \alpha,$$

$$v = \bigwedge_{i\in I} \bigvee_{j\in J} x_{ij}, \tag{1}$$

$$u = \bigvee_{\varphi\in J^I} \bigwedge_{i\in I} x_{i,\varphi(i)}, \tag{2}$$

then $u = v$.

Let us remark that if (1) and (2) are valid, then clearly $u \leqslant v$. Also, in the above definition we can suppose, without loss of generality, that $\{x_{ij}\}_{i\in I, j\in J}$ is a subset of $[u, v]$. In fact, the elements x_{ij} in (1) and (2) can be replaced by $x^1_{ij} = (x_{ij} \vee u) \wedge v$. Next, without loss of generality it suffices to consider only the case when $u = 0$ (since the interval $[u, v]$ is isomorphic to the interval $[0, v_1]$, where v_1 is the relative complement of u in the interval $[0, v]$). Finally, we remark that the condition expressed in 3.7 is equivalent to the corresponding dual condition.

3.8. Lemma. *Let α be an infinite cardinal and let $A \in \mathcal{A}$, $A_i \in C(A)$ $(i \in I)$. Suppose that all A_i are α-distributive and that $A = \bigvee_{i\in I} A_i$. Then A is α-distributive.*

Proof. By way of contradiction, assume that A is not α-distributive. Then there are $u, v \in A$ and $\{x_{ij}\}_{(i,j)\in I\times J} \subseteq A$ such that $\operatorname{card} I \leqslant \alpha$, $\operatorname{card} J \leqslant \alpha$, the relations (1), (2) are valid and $u = 0 < v$.

Let $\{y_1, y_2, \ldots, y_n\}$ be as in the proof of 3.3, where we put $b = v$. We can again apply the isomorphism φ from 3.2. All intervals $[0, y_i]$ are α-distributive, hence the interval $[0, v]$ is α-distributive as well; we have arrived at a contradiction. □

Let $\mathcal{A}_{d(\alpha)}$ be the class of all $A \in \mathcal{A}$ such that A is α-distributive.

3.9. Proposition. *Let α be an infinite cardinal. Then $\mathcal{A}_{d(\alpha)}$ is a radical class.*

Proof. The corresponding conditions (i) and (ii) are obviously valid; the condition (iii) holds in view of 3.8. □

3.10. Corollary. *Let $\mathcal{A}_c$ be the class of all $A \in \mathcal{A}$ such that A is completely distributive. Then $\mathcal{A}_c$ is a radical class.*

Let α be an infinite cardinal. We denote by

$\mathcal{A}_{e(\alpha)}$—the class of all $A \in \mathcal{A}$ such that for each interval $[a_1, a_2]$ of A the relation $\operatorname{card}[a_1, a_2] \leqslant \alpha$ is valid;

$\mathcal{A}'_{e(\alpha)}$—the class of all $A \in \mathcal{A}$ such that for each interval $[a_1, a_2]$ of A the relation $\operatorname{card}[a_1, a_2] < \alpha$ is valid.

3.11. Proposition. *Let α be an infinite cardinal. Then $\mathcal{A}_{e(\alpha)}$ is a radical class.*

Proof. The conditions (i) and (ii) from 1.2 obviously hold. Let the assumptions from (iii) be valid, where $\mathcal{A}_1 = \mathcal{A}_{e(\alpha)}$.

Let $0 < b \in A$. Next, let $y_1, y_2, \ldots, y_n$ be as in the proof of 3.3. Since $\operatorname{card}[0, y_i] \leqslant \alpha$ for $i = 1, 2, \ldots, n$, in view of 3.2 we infer that $\operatorname{card}[0, b] \leqslant \alpha$, whence (iii) is valid as well. □

3.12. Proposition. *Let α be an infinite cardinal. Then $\mathcal{A}'_{e(\alpha)}$ is a radical class.*

P r o o f. The proof is similar to that of 3.11. The modification consists in putting $\alpha_1 = \max\{\operatorname{card}[0, y_i]\}_{i=1,2,\ldots,n}$. Then $\operatorname{card}[0, b] \leqslant \alpha_1^n < \alpha$. □

In particular, for $\alpha = \aleph_0$ we obtain from 3.12

3.13. Corollary. *$\mathcal{A}_e$ is a radical class.*

4. On some radical classes defined by cardinal functions

We recall some notions and notation from [8]. Let B be a Boolean algebra.

A subset of B is called *disjointed* if it consists of non-zero elements which are pairwise disjoint, i.e. $a \wedge b = 0$ if $a \neq b$, where $a, b \in B$.

A subset D of B is said to be *dense* in B if for every $b \in B$ with $b > 0$ there is $d \in D$ such that $0 < d \leqslant b$.

Let α be a cardinal. A subset D of B is called *α-compact* if there is a non-zero lower bound to every subset C of D possessing the properties (i) $\operatorname{card} C < \alpha$, and (ii) g.l.b. $F \neq 0$ for every finite $F \subseteq C$.

Let $\mathcal{B}$ be the class of all Boolean algebras and let $\mathcal{B}_1$ be a subclass of $\mathcal{B}$ which is closed with respect to isomorphisms.

By a cardinal function f on the class $\mathcal{B}_1$ we understand a rule that assigns to each $B \in \mathcal{B}_1$ a cardinal $f(B)$ such that if B is isomorphic to B' then $f(B) = f(B')$.

In [8] the following cardinal functions were investigated:

$\pi_1(B) = \min\{\alpha\colon\ D \subseteq B,\ D$ disjointed implies $\operatorname{card} D \leqslant \alpha\}$.

$\pi_1'(B) = \min\{\alpha\colon\ D \subseteq B,\ D$ disjointed implies $\operatorname{card} D < \alpha\}$.

$\pi_2(B) = \min\{\operatorname{card} D\colon\ D$ is dense in $B\}$.

$\pi_3(B) = \sup\{\alpha\colon\ B$ contains a dense α-compact subset$\}$.

$\pi_4(B) = \sup\{\alpha\colon\ B$ is α-distributive$\}$.

(The radical functions π_1, π_1' and π_2 are defined on the class of all Boolean algebras; π_3 and π_4 are defined whenever the corresponding suprema exist.)

For each π of the above mentioned cardinal functions and each infinite cardinal β we denote by $\mathcal{A}(\pi, \beta)$ the class of all $A \in \mathcal{A}$ such that if $[0, b]$ is a subalgebra of A, then $\pi([0, b]) \leqslant \beta$.

Our aim is to investigate the question when $\mathcal{A}(\pi, \beta)$ is a radical class.

The method is analogous to that applied in the previous section. In all cases we first verify whether the class $\mathcal{A}(\pi, \beta)$ is closed with respect to joins, i.e., whether the condition (iii) from 1.2 is valid; with this verification we proceed as in the proof of 3.3. Also, we use the notation from the beginning of the proof of 3.3.

4.1. Lemma. *Let β be an infinite cardinal. Then $\mathcal{A}(\pi_1, \beta)$ satisfies the condition* (iii) *from* 1.2.

Proof. In view of the assumption we have $\pi_1([0, y_i]) \leqslant \beta$ for $i = 1, 2, \ldots, n$. We have to verify whether $\pi_1([0, b]) \leqslant \beta$ is valid. By way of contradiction, suppose that $\pi_1([0, b]) > \beta$.

Hence there exists a subset D of $[0, b]$ such that D is disjointed and $\operatorname{card} D > \beta$. Let $d \in D$. Then

$$d = d \wedge b = (d \wedge y_1) \vee \ldots \vee (d \wedge y_n).$$

For $i \in \{1, 2, \ldots, n\}$ denote $d_i = d \wedge y_i$, $D_i = \{d_i\colon\ d \in D\}$. The mapping

$$\psi\colon\ D \longrightarrow D_1 \times D_2 \times \ldots \times D_n$$

defined by $\psi(d) = (d_1, d_2, \ldots, d_n)$ for each $d \in D$ is injective. If $\operatorname{card} D_i \leqslant \beta$ for $i = 1, 2, \ldots, n$, then $\operatorname{card} D \leqslant \beta$, which is impossible. Hence there exists $i \in \{1, 2, \ldots, n\}$ such that $\operatorname{card} D_i > \beta$ and thus $\operatorname{card}(D_i \setminus \{0\}) > \beta$. Next, $D_i \setminus \{0\}$ is a disjointed subset of $[0, y_i]$. This yields that $\pi_1([0, y_i]) > \beta$, which is a contradiction. □

4.2. Proposition. *Let β be an infinite cardinal. Then $\mathcal{A}(\pi_1, \beta)$ is a radical class.*

Proof. The conditions (i) and (ii) of 1.2 are obviously satisfied and the condition (iii) is valid in view of 4.1. □

4.3. Lemma. *Let β be an infinite cardinal. Then $\mathcal{A}(\pi_1', \beta)$ satisfies the condition (iii) from 1.2.*

Proof. The proof is the same as in 4.1 with the distinction that in the relations $\operatorname{card} D > \beta$, $\pi_1([0, y_i]) > \beta$ (and in other corresponding relations) the symbol $>$ is replaced by $\geqslant$. □

As a consequence we obtain

4.4. Proposition. *Let β be an infinite cardinal. Then $\mathcal{A}(\pi_1', \beta)$ is a radical class.*

4.5. Lemma. *Let β be an infinite cardinal. Then $\mathcal{A}(\pi_2, \beta)$ satisfies the condition* (iii) *from* 1.2.

Proof. In view of the above notation, the relation $\pi_2([0, y_i]) \leqslant \beta$ is valid for $i = 1, 2, \ldots, n$. Hence for each $[0, y_i]$ there exists a dense subset D_i with $\operatorname{card} D_i \leqslant \beta$. Put $D = D_1 \cup D_2 \cup \ldots \cup D_n$. Then $\operatorname{card} D \leqslant \beta$. Let $x \in [0, b]$, $x > 0$. We have

$$x = (x \wedge y_1) \vee (x \wedge y_2) \vee \ldots \vee (x \wedge y_n).$$

There exists $i \in I$ such that $x \wedge y_i > 0$. Next, there exists $d_i \in D_i$ such that $0 < d_i \leqslant x \wedge y_i$. Hence D is a dense subset of $[0, b]$. Therefore $\pi_2[0, b]) \leqslant \beta$. □

4.6. Proposition. *Let β be an infinite cardinal. Then $\mathcal{A}(\pi_2, \beta)$ is a radical class.*

The proof is as in 4.2 with the distinction that 4.1 is replaced by 4.5.

4.7. Proposition. *Let β be an infinite cardinal. Then $\mathcal{A}(\pi_3, \beta)$ satisfies the condition* (iii) from 1.2.

Proof. Under the notation as above let $\pi_3([0, y_i]) \leqslant \beta$ for $i = 1, 2, \ldots, n$. We have to verify that $\pi_3([0, b]) \leqslant \beta$. By way of contradiction, suppose that $\pi_3([0, b]) > \beta$. Hence there exists a subset D of $[0, b]$ such that it is dense in $[0, b]$ and α-compact for some $\alpha > \beta$. Let $i \in \{1, 2, \ldots, n\}$. Put $D_i = D \cap [0, y_i]$. Then D_i is dense in $[0, y_i]$. Let $C \subseteq D_i$, $\operatorname{card} C \subseteq \alpha$ and suppose that for each finite subset F of C the relation $\inf F > 0$ is valid. Then $C \subseteq D$ and hence there is $0 < z \in [0, b]$ such that $z \leqslant c$ for each $c \in C$. In view of $b = y_1 \vee y_2 \vee \ldots \vee y_n$ we obtain that $z = (z \wedge y_1) \vee (z \wedge y_2) \vee \ldots \vee (z \wedge y_n)$. In 3.1 we verified that without loss of generality we can suppose that whenever $i(1)$ and $i(2)$ are distinct elements of $\{1, 2, \ldots, n\}$ then $y_{i(1)} \wedge y_{i(2)} = 0$. Since $c \leqslant y_i$ for each $c \in C$ we get that $z \in [0, y_i]$. Hence D_i is α-compact with respect to $[0, y_i]$; therefore $\pi_3[0, y_i] \geqslant \alpha > \beta$, which is a contradiction. □

4.8. Lemma. *Let β be an infinite cardinal. Then there exists a Boolean algebra B such that $\pi_3(B) = \beta$.*

Proof. This is a consequence of [8], Theorem 3.1. □

4.9. Lemma. *Let B be a finite Boolean algebra. Then $\pi_3(B)$ is not defined.*

Proof. Let α be an infinite cardinal. Put $D = B$. Then D is α-compact and dense in B. Hence $\pi_3(B)$ does not exist. □

4.10. Lemma. *Let β_1 be an infinite cardinal. Let B_1 and B_2 be Boolean algebras such that $\pi_3(B_1) = \beta_1$ and B_2 is finite. Put $B = B_1 \times B_2$. Then $\pi_3(B) = \beta_1$.*

Proof. Let b_1, b_2 and b be the greatest element of B_1, B_2 or B, respectively. Hence $b = b_1 \vee b_2$. Let α be a cardinal and suppose that D is a dense subset in B which is α-compact. By the same method as in the proof of 4.7 we obtain that the relation $\alpha > \beta_1$ leads to a contradiction. Thus $\alpha \leqslant \beta_1$. Hence $\pi_3(B)$ does exist and $\pi_3(B) \leqslant \beta_1$.

There exists a set $D_1 \subseteq B_1$ such that D_1 is dense in B_1 and β_1-compact. Put $D_2 = B_2$, $D = D_1 \cup D_2$. Then D is a dense subset of B. Let C be a subset of D with $\operatorname{card} C \leqslant \beta_1$ such that, whenever F is a finite subset of C, then $\inf F > 0$. In such a case we must have either $C \subseteq D_1$ or $C \subseteq D_2$. In both these cases there exists $0 < b' \in B$ such that $b' < c$ for each $c \in C$. Therefore $\pi_3(B) \geqslant \beta_1$. Summarizing, we conclude that $\pi_3(B) = \beta_1$. □

By the same method as in the proof of 4.10 we can show that the following result is valid.

4.10.1. Lemma. *Let β_1 and β_2 be infinite cardinals, $\beta_1 < \beta_2$. Next, let B_1 and B_2 be Boolean algebras with $\pi_3(B_i) = \beta_i$ $(i = 1, 2)$. Put $B = B_1 \times B_2$. Then $\pi_3(B) = \beta_1$.*

4.11. Proposition. *Let β be an infinite cardinal. Then $\mathcal{A}(\pi_3, \beta)$ fails to be a radical class.*

Proof. In view of 4.8 there exists a Boolean algebra B_1 such that $\pi_3(B_1) = \beta$. Let B_2 be a finite Boolean algebra, $B = B_1 \times B_3$. Hence in view of 4.10, $\pi_3(B) = \beta$, thus $B \in \mathcal{A}(\pi_3, \beta)$. We have $B_2 \in C(B)$ and according to 4.9, B_2 does not belong to $\mathcal{A}(\pi_3, \beta)$. Thus $\mathcal{A}(\pi_3, \beta)$ does not satisfy the condition (ii) from 1.2. □

4.12. Proposition. *Let β be an infinite cardinal. Then $\mathcal{A}(\pi_4, \beta)$ satisfies the condition* (iii) *from* 1.2.

Proof. We apply the notation as above. Let $\pi_4([0, y_i]) \leqslant \beta$ for $i = 1, 2, \ldots, n$. By way of contradiction, suppose that the relation $\pi_4([0, b]) \leqslant \beta$ does not hold. Hence there exists a cardinal $\alpha > \beta$ such that $[0, b]$ is α-distributive. Then all $[0, y_i]$ are α-distributive, which is impossible. □

4.13. Proposition. *Let β be an infinite cardinal. Then $\mathcal{A}(\pi_4, \beta)$ fails to be a radical class.*

Proof. There exists a Boolean algebra B_1 which is not $\aleph_0$-distributive. Hence $\pi_4(B_1) \leqslant \beta$. Let B_2 be a finite Boolean algebra, $B = B_1 \times B_2$. Then $\pi_4(B) \leqslant \beta$, $B \in \mathcal{A}(\pi_4, \beta)$. At the same time, $B_2 \in C(B)$ and $B \notin \mathcal{A}(\pi_4, \beta)$. Therefore $\mathcal{A}(\pi_4, \beta)$ does not satisfy the condition (ii) from 1.2. □

The following lemma will be applied in the subsequent section.

4.14. Lemma. *Let B and B_1 be Boolean algebras such that $(S_c\{B\})^* = (S_c\{B_1\})^*$. Suppose that both $\pi_1(B)$ and $\pi_1(B_1)$ are infinite. Then $\pi_1(B) = \pi_1(B_1)$.*

P r o o f. Let b^1 be the maximal element of B_1. We have $b^1 \in (S_c\{B\})^*$. Hence there are $x_1, x_2, \ldots, x_n \in B$ and $y_1, y_2, \ldots, y_n \in B_1$ such that $[0, x_i] \cong [0, y_i]$ for $i = 1, 2, \ldots, n$ and $y_1 \vee y_2 \vee \ldots y_n = b^1$. Applying the analogous method as in the proof of 4.1 we obtain that $\pi_1(B_1) \leqslant \pi_1(B)$ is valid. Similarly, $\pi_1(B) \leqslant \pi_1(B_1)$. $\square$

5. Atoms of the lattice $\mathfrak{A}$

The collection of all atoms of $\mathfrak{A}$ will be denoted by $\mathfrak{A}_a$.

If $\mathcal{A}_1$ is a radical class such that all generalized Boolean algebras belonging to $\mathcal{A}_1$ are complete or conditionally complete, then $\mathcal{A}_1$ will be called complete or conditionally complete, respectively.

The only complete radical class is $\mathcal{A}_0$. Namely, if $\mathcal{A}_1$ is a radical class distinct from $\mathcal{A}_0$, then there is $A \in \mathcal{A}_1$ with $A \neq \{0\}$. Let I be an infinite set and for each $i \in I$ let $A_i = A$. Put $A' = \sum\limits_{i \in I} A_i$. Then $A' \in \mathcal{A}_1$ and A' fails to be complete.

We can use analogous terminology for the partially ordered collection $\mathcal{L}$ consisting of all radical classes of lattice ordered groups, but a certain terminological distinction must be observed.

For a lattice ordered group G we denote by $\overline{G}$ the underlying lattice. If $G \neq \{0\}$, then the lattice $\overline{G}$ cannot be complete. The terminology commonly used in the theory of lattice ordered groups is as follows: a lattice ordered group G is said to be complete if the lattice $\overline{G}$ is conditionally complete.

Let R_1 be a radical class of lattice ordered groups. We call R_1 conditionally complete if, whenever $G \in R_0$, then the lattice $\overline{G}$ is conditionally complete.

In [1], Proposition 3.3 it is proved that there exists an injective mapping φ of the class of all infinite cardinals into the collection of all atoms of $\mathcal{L}$. By looking at the construction of this mapping we easily obtain that whenever α is an infinite cardinal, then the corresponding radical class $\varphi(\alpha)$ fails to be conditionally complete.

In the present section the following result will be proved.

5.1. Theorem. *There exists an injective mapping ψ of the class of all infinite cardinals into the collection $\mathfrak{A}_a$ such that for each infinite cardinal α the radical class $\psi(\alpha)$ is conditionally complete.*

We start by giving some definitions and lemmas.

5.2. Definition. Let $\emptyset \neq X \subseteq \mathcal{A}$. The radical class $(S_cX)^*$ is said to be generated by X. If $A \in \mathcal{A}$ and $X = \{A\}$, then $(S_cX)^*$ is called a principal radical class generated by A.

5.3. Definition. A Boolean algebra B is called *homogeneous* if for each $b \in B$ with $b > 0$ the Boolean algebra $[0, b]$ is isomorphic to B.

5.4. Definition. A Boolean algebra B is said to be *weakly homogeneous* if for each $b \in B$ with $b > 0$ there exist $b_i \in [0, b]$ and $b'_i \in B$ $(i = 1, 2, \ldots, n)$ such that $[0, b_i] \cong [0, b'_i]$ for $i = 1, 2, \ldots, n$ and $b_1 \vee b_2 \vee \ldots \vee b_n$ is the greatest element of B.

5.5. Lemma. *Let $B \neq \{0\}$ be a weakly homogeneous Boolean algebra and let $\mathcal{A}_1$ be the principal radical class generated by B. Then $\mathcal{A}_1$ is an atom of $\mathfrak{A}$.*

Proof. Since $B \in \mathcal{A}_1$ we have $\mathcal{A}_1 \neq \mathcal{A}_0$. Let $\mathcal{A}_2 \in \mathfrak{A}$, $\mathcal{A}_0 < \mathcal{A}_2 \leqslant \mathcal{A}_1$. Thus there is $B_2 \in \mathcal{A}_2$ with $B_2 \neq \{0\}$. Choose $b_2 \in B_2$, $b_2 > 0$. Then $B_2 \in \mathcal{A}_1 = (S_c\{B\})^*$. Let b_2^m be the greatest element of B_2. There exist elements $c_1, c_2, \ldots, c_n$ in B_2 and $c'_1, c'_2, \ldots, c'_n$ in B such that $[0, c_i] \cong [0, c'_i]$ for $i = 1, 2, \ldots, n$ and $c_1 \vee c_2 \vee \ldots \vee c_n = b_2^m$. Hence there is $i(1) \in \{1, 2, \ldots, n\}$ such that $c_{i(1)} > 0$. Then we have $c'_{i(1)} > 0$ as well. In view of the weak homogeneity of B there are elements $d'_j \in [0, c'_{i(j)}]$ and $d_j \in B$ $(j = 1, 2, \ldots, m)$ such that $[0, d'_j] \cong [0, d_j]$ for $j = 1, 2, \ldots, m$ and $d_1 \vee d_2 \vee \ldots \vee d_m$ is the greatest element of B. For each $j \in \{1, 2, \ldots, m\}$ there exists $e_j \in [0, c_{i(1)}]$ with $[0, e_j] \cong [0, d'_j]$. This yields that $B \in (S_c\{B_2\})^*$ and therefore $\mathcal{A}_1 \leqslant \mathcal{A}_2$, completing the proof. □

In the above proof we applied the obvious fact that if $\mathcal{A}_1$ is a radical class distinct from $\mathcal{A}_0$, then there exists a nonzero Boolean algebra belonging to $\mathcal{A}_1$. This fact will be used also in the following lemma.

5.6. Lemma. *Let $\mathcal{A}_1$ be an atom of $\mathfrak{A}$. Then there exists a nonzero Boolean algebra B in $\mathcal{A}_1$ and for each such B the following conditions are valid:*

(i) *B is weakly homogeneous;*
(ii) *$\mathcal{A}_1$ is a principal radical class generated by B.*

Proof. Denote $\mathcal{A}_2 = (S_c\{B\})^*$. Thus $\mathcal{A}_0 < \mathcal{A}_2 \leqslant \mathcal{A}_1$. Since $\mathcal{A}_1$ is an atom we obtain that $\mathcal{A}_2 = \mathcal{A}_1$. Therefore (ii) holds.

Let $0 < b_1 \in B$. Put $[0, b_1] = B_1$ and $(S_c\{B_1\})^* = \mathcal{A}_3$. We must have $\mathcal{A}_3 = \mathcal{A}_1$. Thus there are $c_1, c_2, \ldots, c_n \in B_1$ and $c'_1, c'_2, \ldots, c'_n \in B$ such that $[0, c_i] \cong [0, c'_i]$ is valid for $i = 1, 2, \ldots, n$ and $c'_1 \vee c'_2 \vee \ldots c'_n$ is the greatest element of B. Hence B is weakly homogeneous. □

5.7. Proposition. *Let α, β and γ be infinite cardinals. There exists a Boolean algebra $B_{\alpha\beta\gamma}$ such that*

(i) *$B_{\alpha\beta\gamma}$ is complete;*
(ii) *if $\alpha \leqslant \beta$, then $B_{\alpha\beta\gamma}$ is homogeneous;*
(iii) *if $\alpha = \aleph_0 < \beta = \gamma$, then $\pi_1(B_{\alpha\beta\gamma}) = \gamma$.*

P r o o f. Consider the Boolean algebra $B_{\alpha\beta\gamma}$ constructed in [8]. According to [8], p. 131, $B_{\alpha\beta\gamma}$ is complete. Next, in view of [8], 3.12, the condition (ii) is valid. Finally, in view of 3.14 in [8] (the first line of the table in 3.14) the condition (iii) is satisfied. □

Let α be a cardinal, $\alpha > \aleph_0$. Denote $B^\alpha = B_{\aleph_0\alpha\alpha}$ and let $\mathcal{A}_\alpha$ be the principal radical class generated by B^α.

5.8. Lemma. *Let $\alpha(1)$ and $\alpha(2)$ be distinct cardinals, $\alpha(i) > \aleph_0$ $(i = 1, 2)$. Then $\mathcal{A}_{\alpha(1)} \neq \mathcal{A}_{\alpha(2)}$.*

P r o o f. In view of 5.7 (iii) we have $\pi_1(B^{\alpha(i)}) = \alpha(i)$ for $i = 1, 2$. By way of contradiction, suppose that $\mathcal{A}_{\alpha(1)} = \mathcal{A}_{\alpha(2)}$. Then 4.14 yields that $\pi_1(B^{\alpha(1)}) = \pi_1(B^{\alpha(2)})$, which is a contradiction. □

5.9. Lemma. *For each cardinal α with $\alpha > \aleph_0$ put $\psi_1(\alpha) = \mathcal{A}_\alpha$. Then ψ_1 is an injective mapping of the class of all cardinals greater than $\aleph_0$, into $\mathfrak{A}_a$.*

P r o o f. In view of 5.7 (ii) and 5.5, $\psi_1(\alpha)$ belongs to $\mathfrak{A}_a$ whenever α is a cardinal with $\alpha > \aleph_0$. Next, according to 5.8, the mapping ψ_1 is injective. □

5.10. Lemma. *$\mathcal{A}_e \in \mathfrak{A}_a$ and $\mathcal{A}_e$ is conditionally complete.*

P r o o f. Clearly $\mathcal{A}_e \neq \mathcal{A}_0$. Let $\mathcal{A}_1 \in \mathfrak{A}$, $\mathcal{A}_0 < \mathcal{A}_1 \leqslant \mathcal{A}_e$. Thus there exists a Boolean algebra $B \in \mathcal{A}_1$ such that $B \neq \{0\}$ and B is finite. Hence there exists $0 < b_1 \in B$ such that $[0, b_1]$ is a two-element set. If $A \in \mathcal{A}_e$ and $0 < b \in A$, then the interval $[0, b]$ can be expressed as a join of two-element intervals; therefore $\mathcal{A}_e \leqslant (S_c(B))^* \leqslant \mathcal{A}_1$. This shows that $\mathcal{A}_1 = \mathcal{A}_e$. The conditional completeness follows from 3.4 and from the fact that B is complete. □

P r o o f o f 5.1. Let B be as in the proof of 5.10. Then $\pi_1(B_0) < \aleph_0$ and hence according to 5.7 (iii) and 4.14 we infer that $B \notin \mathcal{A}_\alpha$ whenever $\alpha > \aleph_0$. We define a mapping ψ of the class of all infinite cardinals as follows: $\psi(\aleph_0) = \mathcal{A}_e$, $\psi(\alpha) = \mathcal{A}_\alpha$ if $\alpha > \aleph_0$. In view of 5.9 and 5.10, $\psi(\beta) \in \mathfrak{A}_a$ for each infinite cardinal α, and in view of 5.7 and 5.10, all $\psi(\beta)$ are conditionally complete. Now it suffices to apply 5.8 and the fact that $\psi(\aleph_0) \neq \psi(\beta)$ for $\beta > \aleph_0$; we obtain that ψ is injective. □

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