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DOUBLE CONVERGENCE AND PRODUCTS OF FRÉCHET SPACES

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Abstract. The paper is devoted to convergence of double sequences and its application to products. In a convergence space we recognize three types of double convergences and points, respectively. We give examples and describe their structure and properties. We investigate the relationship between the topological and convergence closure product of two Fréchet spaces. In particular, we give a necessary and sufficient condition for the topological product of two compact Hausdorff Fréchet spaces to be a Fréchet space.

I

In this section we recall definitions and some properties of convergence closure spaces. Let X be a set of points. Let $\mathcal{L}$ be a collection of pairs $(\langle x_n\rangle, x)$, $\langle x_n\rangle \in X^{\mathbb{N}}$, $x \in X$, such that the Fréchet axioms of convergence are satisfied. Then $\mathcal{L}$ is called a convergence for X. Instead of $(\langle x_n\rangle, x) \in \mathcal{L}$ we write $\mathcal{L}\text{-}\lim x_n = x$, or simply, $\lim x_n = x$. We say that the sequence $\langle x_n\rangle$ converges or, more precisely, $\mathcal{L}$-converges to x.

1. For each $A \subset X$ define its closure λA: $x \in \lambda A$ if there is a sequence of points of A converging to x.

2. $\lambda\{x\} = \{x\}$ for each $x \in X$ and $A \subset \lambda A$, $\lambda(A \cup B) = \lambda A \cup \lambda B$ for each $A, B \subset X$.

3. A set X carrying a convergence $\mathcal{L}$ and the corresponding closure λ is called a convergence closure (topological, if λ is a topology) space or a convergence space. It is denoted $(X, \mathcal{L}, \lambda)$ or (X, λ).

4. A set $L(x) \subset X$ is said to be a closure neighbourhood of $x \in X$ if there is no sequence of points of $X \setminus L(x)$ converging to x. Then $x \in \lambda A$ iff each closure neighbourhood $L(x)$ contains points of A.

5. The maximal convergence $\mathcal{L}^*$ for $(X, \mathcal{L}, \lambda)$ is defined by means of the closure or of closure neighbourhoods. Put $(\langle x_n\rangle, x) \in \mathcal{L}^*$ whenever $x \in \lambda\{x_{n_i}\,;\ i \in \mathbb{N}\}$ for each

subsequence $\langle x_{n_i}\rangle$ of $\langle x_n\rangle$. Equivalently: $(\langle x_n\rangle, x) \in \mathcal{L}^*$ if each convergence closure niehgbourhood $L(x)$ of x contains all points x_n except for a finite number of indexes $n \in \mathbb{N}$.

6. If $\mathcal{L} = \mathcal{L}^*$ and a sequence $\langle x_n\rangle$ does not $\mathcal{L}$-converge to a point $x \in X$ then there is a subsequence $\langle x_{n_i}\rangle$ no subsequence of which $\mathcal{L}$-converges to x.

7. λ is a topology for X provided $\lambda A = \lambda(\lambda A)$ for each $A \subset X$, i.e. $\lambda = \lambda^2$. Further, $(X, \mathcal{L}, \lambda)$ is not a topological space iff there are points x, x_m, x_{mn} of X such that $\lim x_m = x$, $\lim x_{mn} = x_m$, $m \in \mathbb{N}$, and there is no sequence in $\{x_{mn};\ m, n \in \mathbb{N}\}$ converging to x.

8. A convergence topological space $(X, \mathcal{L}, \lambda)$ is called a Fréchet space if $\mathcal{L} = \mathcal{L}^*$.

9. A convergence closure space $(X, \mathcal{L}, \lambda)$ is called an H space if for each two distinct points x_1 and x_2 of X there are their respective disjoint closure neighbourhoods $L(x_1)$ and $L(x_2)$.

II

In some papers on sequential convergence ([KR77], [NO77], [AR79], [FV85], [KO85], [NO85]) generalizations of simple sequences to multiple sequences are used to tackle various problems. In this section we investigate properties of double sequences which we apply later to products of Fréchet spaces. We start with a generalization which we hope to be both interesting and useful.

Let ω denote the first infinite ordinal number. For each ordinal number $\alpha > 0$, let Ω_α denote the set of its predecessors.

Definition 1. Let $\alpha > 0$ be an ordinal number and let X be a non-void set. Let φ be a map of Ω_α into X considered as a subset of $\Omega_\alpha \times X$. Then the set φ, ordered by $(\xi, \varphi(\xi)) < (\eta, \varphi(\eta))$ iff $\xi < \eta < \alpha$, is called a *sequence* of type α (or simply a sequence). Denote $\varphi(\xi) = x_\xi$, $\xi \in \Omega_\alpha$, and $\varphi = \langle x_\xi\rangle$.

Let $0 < \beta \leqslant \alpha$, let A be a subset of Ω_α such that Ω_β and A are isotonic and let $h\colon \Omega_\beta \to A$ be an isotonic map. Define $\psi\colon \Omega_\beta \to X$ by $\psi(\eta) = \varphi(h(\eta))$, $\eta \in \Omega_\beta$. Denote $h(\eta) = \xi_\eta$, $\eta \in \Omega_\beta$. Then $\psi = \langle x_{\xi_\eta}\rangle$ is said to be a *subsequence* of the sequence $\langle x_\xi\rangle$.

We say that $\varphi = \langle x_\xi\rangle$ is *finite*, *simple*, or *transfinite* if $\alpha < \omega$, $\alpha = \omega$, or $\alpha > \omega$, respectively.

Let $\mathcal{S}$ denote the set of all subsequences ψ of φ such that $\beta = \alpha$ and let $\mathcal{S}_\varphi$ be a subset of $\mathcal{S}$. We say that $\mathcal{S}_\varphi$ is a *complete system of subsequences* of φ provided for each $\psi \in \mathcal{S}$ there exists a subsequence χ of ψ such that $\chi \in \mathcal{S}_\varphi$.

Proposition 1. *Let φ be a sequence of points of X. Let ψ be a subsequence of φ and let χ be a subsequence of ψ. Then χ is a subsequence of φ.*

P r o o f. Trivial.

According to Definition 1, a sequence $\langle x_\xi \rangle$ of type ω is a linearly ordered set of pairs $(0, x_0) < (1, x_1) < \ldots$. Usually a sequence $\langle x_n \rangle$ is indexed by the set of natural numbers $\mathbb{N}$. With the obvious abuse of formalism, $\langle x_n \rangle$ can be identified with the set of ordered pairs $(1, x_1) < (2, x_2) < \ldots$ and its subsequence $\langle x_{n_i} \rangle$ with the set of ordered pairs $(1, x_{n_1}) < (2, x_{n_2}) < \ldots$. Hence usual sequences and their subsequences can be considered as simple sequences and their subsequences from Definition 1.

Analogously (since ω^2 is isotonic to the set $\mathbb{N} \times \mathbb{N}$ carrying the lexicographic order: $(m, n) < (r, s)$ iff either $m < r$ or $m = r$ and $n < s$), a transfinite sequence $\langle x_\xi \rangle$ of type ω^2 can be identified with a double sequence defined as follows.

Definition 2. Let X be a non-void set. Let φ be a map of the lexicographically ordered set $\mathbb{N} \times \mathbb{N}$ into X, considered as a subset of $\mathbb{N} \times \mathbb{N} \times X$. Then the set φ, ordered by $((m, n), \varphi((m, n))) < ((r, s), \varphi((r, s)))$ iff $(m, n) < (r, s)$, is called a *double sequence*. Denote $\varphi((m, n)) = x_{mn}$ and $\varphi = \langle x_{mn} \rangle$.

For a fixed $m \in \mathbb{N}$, the simple sequence $\langle x_{mn} \rangle_{n=1}^{\infty}$ is called the *m-th straight-sequence* and its subsequences are called *straight-subsequences* in $\varphi = \langle x_{mn} \rangle$; we condense $\langle x_{mn} \rangle_{n=1}^{\infty}$ to $\langle x_{mn} \rangle_n$.

For a map $g\colon \mathbb{N} \to \mathbb{N}$, the simple sequence $\langle x_{mg(m)} \rangle_m$ is called a *cross-sequence* and its subsequences are called *cross-subsequences* in $\varphi = \langle x_{mn} \rangle$.

Let A be a subset of $\mathbb{N} \times \mathbb{N}$ isotonic to $\mathbb{N} \times \mathbb{N}$ and let $h\colon \mathbb{N} \times \mathbb{N} \to A$ be an isotonic map. Define $\psi\colon \mathbb{N} \times \mathbb{N} \to X$ by $\psi((i, j)) = \varphi(h((i, j)))$, $(i, j) \in \mathbb{N} \times \mathbb{N}$. Denote $\psi((i, j)) = y_{ij}$ and $\psi = \langle y_{ij} \rangle$. Then the double sequence $\psi = \langle y_{ij} \rangle$ is called a *double subsequence* of $\varphi = \langle x_{mn} \rangle$.

Note. The double subsequence $\langle y_{ij} \rangle$ is the ordered set of pairs $((1, 1), x_{h((1,1))}) < ((1, 2), x_{h((1,2))}) < \ldots < ((i, j), x_{h((i,j))}) < \ldots$. If we write $h(p, q)$ or $h(r, s)$ or $h(m, n)$ instead of $h(i, j)$, then we have a subsequence $\langle y_{pq} \rangle$, $\langle y_{rs} \rangle$, $\langle y_{mn} \rangle$, respectively. We mostly use the notation $h(i, j)$ and $\langle y_{ij} \rangle$.

A general map $H\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ can be denoted $h((i, j)) = (f^i(j), g^i(j))$ where f^i, g^i are functions on $\mathbb{N}$ into $\mathbb{N}$.

Lemma 1. *Let h be a map of $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N} \times \mathbb{N}$, $h((i, j)) = (f^i(j), g^i(j))$. Then h is isotonic iff the following condition is satisfied:*

(1) *$\langle f^i(j), g^i(j) \rangle_j$ is a simple sequence increasing in the lexicographically ordered set $\mathbb{N} \times \mathbb{N}$ and, further, there is a function $e\colon \mathbb{N} \to \mathbb{N}$ such that $f^i(j) = f^i(e(i))$ for all $j \geqslant e(i)$ and finally $f^i(1) < f^{i+1}(1)$ for all $i \in \mathbb{N}$.*

P r o o f. Necessity. Let h be an isotonic map. Let $i \in \mathbb{N}$ be fixed. Then $\langle h((i, j)) \rangle_j$ is a simple sequence increasing in the lexicographically ordered set $\mathbb{N} \times \mathbb{N}$.

Therefore there are maps f^i, g^i of $\mathbb{N}$ into $\mathbb{N}$ such that $h((i,j)) = (f^i(j), g^i(j))$ for all $j \in \mathbb{N}$ and $\langle (f^i(j), g^i(j)) \rangle_i$ is a simple sequence increasing in the lexicographically ordered set $\mathbb{N} \times \mathbb{N}$. Suppose that, on the contrary, there is a subsequence $\langle j_r \rangle$ of $\langle j \rangle$ such that $f^i(j_1) < f^i(j_2) < \ldots$. Then the set of pairs $(f^i(j_r), g^i(j_r))$, $r \in \mathbb{N}$, is cofinal in $\mathbb{N} \times \mathbb{N}$. Thus the set of pairs (i, j_r), $r \in \mathbb{N}$, is cofinal in $\mathbb{N} \times \mathbb{N}$, because h is isotonic. This is a contradiction. Therefore there is a number $e(i) \in \mathbb{N}$ such that $f^i(j) = f(e(i))$ for all $j \geqslant e(i)$. Because $(i, e(i)) < (i+1, 1)$, it follows that $f^i(e(i)) < f^{i+1}(1)$ and also $f^i(1) < f^{i+1}(1)$.

Sufficiency. Suppose that condition (1) holds. Let A denote the set of all pairs $(f^i(j), g^i(j))$, $(i,j) \in \mathbb{N} \times \mathbb{N}$. It is a subset of the lexicographically ordered $\mathbb{N} \times \mathbb{N}$. Put $\overline{h}(f^i(j), g^i(j)) = (i,j)$ and prove that $\overline{h}\colon A \to \mathbb{N} \times \mathbb{N}$ is an isotonic map. Let $(f^i(j), g^i(j)) < (f^r(s), g^r(s))$. Two cases are possible. Either $f^i(f) < f^r(s)$ or $f^i(j) = f^r(s)$ and $g^i(j) < g^r(s)$. In the first case, it follows from (1) that $f^i(j) \leqslant f^i(e(i)) < f^{i+1}(1) \leqslant f^r(1) \leqslant f^r(s)$. Consequently, $i < r$ and so $(i,j) < (r,s)$. In the second case we deduce from (1) that $i = r$. Hence $g^i(j) < g^i(s)$. Since $\langle (f^i(n), g^i(n)) \rangle_i$ is an increasing sequence of pairs of $\mathbb{N} \times \mathbb{N}$, we have $g^i(j) < g^i(j+1) < \ldots < g^i(s)$ and $f^i(j) = f^i(j+1) = \ldots = f^i(s)$. Therefore $j < s$. Hence $(i,j) < (r,s)$. Thus the inverse map $\overline{h}^{-1} = (f^i(j), g^i(j)) = h(i,j)$ is isotonic. This completes the proof because the map $\overline{h}^{-1} = h$, by (1). □

From Lemma 1 it follows: $\langle y_{ij} \rangle$ is a subsequence of $\langle x_{mn} \rangle$ iff (1) holds.

Definition 3. Let $\langle x_{mn} \rangle$ be a double sequence of points of a set X. Let $\langle y_{ij} \rangle$ be a double subsequence of $\langle x_{mn} \rangle$. We say that $\langle y_{ij} \rangle$ is a *two-fold subsequence* of $\langle x_{mn} \rangle$ if each straight-sequence in $\langle y_{ij} \rangle$ is a straight-subsequence in $\langle x_{mn} \rangle$.

Example 1. Let $X = R$. Denote $x_{mn} = m + n(n+1)^{-1}$. Then $\langle x_{mn} \rangle$ is a double sequence of real numbers. Let $e(i) = i$, and $f^i(j) = 2i+1$, $j \geqslant i$, $f^i(j) = 2i$, $j < i$, and $g^i(j) = 2j$, $j \geqslant i$, $g^i(j) = 3j$, $j < i$. Denote $y_{ij} = x_{f^i(j) g^i(j)}$. It follows from Lemma 1 and Definition 2 that $\langle y_{ij} \rangle$ is a double subsequence of $\langle x_{mn} \rangle$. It is not a two-fold subsequence of $\langle x_{mn} \rangle$ because no straight-sequence $\langle y_{ij} \rangle_j$, $i \geqslant 2$, is a straight-subsequence in $\langle x_{mn} \rangle$.

Definition 4. Let $\langle x_{mn} \rangle$ be a double sequence of points of a set X. Let $k \in \mathbb{N}$, let $f\colon \mathbb{N} \to \mathbb{N}$ be a map and let $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ an isotonic map such that $h(i,j) = (k - 1 + i, f(i) - 1 + j)$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. Put $y_{ij} = x_{h(i,j)}$. Then $\langle y_{ij} \rangle$ is a double subsequence of $\langle x_{mn} \rangle$. It is called the (k,f)-*subsequence* of the sequence $\langle x_{mn} \rangle$.

Corollary 1. *Let $\langle x_{mn} \rangle$ be a double sequence of points of a set X. The system of all two-fold subsequences of the sequence $\langle x_{mn} \rangle$ is a complete system of double subsequences of $\langle x_{mn} \rangle$.*

Proof. Let $\langle y_{ij}\rangle$ be a double subsequence of $\langle x_{mn}\rangle$. Let $e\colon \mathbb{N}\to\mathbb{N}$ be a map with property (1) in Lemma 1. Then the $(1,e)$-subsequence of $\langle y_{ij}\rangle$ is a two-fold subsequence of $\langle x_{mn}\rangle$. $\square$

Remark. A suitable notation for two-fold subsequences of a double sequence follows directly from Definition 3. Let $\langle z_{ij}\rangle$ be a two-fold subsequence of a sequence $\langle x_{mn}\rangle$. There is a subsequence $\langle m_i\rangle$ and subsequences $\langle n^i_j\rangle_j, i\in\mathbb{N}$, of $\langle n\rangle$ such that $z_{ij}=x_{m_i n^i_j}$. Hence $\langle z_{ij}\rangle=\langle x_{m_i n^i_j}\rangle$. This notation has been used in [NO77] and [NO85].

Lemma 2. *Let $\langle x_{mn}\rangle$ be a double sequence of points of a set X. Then there exists a one-to-one double subsequence of $\langle x_{mn}\rangle$ iff there is a double subsequence $\langle y_{ij}\rangle$ such that $\{y_{ij};\ j\in\mathbb{N}\}$ are infinite sets for all $i\in\mathbb{N}$.*

Proof. The necessity is trivial.

Sufficiency. Now let $\langle y_{ij}\rangle$ be a double subsequence of $\langle x_{mn}\rangle$ with infinite $\{y_{ij};\ j\in\mathbb{N}\}$ for all $i\in\mathbb{N}$. With respect to Lemma 2, we can suppose that $\langle y_{ij}\rangle$ is a two-fold subsequence of $\langle x_{mn}\rangle$. Denote $n^1_1=1$ and put $z_{11}=y_{1n^1_1}$. Let $p>1$ be an integer number. Suppose that we have chosen distinct points $z_{ij}=y_{in^j_j}, j\leqslant p, i\leqslant p$, $n^i_1<n^i_2<\ldots<n^i_p, i\leqslant p$. Since $\{y_{ij};\ j\in\mathbb{N}\}$ for all $i\in\mathbb{N}$ are infinite sets, we deduce that there are numbers $n^i_{p+1}>n^i_p$, $i\leqslant p$, and numbers $n^{p+1}_1<n^{p+1}_2<\ldots<n^{p+1}_{p+1}$ such that the points $y_{in^i_j}, i\leqslant p+1, j\leqslant p+1$, are distinct. Put $z_{ij}=y_{in^i_j}$, $i\leqslant p+1$, $j\leqslant p+1$. In such a way, by mathematical induction, we have constructed a one-to-one two-fold subsequence $\langle z_{ij}\rangle$ of $\langle y_{ij}\rangle$. It is also a two-fold subsequence of $\langle x_{mn}\rangle$ because $\langle y_{ij}\rangle$ is a two-fold subsequence of $\langle x_{mn}\rangle$. $\square$

III

In this section we define the notion of a double convergence in a convergence closure space $(X,\mathcal{L},\lambda)$ with the maximal convergence $\mathcal{L}=\mathcal{L}^*$. The definition of $\lim x_{mn}=x$ is analogous to the definition of maximal convergence of simple sequences, see 5. in Section I.

Definition 5. Let $(X,\mathcal{L},\lambda)$ be a convergence closure space. Let $\mathcal{D}$ denote the collection of all pairs $(\langle x_{mn}\rangle,x)$, $x_{mn}\in X$, $x\in X$, such that the following property is satisfied:

(2) If $\langle y_{ij}\rangle$ is a double subsequence of $\langle x_{mn}\rangle$ then $x\in\lambda\{y_{ij};\ i,j\in\mathbb{N}\}$.

Call $\mathcal{D}$ the *double convergence* for the space $(X,\mathcal{L},\lambda)$. Instead of $(\langle_{mn}\rangle,x)\in\mathcal{D}$ we write $\mathcal{D}$-$\lim x_{mn}=x$ or, simply, $\lim x_{mn}=x$, and say that the double sequence $\langle x_{mn}\rangle$ converges to the point x or, that x is a limit of the double sequence $\langle x_{mn}\rangle$.

Lemma 3. $\lim x_{mn} = x$ *iff the following property is satisfied:*

(3) *If* $\langle y_{ij}\rangle$ *is a two-fold subsequence of* $\langle x_{mn}\rangle$ *then* $x \in \lambda\{y_{ij}\,;\ i,j \in \mathbb{N}\}$.

P r o o f. If $\lim x_{mn} = x$ then (3) holds because each two-fold subsequence is a double subsequence. Suppose that (3) is true. Let $\langle y_{ij}\rangle$ be a double subsequence of $\langle x_{mn}\rangle$. According to Corollary 1 there is a two-fold subsequence $\langle z_{rs}\rangle$ of $\langle x_{mn}\rangle$ which is a double subsequence of $\langle y_{ij}\rangle$. Hence $x \in \lambda\{z_{rs}\,;\ r,s \in \mathbb{N}\}$ and also $x \in \lambda\{y_{ij}\,;\ i,j \in \mathbb{N}\}$. Therefore $\lim x_{mn} = x$. $\square$

Lemma 4. $\lim x_{mn} = x$ *iff the following implication is true:*

(4) *If* $L(x)$ *is a closure neighbourhood of the point* x*, then there are a number* $k \in \mathbb{N}$ *and a function* $g\colon \mathbb{N} \to \mathbb{N}$ *such that* $x_{mn} \in L(x)$ *whenever* $m \geqslant k$*,* $n \geqslant g(m)$.

P r o o f. Suppose that $\lim x_{mn} = x$ and (4) does not hold. There is $m_1 \in \mathbb{N}$ and a straight-subsequence $\langle x_{m_1 m^1_j}\rangle$ no point of which belongs to $L(x)$. Inductively, let $p \in \mathbb{N}$. Suppose that we have chosen numbers $m_1 < m_2 < \ldots < m_{p-1}$ and straight-subsequences $\langle x_{m_i n^i_j}\rangle$, $i < p$, such that no point $x_{m_i n^i_j}$ belongs to $L(x)$. Since (4) is not true, there are $m_p > m_{p-1}$ and a row subsequence $\langle x_{m_p n^p_j}\rangle$ no point of which belongs to $L(x)$. This way we can construct a double subsequence $\langle x_{m_i n^i_j}\rangle$ of $\langle x_{mn}\rangle$ no point of which belongs to the neighbourhood $L(x)$. Therefore $x \notin \lambda\{x_{m_i n^i_j}\,;\ i,j \in \mathbb{N}\}$. This is a contradiction with Definition 5.

Now, suppose that (4) holds. Let $\langle y_{ij}\rangle$ be a double subsequence of $\langle x_{mn}\rangle$ and let $L(x)$ be a neighbourgood of x. Clearly $\emptyset \neq L(x) \cap \{y_{ij}\,;\ i,j \in \mathbb{N}\}$ and $x \in \lambda\{y_{ij}\,;\ i,j \in \mathbb{N}\}$. Hence $\lim x_{mn} = x$, by Definition 5. $\square$

Notice that the implication (4) can be reformulated as follows: if $L(x)$ is closure neighbourhood of x, then there is a (k,g)-subsequence $\langle z_{rs}\rangle$ of $\langle x_{mn}\rangle$ such that $\{z_{rs}\,;\ r,s \in \mathbb{N}\} \subset L(x)$.

From Definition 5 and in view of 4. in I, it follows that: (i) if $x_{mn} = x$ for all $m,n \in \mathbb{N}$, then $\lim x_{mn} = x$, (ii) if $\lim x_{mn} = x$ and $\langle y_{ij}\rangle$ is a double subsequence of $\langle x_{mn}\rangle$ then $\lim y_{ij} = x$. A double convergence need not be single-valued. This is shown by a well-known example of a compact Fréchet space which is not Hausdorff. Let X be a set of points $x, y, x_{mn},\ m,n \in \mathbb{N}$. Let $\mathcal{L}$ be a convergence for X such that each straight-sequence in $\langle x_{mn}\rangle$ $\mathcal{L}$-converges to x and each cross-sequence in $\langle x_{mn}\rangle$ $\mathcal{L}$-converges to y. According to Definition 5, $\lim x_{mn} = x$ and $\lim x_{mn} = y$, as well. Notice that if $(X, \mathcal{L}, \lambda)$ is a convergence closure H-space, then (by Lemma 4) the double convergence for X is single-valued.

Lemma 5. *Let $\mathcal{D}$ be the double convergence for a convergence closure space $(X, \mathcal{L}, \lambda)$. Then $\mathcal{D}$ satisfies the following maximality condition:*

($*$) *Let $\langle x_{mn} \rangle$ be a double sequence of points and x a point of X. If each double subsequence $\langle y_{ij} \rangle$ of $\langle x_{mn} \rangle$ has a double subsequence $\langle z_{rs} \rangle$ of $\langle y_{ij} \rangle$ converging to x, then $\lim x_{mn} = x$.*

P r o o f. Trivial. $\square$

Lemma 6. *Let $\langle z_{mn} \rangle$ be a double sequence of points of a convergence closure space $(X, \mathcal{L}, \lambda)$. Let $\lim z_{mn} = x$. Then there is a straight-subsequence or a cross-subsequence in $\langle z_{mn} \rangle$ which $\mathcal{L}$-converges to the point x.*

P r o o f. It follows from Definition 5 that $x \in \lambda\{z_{mn};\ m, n \in \mathbb{N}\}$. If $z_{mn} = x$ for infinitely many indexes, then the assertion is trivial. In the opposite case, there is a double subsequence of points $t_{pq} \neq x$ and so, because $x \in \lambda\{t_{pq};\ p, q \in \mathbb{N}\}$, there is a one-to-one simple sequence of points of the set $\{z_{mn};\ m, n \in \mathbb{N}\}$ $\mathcal{L}$-converging to the point x. From this we deduce that there are functions $f, g\colon \mathbb{N} \to \mathbb{N}$ and distinct pairs $(f(i), g(i))$ such that $\langle z_{f(i)g(i)} \rangle$ is a constant or a one-to-one sequence $\mathcal{L}$-converging to x. Denote $z_{f(r)g(r)} = t_r$. Then $\mathcal{L}$-$\lim t_r = x$. Two cases are possible:

1. There is a subsequence $\langle r_i \rangle$ of $\langle r \rangle$ such that $f(r_1) < f(r_2) < \ldots$. Consequently $\langle z_{f(r_i)g(r_i)} \rangle$ is a cross-subsequence in $\langle z_{mn} \rangle$ and a subsequence of $\langle t_r \rangle$. Hence $\mathcal{L}$-$\lim z_{f(r_i)g(r_i)} = x$.
2. There are positive integers p, r_0 such that $f(r) = p$, $r > r_0$. Because the pairs $(f(r), g(r))$ are distinct there is a subsequence $\langle s_i \rangle$ of $\langle r \rangle$ such that $g(s_1) < g(s_2) < \ldots$. Hence we have a sequence $\langle z_{pg(s_i)} \rangle = \langle t_{s_i} \rangle$. It is a straight-subsequence in $\langle z_{mn} \rangle$ and a subsequence of $\langle t_r \rangle$ as well. Therefore $\mathcal{L}$-$\lim z_{pg(s_i)} = x$.

$\square$

Theorem 1. *Let $(X, \mathcal{L}, \lambda)$ be a convergence closure space. Then $\lim x_{mn} = x$ iff in each double subsequence of $\langle x_{mn} \rangle$ there is a straight-subsequence or a cross-subsequence in $\langle x_{mn} \rangle$ $\mathcal{L}$-converging to the point x.*

P r o o f. Necessity. Let $\langle y_{ij} \rangle$ be a double subsequence of $\langle x_{mn} \rangle$. According to Corollary 1, there is a two-fold subsequence $\langle z_{rs} \rangle$ of both sequences $\langle y_{ij} \rangle$ and $\langle x_{mn} \rangle$. From Lemma 6 it follows that there is a straight-subsequence or a cross-subsequence in $\langle z_{rs} \rangle$, hence also in $\langle x_{mn} \rangle$, which $\mathcal{L}$-converges to x.

Sufficiency. Follows directly from Definition 5. $\square$

Corollary 2. $\lim x_{mn} = x$ *iff in each two-fold subsequence of $\langle x_{mn} \rangle$ there is a straight-subsequence or a cross-subsequence in $\langle x_{mn} \rangle$ $\mathcal{L}$-converging to the point x.*

Proof follows instantly from Theorem 1 and Corollary 1.

R e m a r k. In [NO77] and [NO85] I have defined the double convergence by means of double subsequences $\langle x_{m_i n_j^i}\rangle$ of a double sequence $\langle x_{mn}\rangle$ in the same manner as in Corollary 2. These subsequences are now called two-fold subsequences. Theorem 1 and Corollary 2 show that both definitions of double convergence, namely Definition 5 and that one in [NO85], are equivalent. □

Definition 6. Let $(X, \mathcal{L}, \lambda)$ be a convergence closure space and $\mathcal{D}$ the double convergence for X. Let $\mathcal{D}\text{-}\lim x_{mn} = x$.

If each cross-sequence in $\langle x_{mn}\rangle$ $\mathcal{L}$-converges to x, then we say that $(\langle x_{mn}\rangle, x)$ is a π-*element* of $\mathcal{D}$, $\langle x_{mn}\rangle$ is a π-*sequence* and x is a π-*point.*

If no cross-subsequence in $\langle x_{mn}\rangle$ $\mathcal{L}$-converges to x, then we say that $(\langle x_{mn}\rangle, x)$ is a ϱ-*element* of $\mathcal{D}$, $\langle x_{mn}\rangle$ is a ϱ-*sequence* and x is a ϱ-*point.*

If in each double subsequence of $\langle x_{mn}\rangle$ there is a cross-subsequence which $\mathcal{L}$-converges to x and a cross-subsequence no subsequence of which $\mathcal{L}$-converges to x, then we say that $(\langle x_{mn}\rangle, x)$ is a σ-*element* of $\mathcal{D}$, $\langle x_{mn}\rangle$ is a σ-*sequence* and x is a σ-*point.*

Let $\mathcal{D}_\pi$, $\mathcal{D}_\varrho$ and $\mathcal{D}_\sigma$ denote the set of all π-elements, ϱ-elements and σ-elements, respectively. We write $\pi\text{-}\lim x_{mn} = x$, $\varrho\text{-}\lim x_{mn} = x$ and $\sigma\text{-}\lim x_{mn} = x$ instead of $(\langle x_{mn}\rangle, x) \in \mathcal{D}_\pi$, $(\langle x_{mn}\rangle, x) \in \mathcal{D}_\varrho$ and $(\langle x_{mn}\rangle, x) \in \mathcal{D}_\sigma$, respectively.

Definition 7. Let $(X, \mathcal{L}, \lambda)$ be a convergence space. Let $\lim x_{mn} = x$. We say that $\langle x_{mn}\rangle$ is a sequence of the *first* (*second*) *kind* if there is a one-to-one (constant) simple sequence $\langle x_m\rangle$ of points of X such that $\mathcal{L}\text{-}\lim x_m = x$ and $\mathcal{L}\text{-}\lim x_{mn} = x_m$ for all $m \in \mathbb{N}$. We say that $\langle x_{mn}\rangle$ is of the *third kind* if no double subsequence of $\langle x_{mn}\rangle$ is either of the first or of the second kind.

Next we present examples of Fréchet spaces with π-, ϱ- and σ-points, respectively. Clearly, to construct a maximal convergence on a set X we can proceed as follows:

(i) we start with a family $\mathcal{L}_g$ of so-called convergence generating elements $(\langle x_n\rangle, x)$, where each $\langle x_n\rangle$ is a one-to-one simple sequence in X and $x \in X$;

(ii) enlarge $\mathcal{L}_g$ to $\mathcal{L}$ by adding all elements of the form $(\langle x\rangle, x)$, $x \in X$, and $(\langle x_{n_i}\rangle, x)$, where $\langle x_{n_i}\rangle$ is a subsequence of $\langle x_n\rangle$;

(iii) if $\{\{x_n;\ n \in \mathbb{N}\};\ (\langle x_n\rangle, x) \in \mathcal{L}_g\}$ is an almost disjoint family of subsets of X (each two sets have a finite intersection; abbr. a. d.), then $\mathcal{L}$ is a convergence and $\mathcal{L}^*$ is a maximal convergence on X.

Example 2. Let $(Q, \mathcal{L}, \lambda)$ be the usual convergence closure space of rational numbers.

a) Put $x_{mn} = 2^{-m}(1 + 2^{-n})$, $m, n \in \mathbb{N}$. Then $\lim x_{mn} = 0$ and $\langle x_{mn}\rangle$ is a one-to-one π-sequence of the first kind. Notice that $\langle 2^{-1} + 2^{-(n+1)}\rangle_{n=1}^{\infty}$ is the first straight-sequence in $\langle x_{mn}\rangle$ and the sequence $\langle x_{mn}\rangle$ is a set of pairs $((1,1),0.75) < ((1,2),0.3625) < \ldots$ increasing in $(\mathbb{N}\times\mathbb{N})\times\mathbb{Q}$.

b) Put $x_{mn} = 2^{1-m}(2n-1)^{-1}$, $m,n\in\mathbb{N}$. Then $\lim x_{mn} = 0$ and $\langle x_{mn}\rangle$ is a one-to-one π-sequence of the second kind.

c) Let $\langle x_{mn}\rangle$ be a one-to-one double sequence of rational numbers such that $\lim\limits_{n\to\infty} x_{mn} = m^{-1}\sqrt{2}$, $m\in\mathbb{N}$. Then $\langle x_{mn}\rangle$ is a one-to-one π-sequence of the third kind.

d) Put $x_{mn} = m^{-1}$, $m,n\in\mathbb{N}$. Then $\lim x_{mn} = 0$ and $\langle x_{mn}\rangle$ is a π-sequence of the first kind. There is no one-to-one double subsequence of $\langle x_{mn}\rangle$.

Example 3. Put $X = \{p\}\cup\mathbb{N}\times\mathbb{N}$. Define a convergence $\mathcal{L}^*$ by means of convergence generating elements $\mathcal{L}_g$ as follows:

$\mathcal{L}_g$ consists of all elements $(\langle(m,n)\rangle, p)$, $m\in\mathbb{N}$. Then $(X,\mathcal{L}^*,\lambda)$ is a Fréchet space and $\langle(m,n)\rangle$ is a one-to-one ϱ-sequence of the second kind.

Following W. Sierpiński, define an order on the set $\mathbb{N}^{\mathbb{N}}$ of all functions of $\mathbb{N}$ into $\mathbb{N}$ as follows: $f\prec g$ iff $f(n) < g(n)$ for all but finitely many $n\in\mathbb{N}$. It is known that if $\{f^\xi;\ \xi<\alpha<\omega_1\}$ is a subset of $\mathbb{N}^{\mathbb{N}}$, then there exists $f^\alpha\in\mathbb{N}^{\mathbb{N}}$ such that $f^\xi\prec f^\alpha$ for all $\xi<\alpha$; further, under the continuum hypothesis there is a well-ordered cofinal subset of $\mathbb{N}^{\mathbb{N}}$ (of the cardinality of continuum). Using this fact, we construct a Fréchet space with a σ-point.

Let $\langle x_{mn}\rangle$ be a one-to-one double sequence; define an order on the set of all its cross-sequences: for $f,g\in\mathbb{N}^{\mathbb{N}}$ put $\langle x_{mf(m)}\rangle\prec\langle x_{mg(m)}\rangle$ iff $f\prec g$ (note: $\{x_{mf(m)};\ m\in\mathbb{N}\}$ and $\{x_{mg(m)};\ m\in\mathbb{N}\}$ are almost disjoint sets).

Proposition 2. *Under the continuum hypothesis there is a well-ordered family $\mathfrak{F}$ of cross-sequences in $\langle x_{mn}\rangle$ such that if $\langle x_{m_i n^i_j}\rangle$ is a two-fold subsequence of $\langle x_{mn}\rangle$, then there are two different cross-sequences in $\mathfrak{F}$ and their subsequences such that each subsequence is a cross-sequence in $\langle x_{m_i n^i_j}\rangle$.*

Proof. Observe that the cardinality of the set of all infinite subsequences of the sequence $\langle x_{1n}\rangle$ is $\aleph_1$. Let T denote the set of all two-fold subsequences of $\langle x_{mn}\rangle$. Then $|T| = \aleph_1$. Let $T_\xi, \xi<\omega_1$, be the elements of T. We say that a cross-sequence $\langle x_{mf(m)}\rangle$ has the property $p(T_\xi)$ if there is a subsequence of it which is also a cross-subsequence in the two-fold sequence T_ξ. Suppose that we have already chosen cross-sequences having the property $p(T_\xi)$ such that

$$\langle x_{mg^0(m)}\rangle\prec\langle x_{mh^0(m)}\rangle\prec\langle x_{mg^1(m)}\rangle\prec\ldots\prec\langle x_{mg^\xi(m)}\rangle\prec\langle x_{mh^\xi(m)}\rangle, \tag{5}$$

$\xi<\alpha<\omega_1$. We have to prove that there are functions g^α and h^α such that $\langle x_{mh^\xi(m)}\rangle\prec\langle x_{mg^\alpha(m)}\rangle\prec\langle x_{mh^\alpha(m)}\rangle$, where both cross-subsequences $\langle x_{mg^\alpha(m)}\rangle$ and

$\langle x_{mh^\alpha(m)}\rangle$ have the property $p(T_\alpha)$. It is known (see above) that there are functions g, h such that $\langle x_{mh^\xi(m)}\rangle \prec \langle x_{mg(m)}\rangle \prec \langle x_{mh(m)}\rangle$, $\xi < \alpha$. Let $T_\alpha = \langle x_{m_i n_j^i}\rangle$ and define functions g^α, h^α as follows. If $m \neq m_i$ put $g^\alpha(m) = g(m)$ and $h^\alpha(m) = h(m)$. Let $m = m_i$. Notice that $\langle n_j^i\rangle$ is an increasing sequence of positive integers. Choose $p < q$ among them such that $h(m_i) < p$. Then put $g^\alpha(m_i) = p$ and $h^\alpha(m_i) = q$. Hence we have functions g^α, h^α and cross-sequences $\langle x_{mg^\alpha(m)}\rangle$, $\langle x_{mh^\alpha(m)}\rangle$ having the property $p(T_\alpha)$ and such that (5) holds for each $\xi \leqslant \alpha$.

We have just constructed, by transfinite induction, an almost disjoint family of cross-sequences in $\langle x_{mn}\rangle$ with properties $p(T_\xi)$, $\xi < \omega_1$. Denote $\mathfrak{F}_1$ ($\mathfrak{F}_2$) the family of all cross-sequences $\langle x_{mg^\xi(m)}\rangle$ ($\langle x_{mh^\xi(m)}\rangle$), $\xi < \omega_1$. Both families are a. d. and $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2$ is an a. d. family, too. □

Example 4. a) Let X be a set of points $x = (1,1)$ and $x_{mn} = (m(m+1)^{-1}, n(n+1)^{-1})$, $m,n \in \mathbb{N}$. Let $\mathcal{L}^*$ be the convergence for X defined by the following generating convergence elements: $(\langle x_{mg^\xi(m)}\rangle, x)$, $\xi < \omega_1$. We get a Fréchet space $(X, \mathcal{L}^*, \lambda)$. Then $\lim x_{mn} = x$, by Definition 5. Let $\langle x_{mh^\xi(m)}\rangle$ be a cross-sequence of the system $\mathfrak{F}_2$. No subsequence of it $\mathcal{L}$-converges to x because $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are disjoint systems. Consequently, $\langle x_{mn}\rangle$ is a one-to-one σ-sequence of the third kind.

b) Let X be a set of points $x = (1,1)$, $x_{mn} = (m(m+1)^{-1}, n(n+1)^{-1})$, $m,n \in \mathbb{N}$. Let $(\langle x_{mh^\xi(m)}\rangle, x)$, $\xi < \omega_1$ and $(\langle x_{mn}\rangle_n, x)$ $m \in \mathbb{N}$, be generating convergence elements. Then we get a Fréchet space $(X, \mathcal{L}^*, \lambda)$ and a σ-sequence $\langle x_{mn}\rangle$ of the second kind.

c) Let X be a set of points $x = (1,1)$, $x_m = (m(m+1)^{-1}, 1)$, $x_{mn} = (m(m+1)^{-1}, n(n+1)^{-1})$, $m,n \in \mathbb{N}$. Consider the following generating convergence elements $(\langle x_m\rangle, x)$, $(\langle x_{mn}\rangle_n, x_m)$, $m \in \mathbb{N}$, and $(\langle x_{mg^\xi(m)}\rangle, x)$, $\xi < \omega_1$. We get a Fréchet space $(X, \mathcal{L}^*, \lambda)$. The point $x = \lim x_{mn}$ is a σ-point and $\langle x_{mn}\rangle$ a one-to-one σ-sequence of the first kind.

Example 5. Let X consist of points $x = (1,1)$, $x_{mn} = (m(m+1)^{-1}, n(n+1)^{-1})$, $m,n \in \mathbb{N}$. Denote $y = (2^{-1}, 2^{-1})$. Let $\langle x_{mg^\xi(m)}\rangle, x)$ and $(\langle x_{mh^\xi(m)}\rangle, y)$ be generating convergence elements. Let $(X, \mathcal{L}^*, \lambda)$ denote the resulting Fréchet space. Then, according to Definition 5, $\mathcal{D}$-$\lim x_{mn} = x$ and $\mathcal{D}$-$\lim y_{mn} = y$. Hence, by Definition 6, both points x and y are σ-points. Let $L(x)$ and $L(y)$ be neighbourgoods of x and y. Hence $L(x) \cap L(y) \neq \emptyset$, by Lemma 4. Therefore $(X, \mathcal{L}^*, \lambda)$ is not Hausdorff. Notice that the double convergence $\mathcal{D}$ for X is multivalued even though $\mathcal{L}$ is single-valued.

Lemma 7. *Let $\langle y_{ij}\rangle$ be a double subsequence of a π-sequence (ϱ-sequence, σ-sequence) of $\langle x_{mn}\rangle$. Then $\langle y_{ij}\rangle$ is a π-sequence (ϱ-sequence, σ-sequence).*

Proof. Follows directly from Definition 6. □

Lemma 8. *Let ϱ-$\lim x_{mn} = x$. Then there are $k \in \mathbb{N}$ and straight-subsequences $\langle x_{in^i_j} \rangle_i$ in $\langle x_{mn} \rangle$ with $\mathcal{L}$-$\lim x_{in^i_j} = x$ for each $i \geqslant k$.*

Proof. Otherwise there would be a double subsequence $\langle t_{rs} \rangle$ of $\langle x_{mn} \rangle$ such that $x \notin \lambda\{t_{rs};\ r, s \in \mathbb{N}\}$. This is not possible because $\lim x_{mn} = x$. □

Lemma 9. *Let ϱ-$\lim x_{mn} = x$ (σ-$\lim x_{mn} = x$). There is a one-to-one double subsequence $\langle z_{rs} \rangle$ of $\langle x_{mn} \rangle$ such that ϱ-$\lim z_{rs} = x$ (σ-$\lim z_{rs} = x$).*

Proof. Suppose that there is no one-to-one double subsequence of $\langle x_{mn} \rangle$. It follows from Lemma 3 that there is a double subsequence $\langle t_{rs} \rangle$ of $\langle x_{mn} \rangle$ with constant straight-sequences $\langle t_{rs} \rangle_s = \langle t_{r1} \rangle$. Since $\lim t_{rs} = x$ it follows that there is a cross-subsequence in $\langle t_{rs} \rangle$ $\mathcal{L}$-converging to x. This is a contradiction in view of Lemma 7. Consequently, there is a one-to-one double subsequence $\langle z_{rs} \rangle$ of $\langle x_{mn} \rangle$. It is a ϱ-sequence (σ-sequence), by Lemma 7. □

The following Corollary can be easily proved by Lemmas 8 and 9.

Corollary 3. *Let ϱ-$\lim x_{mn} = x$. There is a one-to-one double subsequence $\langle y_{ij} \rangle$ of $\langle x_{mn} \rangle$ such that ϱ-$\lim y_{ij} = x$ and $\mathcal{L}$-$\lim y_{ij} = x$, $i \in \mathbb{N}$.*

Theorem 2. *Let $(X, \mathcal{L}, \lambda)$ be a convergence closure space. Let $\lim x_{mn} = x$. Then there is a π-subsequence or a ϱ-subsequence or a σ-subsequence of the sequence $\langle x_{mn} \rangle$.*

Proof. If there is a π-subsequence, the proof is finished. Let there be no π-subsequence of $\langle x_{mn} \rangle$. Then there is in each subsequence of $\langle x_{mn} \rangle$ at least one cross-subsequence which does not $\mathcal{L}$-converge to the point x. With respect to 6. in Section I we can suppose that no subsequence of it $\mathcal{L}$-converges to x. Let $\langle y_{ij} \rangle$ be any of these subsequences. If there is a ϱ-subsequence of $\langle y_{ij} \rangle$ we have nothing to prove. Hence suppose that there is no ϱ-subsequence of $\langle y_{ij} \rangle$. It follows that there is at least one cross-subsequence in each double subsequence of $\langle y_{ij} \rangle$ which $\mathcal{L}$-converges to the point x. Therefore $\langle y_{ij} \rangle$ is a σ-subsequence of $\langle x_{mn} \rangle$, by Definition 6. □

Now, we use Theorem 2 to introduce a classification of points of convergence closure spaces.

Definition 8. Let $(X, \mathcal{L}, \lambda)$ be a convergence closure space. Let X_ϱ denote the set of all ϱ-points of the space X and let $X_{\varrho'} = X - X_\varrho$. Points of $X_{\varrho'}$ are called *non-ϱ-points* (or *ϱ'-points*). Let X_σ denote the set of all σ-points of X and let $X_{\sigma'} = X - X_\sigma$. Points of the set $X_{\sigma'}$ are called *non-σ-points* (or *σ'-points*). Denote $X_{\varrho\sigma} = X_\varrho \cap X_\sigma$, $X_{\varrho'\sigma} = X_{\varrho'} \cap X_\sigma$, $X_{\varrho\sigma'} = X_\varrho \cap X_{\sigma'}$, $X_{\varrho'\sigma'} = X_{\varrho'} \cap X_{\sigma'}$.

Lemma 10. *Let $(X, \mathcal{L}, \lambda)$ be a convergence closure space. Then $X = X_{\varrho\sigma} \cup X_{\varrho'\sigma} \cup X_{\varrho\sigma'} \cup X_{\varrho'\sigma'}$, where the four components are mutually disjoint.*

P r o o f. Follows instantly from Definition 8. □

Now, we are going to give some Fréchet spaces to prove that $\varrho\sigma$-, $\varrho'\sigma$-, $\varrho\sigma'$-, $\varrho'\sigma'$-points do exist.

Example 6. Let $(X, \mathcal{L}, \lambda)$ be a discrete Fréchet space. It is clear that $X = X_{\varrho'\sigma'}$.

Example 7. Let X be a set of points x, x_{mn}, y_{mn}, $m, n \in \mathbb{N}$. Let $(\langle x_{mh^{\xi}(m)}\rangle, x)$, $\xi < \omega_1$, and $(\langle y_{mn}\rangle_n, x)$, $m \in \mathbb{N}$, be generating convergence elements. Then we have a Fréchet space $(X, \mathcal{L}^*, \lambda)$, a σ-sequence $\langle x_{mn}\rangle$ and a ϱ-sequence $\langle y_{mn}\rangle$. Hence x is a $\varrho\sigma$-point.

Example 8. Let X consist of points x, x_{mn}, $m, n \in \mathbb{N}$. Let $(\langle x_{mn}\rangle_n, x)$, $m \in \mathbb{N}$, be the generating convergence elements. Then $(X, \mathcal{L}^*, \lambda)$ is a Fréchet space no point of which is a σ-point.

Clearly, $\langle x_{mn}\rangle$ is a one-to-one ϱ-sequence, x a ϱ-point and each x_{mn} is an isolated point. Contrariwise, suppose that x is a σ-point. There is a one-to-one σ-sequence $\langle a_{mn}\rangle$, $x \neq a_{mn}$, with $\lim a_{mn} = x$. Denote $A_i = \{a_{in};\ n \in \mathbb{N}\}$, $X_i = \{x_{in};\ n \in \mathbb{N}\}, i \in \mathbb{N}$. Notice that $x \in \lambda A_m$ iff there is $p \in \mathbb{N}$ such that $A_m \cap X_p$ is an infinite set. Three cases are possible.

1. There is $p \in \mathbb{N}$ and a subsequence $\langle m_i\rangle$ of $\langle m\rangle$ such that $A_{m_i} \cap X_p$, $i \in \mathbb{N}$, are infinite sets. There are subsequences $\langle n^i_j\rangle_j$ of $\langle n\rangle$ such that $a_{m_i n^i_j} \in X_p$. Since $\varrho\text{-}\lim x_{pn} = x$, we have a subsequence $\langle a_{m_i n^i_j}\rangle$ of $\langle a_{mn}\rangle$ which is a π-sequence. In view of Lemma 7 we have a contradiction.

2. There is a subsequence $\langle p_i\rangle$ and a subsequence $\langle m_i\rangle$ of $\langle m\rangle$ such that $A_{m_i} \cap X_{p_i}$, $i \in \mathbb{N}$, are infinite sets. Analogously as above, we deduce that there are subsequences $\langle n^i_j\rangle_j$, $i \in \mathbb{N}$, of $\langle n\rangle$ such that $a_{m_i n^i_j} \in A_{m_i} \cap X_{p_i}$. Since X_{p_i} are disjoint sets it follows that $\langle a_{m_i n^i_j}\rangle$ is a ϱ-subsequence of the σ-sequence $\langle a_{mn}\rangle$. This is a contradiction with Lemma 7.

3. $A_m \cap X_n$, $m, n \in \mathbb{N}$, are finite sets. Then $x \notin \lambda A_m$, $m \in \mathbb{N}$. Since $\lim a_{mn} = x$, there is a one-to-one sequence of points $a_r \in \{a_{mn};\ m, n \in \mathbb{N}\}$ which $\mathcal{L}$-converges to x. It follows that there are increasing sequences $\langle r_i\rangle$, $\langle n_i\rangle$ such that $a_{r_i} \in X_{n_i}$. It means that $\langle a_{n_i}\rangle$ is a cross-subsequence in $\langle x_{mn}\rangle$ and so it does not $\mathcal{L}$-converge to x. This is a contradiction, because $\mathcal{L}\text{-}\lim a_r = x$.

A proof of the existence of $\varrho'\sigma$-points is given in Section IV.

Now we are interested in a question what the weights at π-, ϱ-, and σ-points in convergence closure spaces are.

Lemma 11. *Let $(X, \mathcal{L}, \lambda)$ be a convergence closure space. Let x be a point, $w(x) = \aleph_0$ the weight at x and $\langle x_{mn} \rangle$ a double sequence of points converging to x. Then there is a $(1, f)$-subsequence of $\langle x_{mn} \rangle$ which is a π-sequence.*

P r o o f. Denote $L_1(x) \subset L_2(x) \subset \ldots \subset L_i(x) \subset \ldots$ a complete system of neighbourhoods of the point x. It follows from Lemma 4 that there is an increasing sequence $\langle k_i \rangle$ and functions $f^i \colon \mathbb{N} \to \mathbb{N}$ such that $x_{mn} \in L_i(x)$, $m \geqslant k_i$, $n \geqslant f^i(m)$. Put $f(m) = \sum \sum f^j(i)$, $j \leqslant i$, $i \leqslant m$. Then $\langle x_{mf(m)+n} \rangle$ is the $(1, f)$-subsequence of$\langle x_{mn} \rangle$. Suppose that there is a cross-subsequence in it no subsequence of which $\mathcal{L}$-converges to x. Denote it $\langle x_{m_i g(m_i)} \rangle$ and suppose $x \notin \lambda\{x_{m_i g(m_i)} ;\ i \in \mathbb{N}\}$. Then $L(x) = X - \{x_{m_i g(m_i)} ;\ i \in \mathbb{N}\}$ is a neighbourgood of x. Notice that $f(m_i) < g(m_i)$,, $i \in \mathbb{N}$. It follows that no $L_i(x)$ is a subset of $L(x)$. This is a contradiction. Hence $\langle x_{mf(m)+n} \rangle$ is a π-sequence. □

Lemma 12. *Let $(X, \mathcal{L}, \lambda)$ be a convergence closure space. Let $\langle x_{mn} \rangle$ be a ϱ- or σ-sequence $\mathcal{D}$-converging to a point x. Then the weight $w(x) \geqslant \aleph_1$.*

P r o o f. Otherwise, by Lemma 11, there would be a π-subsequence of $\langle x_{mn} \rangle$, which is imposible in view of Lemma 7. □

IV

In this section we apply the convergence of double sequences to products of Fréchet spaces.

Let (X, u) be a topological space. Let $\mathcal{L}_u$ denote the collection of all pairs $(\langle x_n \rangle, x)$ such that x is a point, $\langle x_n \rangle$ is a sequence and each neighbourhood of x contains x_n for all but finitely many $n \in \mathbb{N}$. If (X, u) is Hausdorff, then $\mathcal{L}_u$ is a single-valued convergence. It is known that $\mathcal{L}_u$ can be a single-valued convergence even if (X, u) fails to be Hausdorff. In the sequel we shall always assume that $\mathcal{L}_u$ is single-valued. Observe that $\mathcal{L}_u$ is maximal. The resulting convergence closure space $(X, \mathcal{L}_u, \lambda_u)$ is called the adjoining convergence space. If $u = \lambda_u$, then also (X, u) is called a Fréchet space (cf. 8. in Section 1).

Let $(X, \mathcal{L}_1, \lambda_1)$ and $(Y, \mathcal{L}_2, \lambda_2)$ be convergence closure spaces. Let $\mathcal{L}_{12}$ be the collection of all pairs $(\langle (x_n, y_n) \rangle, (x, y))$ such that $x = \mathcal{L}_1\text{-}\lim x_n$ and $y = \mathcal{L}_2\text{-}\lim y_n$. It is known that $\mathcal{L}_{12}$ is a single-valued convergence on $X \times Y$ and if $\mathcal{L}_1$ and $\mathcal{L}_2$ are maximal, then $\mathcal{L}_{12}$ is maximal, too. Instead of $(\langle (x_m, y_m) \rangle, (x, y)) \in \mathcal{L}_{12}$ we write $\mathcal{L}_{12}\text{-}\lim(x_m, y_m) = (x, y)$ or, simply, $\lim(x_m, y_m) = (x, y)$. Let λ_{12} denote the corresponding closure.

Definition 9. The space $(X \times Y, \mathcal{L}_{12}, \lambda_{12})$ is called the *convergence closure product* of $(X, \mathcal{L}_1, \lambda_1)$ and $(Y, \mathcal{L}_2, \lambda_2)$.

Now, let us turn to the relationship between the topological product and the convergence closure product of two Fréchet spaces. Let $(X, \mathcal{L}_1, \lambda_1)$ and $(Y, \mathcal{L}_2, \lambda_2)$ be Fréchet spaces. Let $(X \times Y, w)$ be their topological product and let $(X \times Y, \mathcal{L}_{12}, \lambda_{12})$ be their convergence closure product.

Proposition 3. *$\mathcal{L}_w = \mathcal{L}_{12}$ and $\lambda_w = \lambda_{12}$.*

Proof. The straightforward proof is omitted. □

Lemma 13. *$(X \times Y, w)$ is a Fréchet space iff $w = \lambda_{12}$.*

Proof. Obvious. □

Corollary 4. *$(X \times Y, w)$ is not a Fréchet space iff either $\lambda_{12} \neq \lambda_{12}^2$ or $\lambda_{12} = \lambda_{12}^2 \neq w$.*

Definition 10. Let $(X, \mathcal{L}_1, \lambda_1)$ and $(Y, \mathcal{L}_2, \lambda_2)$ be Fréchet spaces. Let $\lim x_{mn} = x$ in X and $\lim y_{mn} = y$ in Y. We say that the points x, y are *coupled by sequences* $\langle x_{mn}\rangle, \langle y_{mn}\rangle$ if the following two implications are true:

1. If $\langle x_{m_i f(m_i)}\rangle$ is a cross-subsequence in $\langle x_{mn}\rangle$ $\mathcal{L}_1$-converging to the point x, then the corresponding cross-subsequence $\langle y_{m_i f(m_i)}\rangle$ in $\langle y_{mn}\rangle$ does not $\mathcal{L}_2$-converge to the point y;

2. If $\langle y_{m_i g(m_i)}\rangle$ is a cross-subsequence in $\langle y_{mn}\rangle$ $\mathcal{L}_2$-converging to the point y, then the corresponding cross-subsequence $\langle x_{m_i g(m_i)}\rangle$ in $\langle x_{mn}\rangle$ does not $\mathcal{L}_1$-converge to the point x.

Moreover, if one of the sequences $\langle x_{mn}\rangle$, $\langle y_{mn}\rangle$ is of the first kind and the other of the first or second kind, we say that the points x, y are *strongly coupled*.

Example 9. Let X be the set of points $x = (1, 1)$, $x_m = (m(m+1)^{-1}, 1)$, $x_{mn} = (m(m+1)^{-1}, n(n+1)^{-1})$, $m, n \in \mathbb{N}$, and let $(\langle x_m\rangle, x), (\langle x_{mn}\rangle_n, x_m)$, $m \in \mathbb{N}$, $(\langle x_{mg^\xi(m)}\rangle, x)$, $\xi < \omega_1$, be the generating convergence elements (see Example 4). Let Y be the space constructed in Example 4 c); let y and $\langle y_{mn}\rangle$ be the corresponding σ-point and σ-sequence. Then X and Y are Fréchet spaces, x and y are σ-points coupled by sequences $\langle x_{mn}\rangle$ of the first and $\langle y_{mn}\rangle$ of the second kind.

Example 10. Let X be the set consisting of points x, x_m, $m \in \mathbb{N}$. Let $(\langle x_m\rangle, x)$ be the generating convergence element for X. Put $x_{mn} = x_m$, $m, n \in \mathbb{N}$. Let Y be the set consisting of points y, y_{mn}, $m, n \in \mathbb{N}$. Let $(\langle y_{mn}\rangle_n, y)$, $m \in \mathbb{N}$, be the generating convergence elements for Y. Then X and Y are Fréchet spaces, x is a π-point and y is a ϱ-point which are coupled by sequences $\langle x_{mn}\rangle$ and $\langle y_{mn}\rangle$.

Lemma 14. *Let $(X \times Y, w)$ be the topological product of two Fréchet spaces $(X, \mathcal{L}_1, \lambda_1)$ and $(Y, \mathcal{L}_2, \lambda_2)$. Let $x \in X$, $y \in Y$ be points coupled by sequences $\langle x_{mn} \rangle$, $\langle y_{mn} \rangle$. Let there be no ϱ-subsequence either of $\langle x_{mn} \rangle$ or of $\langle y_{mn} \rangle$. Then $\langle x_{mn} \rangle$ and $\langle y_{mn} \rangle$ are σ-sequences.*

Proof. It follows from Definition 10 that there is no π-subsequence either of $\langle x_{mn} \rangle$ or of $\langle y_{mn} \rangle$. By Theorem 2, $\langle x_{mn} \rangle$, $\langle y_{mn} \rangle$ are σ-sequences. $\square$

To avoid trivialities, in the sequel we assume that all Fréchet spaces are not discrete.

Theorem 3. *Let $(X, \mathcal{L}_1, \lambda_1)$ and $(Y, \mathcal{L}_2, \lambda_2)$ be Fréchet spaces. Then their convergence closure product $(X \times Y, \mathcal{L}_{12}, \lambda_{12})$ is a Fréchet space iff there is no ϱ-point of X or of Y and there are no strongly coupled points.*

Proof. Suppose that there is a ϱ-point $x = \lim x_{mn}$ in X. Since the Fréchet space $(Y, \mathcal{L}_2, \lambda_2)$ is not isolated, there are a one-to-one sequence of points $y_m \in Y$ and a point $y = \mathcal{L}_2\text{-}\lim y_m$. Put $A = \{(x_{mn}, y_m);\ m, n \in \mathbb{N}\}$. Then $\mathcal{L}_{12}\text{-}\lim(x_{mn}, y_m) = (x, y_m)$ and so $(x, y) \in \lambda_{12}^2 A$. However, there is no sequence of points of A which $\mathcal{L}_{12}$-converges to the point (x, y). Therefore $\lambda_{12} \neq \lambda_{12}^2$. We get the same result if we suppose that there is a ϱ-point in $(Y, \mathcal{L}_2, \lambda_2)$.

Let $x \in X$, $y \in Y$ be points coupled by σ-sequences $\langle x_{mn} \rangle$, $\langle y_{mn} \rangle$ of the first kind. Then there are a one-to-one sequence $\langle x_m \rangle$ such that $\mathcal{L}_1\text{-}\lim x_m = x$, $\mathcal{L}_1\text{-}\lim x_{nn} = x_m$, $m \in \mathbb{N}$, and a one-to-one sequence $\langle y_m \rangle$ such that $\mathcal{L}_2\text{-}\lim y_m = y$ and $\mathcal{L}_2\text{-}\lim y_{mn} = y_m$, $m \in \mathbb{N}$. It follows that $(x, y) \in \lambda_{12}^2 A \setminus \lambda_{12} A$, where $A = \{(x_{mn}, y_{mn});\ m, n \in \mathbb{N}\}$. Hence $\lambda_{12} \neq \lambda_{12}^2$.

Let $x \in X$, $y \in Y$ be points coupled by a σ-sequence $\langle x_{mn} \rangle$ of the first kind and a σ-sequence $\langle y_{mn} \rangle$ of the second kind. Then there is a one-to-one sequence $\langle x_m \rangle$ such that $\mathcal{L}_1\text{-}\lim x_m = x$ and $\mathcal{L}_1\text{-}\lim x_{mn} = x_m$, $m \in \mathbb{N}$. Since $\mathcal{L}_2\text{-}\lim y_{mn} = y$, $m \in \mathbb{N}$, for $A = \{(x_{mn}, y_{mn});\ m, n \in \mathbb{N}\}$ we have $(x, y) \in \lambda_{12}^2 A \setminus \lambda_{12} A$. Hence $\lambda_{12} \neq \lambda_{12}^2$.

Now, suppose that $\lambda_{12} \neq \lambda_{12}^2$. Then there are distinct points (x, y), (x_m, y_m), (x_{mn}, y_{mn}), $m, n \in \mathbb{N}$, of $X \times Y$ such that $\mathcal{L}_{12}\text{-}\lim(x_m, y_m) = (x, y)$, $\mathcal{L}_{12}\text{-}\lim(x_{mn}, y_{mn}) = (x_m, y_m)$, $m \in \mathbb{N}$, and such that no cross-subsequence in the double sequence $\langle (x_{mn}, y_{mn}) \rangle$ $\mathcal{L}_{12}$-converges to the point (x, y). Since $\langle (x_m, y_m) \rangle$ is one-to-one, there is an increasing sequence $\langle m_i \rangle$ such that either

(a) $\langle x_{m_i} \rangle$, $\langle y_{m_i} \rangle$ are one-to-one sequences, or

(b) one of the sequences, say $\langle x_{m_i} \rangle$, is one-to-one, whereas $\langle y_{m_i} \rangle$ is constant.

Let (a) hold. For simplicity assume that $\langle m_i \rangle = \langle m \rangle$. If $\langle x_{m_i n_j^i} \rangle$ is a subsequence of $\langle x_{mn} \rangle$, then $x \in \lambda_1 \{x_{m_i; n_j^i};\ i, j \in \mathbb{N}\}$, because (X, λ_1) is a topological space. Therefore $\lim x_{mn} = x$, by Definition 5. Similarly, $\lim y_{mn} = y$. Hence x and y are points coupled by sequences $\langle x_{mn}\rangle$ and $\langle y_{mn}\rangle$ of the first kind. It is clear that there is no ϱ-subsequence either of $\langle x_{mn}\rangle$ or of $\langle y_{mn}\rangle$. Therefore x and y are σ-points.

Let (b) hold. Let $\langle x_m\rangle$ be a one-to-one sequence and let $\langle y_m\rangle$ be a constant sequence. Analogously as in (a) above, $\lim x_{mn} = x$ and $\lim y_{mn} = y$. Notice that there is no ϱ-subsequence of $\langle x_{mn}\rangle$, because $\langle x_{mn}\rangle$ is a double sequence of the first kind. If there is a ϱ-subsequence of $\langle y_{mn}\rangle$, the proof is finished. Let there be no ϱ-subsequence of $\langle y_{mn}\rangle$. Then, in view of Lemma 14, $x \in X$ and $y \in Y$ are σ-points which are coupled by the sequence $\langle x_{mn}\rangle$ of the first kind and the sequence $\langle y_{mn}\rangle$ of the second kind. □

Proposition 4. *Let $(X, \mathcal{L}_1, \lambda_1)$ be a first countable Fréchet space and let $(Y, \mathcal{L}_2, \lambda_2)$ be a Fréchet space. Their convergence closure product $(X \times Y, \mathcal{L}_{12}, \lambda_{12})$ is a Fréchet space iff there is no ϱ-point in $(Y, \mathcal{L}_2, \lambda_2)$.*

Proof. According to Lemma 12, a weight at a ϱ-point and at a σ-point is $> \aleph_0$. It follows that the space $(X, \mathcal{L}_1, \lambda_1)$ contains neither a ϱ-point nor a σ-point. Now, the assertion follows from Theorem 3. □

T. K. Boehme and M. Rosenfeld [BR74] proved (under $2^{\aleph_0} = \aleph_1$) that a compact Hausdorff Fréchet space X has the descending property. From this it follows that $X_\varrho = \emptyset$. We offer another proof by means of the double convergence.

Proposition (Boehme, Rosenfeld). *Let $(X, \mathcal{L}, \lambda)$ be a compact Hausdorff Fréchet space. There is no ϱ-point in the space X.*

Proof. Suppose that $x = \lim x_{mn}$ is a ϱ-point in X. Let L be a compact neighbourhood of the point x. According to Lemma 4 there is a cross-subsequence $\langle t_n\rangle$ in $\langle x_{mn}\rangle$ of points $t_n \in L$ such that $\mathcal{L}\text{-}\lim t_n = a \neq x$. Hence $a \in L$. From this it follows that each neighbourhood of the point x contains a limit $a \neq x$ of a cross-subsequence in $\langle x_{mn}\rangle$. Since X is a Hausdorff Fréchet space and $a \neq x$ there is a one-to-one sequence of limits $\langle a_i\rangle$ $\mathcal{L}$-converging to the point x and a one-to-one double sequence $\langle a_{ij}\rangle$ whose straight-sequences are cross-subsequences in $\langle x_{mn}\rangle$ and $\mathcal{L}\text{-}\lim a_{ij} = a_i$, $i \in \mathbb{N}$. Let $\langle a_{ij}\rangle_j$ be the i-th straight-sequence in $\langle a_{ij}\rangle$. There is a number $g(i) \in \mathbb{N}$, such that no point a_{ij}, $j \geqslant g(i)$ belongs to the set $\{x_{mn};\ m \leqslant i, n \in \mathbb{N}\}$. Otherwise there would be a subsequence of $\langle a_{ij}\rangle_i$ converging to the point $x \neq a_i$. We have proved that there is a $(1, g)$ subsequence $\langle t_{rs}\rangle$ of $\langle a_{ij}\rangle$ such that each intersection $\{t_{rs};\ r, s \in \mathbb{N}\} \cap \{x_{in};\ n \in \mathbb{N}\}$ is a finite set for each $i \in \mathbb{N}$. Hence no cross-subsequence in $\langle t_{rs}\rangle$ $\mathcal{L}$-converges to x. This is a contradiction because $x \in \lambda^2$ $\{t_{rs};\ r, s \in \mathbb{N}\}$, $\lambda^2 = \lambda$ and so there is a cross-subseequence in $\langle t_{rs}\rangle$ $\mathcal{L}$-converging to x. □

Proposition 5. *A convergence closure product $(X \times Y, \mathcal{L}_{12}, \lambda_{12})$ of compact Hausdorff Fréchet spaces $(X, \mathcal{L}_1, \lambda_1)$ and $(Y, \mathcal{L}_2, \lambda_2)$ is a Fréchet space iff there are no strongly coupled σ-points $x \in X, y \in Y$.*

P r o o f. The assertion follows instantly from Proposition (Boehme, Rosenfeld) and Theorem 3. □

Lemma 15. *The topological product $(X \times Y, w)$ of Fréchet spaces $(X, \mathcal{L}_1, \lambda_1)$ and $(Y, \mathcal{L}_2, \lambda_2)$ is not Fréchet iff there are a set Z_0 and a point (a, b) of $X \times Y$ such that $(a, b) \in wZ_0 \setminus \lambda_{12} Z_0$, $Z_0 \cap X \times \{b\} = \emptyset = Z_0 \cap \{a\} \times Y$.*

P r o o f. If $(X \times Y, w)$ is not Fréchet, then there is $T \subset X \times Y$ and (a, b) such that $(a, b) \in wT$, $(a, b) \notin \lambda_{12} T$. Hence there is no sequence of points $(a, y_n) \in T$ or $(x_n, b) \in T$ which $\mathcal{L}_{12}$-converges to (a, b). It suffices to put $Z_0 = T \setminus (X \times b \cup a \times Y)$. □

Remark. If $(a, b) \in wZ_0 \setminus \lambda_{12} Z_0$ and U, resp. V, is a neighbourhood of the point a, resp. b, then $(a, b) \in wZ_1 \setminus \lambda_{12} Z_1$, where $Z_1 = Z_0 \cap (U \times V)$. This is true since $(a, b) \in wZ_1$ and $(a, b) \notin \lambda_{12} Z_1$ because $Z_1 \subset Z_0$.

Definition 11. Let F_1 denote the class of all Fréchet spaces $(Y, \mathcal{L}_2, \lambda_2)$ such that the following implication holds: if $(X, \mathcal{L}_1, \lambda_1)$ is a Fréchet space and the convergence closure product $(X \times Y, \mathcal{L}_{12}, \lambda_{12})$ is a Fréchet space, then the topological product space $(X \times Y, w)$ is a Fréchet space, too.

Notice that if $(Y, \mathcal{L}_2, \lambda_2) \in F_1$ then there is at least one Fréchet space $(X, \mathcal{L}_1, \lambda_1)$ such that $(X \times Y, w)$ is not Fréchet, viz. a Fréchet space containing a ϱ-point.

Proposition 6. *Each compact Hausdorff Fréchet space belongs to the class F_1.*

P r o o f. Let $(Y, \mathcal{L}_2, \lambda_2)$ be a compact Hausdorff Fréchet space. Suppose that there is a Fréchet space $(X, \mathcal{L}_1, \lambda_1)$ such that $(X \times Y, w)$ is not Fréchet. We have to prove that $(X \times Y, \mathcal{L}_{12}, \lambda_{12})$ is not Fréchet. According to Lemma 15, there are a subset Z_0 and a point (a, b) of $X \times Y$ with $(a, b) \in wZ_0 \setminus \lambda_{12} Z_0$, $Z_0 \cap X \times \{b\} = \emptyset = Z_0 \cap \{a\} \times Y$. Let A_0, resp. B_0, denote the projection of Z_0 into X, resp. Y. Then $a \in \lambda_1 A_0$ and $b \in \lambda_2 B_0$, because $(a, b) \in wZ_0$.

Assume that V is a neighbourhood of the point b. Let V_1 be a neighbourhood of b such that $\lambda_2 V_1 \subset V$. Denote $Z_1 = Z_0 \cap X \times V_1$. Let A_1 (resp. B_1) be the projection of Z_1 into X (resp. Y). Evidently, $a \in \lambda_1 A_1 \setminus A_1$, $b \in \lambda_2 B_1 \setminus B_1$. Let $\langle x_n \rangle$ be a one-to-one sequence of points of A_1 $\mathcal{L}_1$-converging to the point a and $\langle y_n \rangle$ a sequence of points of B_1 such that $(x_n, y_n) \in Z_1$. Such sequences do exist, because $(a, b) \in wZ_1$. Since $(Y, \mathcal{L}_2, \lambda_2)$ is a Fréchet space and $\lambda_2 V_1$ a compact set, there is a subsequence

$\langle y_{m_i}\rangle$ of $\langle y_m\rangle$ $\mathcal{L}_2$-converging to a point t of $\lambda_2 V_1$. Consequently, $\mathcal{L}_{12}$-$\lim(x_{m_i}, y_{m_i}) = (a,t)$. The point t will be called a special point and the sequence $\langle(x_{m_i}, y_{m_i})\rangle$ a corresponding special sequence. We have proved that each neighbourhood of the point b contains a special point $t \neq b$ (because $(a,b) \notin \lambda_{12} Z_0$). It follows that there is a one-to-one sequence of special points t_m with $\mathcal{L}_2$-$\lim t_m = b$. Denote $\langle(x_{mn}, y_{mn})\rangle_n$ a special sequence corresponding to t_m, $m \in \mathbb{N}$. If $\langle y_{m_i n^i_j}\rangle$ is a subsequence of the sequence $\langle y_{mn}\rangle$ then $b \in \lambda_2\{y_{m_i n^i_j};\ i,j \in \mathbb{N}\}$, because (Y, λ_2) is a Fréchet space. Hence $\lim y_{mn} = b$, by Definition 5. Notice that $\mathcal{L}_1$-$\lim x_{mn} = a$, $m \in \mathbb{N}$, and so $\lim x_{mn} = a$. From this we deduce, becuase $(a,b) \notin \lambda_{12} Z_0$, that the points a, b are coupled by the sequence $\langle x_{mn}\rangle$ of the second and the sequence $\langle y_{mn}\rangle$ of the first kind. According to Theorem 2, there is a π- or ϱ- or σ-subsequence of the sequence $\langle y_{mn}\rangle$. In the first case, the corresponding subsequence of $\langle x_{mn}\rangle$ is a π-sequence, because (a,b) are coupled points. Hence a is a ϱ-point. The point b cannot be a ϱ-point because $(Y, \mathcal{L}_2, \lambda_2)$ is a compact Hausdorff Fréchet space. If there is a σ-subsequence $\langle y_{m_i n^i_j}\rangle$ of $\langle y_{mn}\rangle$ then $\langle x_{m_i n^i_j}\rangle$ is a σ-sequence and a, b are strongly coupled points. All this contradicts Theorem 3. Thus $(X \times Y, \mathcal{L}_{12}, \lambda_{12})$ is not Fréchet.

Proposition 7. *Each first countable Fréchet space belongs to the class F_1.*

P r o o f. Let $(Y, \mathcal{L}_2, \lambda_2)$ be a first countable Fréchet space. It contains a non-isolated point. Suppose that $(X, \mathcal{L}_1, \lambda_1)$ is a Fréchet space and $(X \times Y, w)$ is not Fréchet. Then there is a point (a,b) and a set $Z_0 \subset X \times Y$ such that $(a,b) \in wZ_0 \setminus \lambda_{12} Z_0$. Let $V_1 \supset V_2 \supset \ldots \supset V_m \supset \ldots$ be an infinite countable complete system of neighbourhoods of the point b. Denote $Z_m = Z_0 \cap (X \times V_m)$ and A_m, resp. B_m, the projection of Z_m into X, resp. Y. There is a simple one-to-one sequence of points $x_{mn} \in A_m$ with $\mathcal{L}_1$-$\lim x_{mn} = a$. Let $\langle y_{mn}\rangle_n$ be a sequence of points of B_m such that $(x_{mn}, y_{mn}) \in Z_m$, $n \in \mathbb{N}$. Consider the double sequences $\langle x_{mn}\rangle$, $\langle y_{mn}\rangle$. It is clear that $\lim x_{mn} = a$ (because $\mathcal{L}_1$-$\lim x_{mn} = a$, $m \in \mathbb{N}$). Let V be a neighbourhood of the point b. There is $k \in \mathbb{N}$ such that $V_k \subset V$. Notice that $V_k \supset V_{k+1} \supset \ldots$ and $y_{mn} \in V_m, n \in \mathbb{N}$. From this it follows that $y_{mn} \in V_k \subset V$, $m \geqslant k, n \geqslant 1$. Hence $\lim y_{mn} = b$, by Lemma 4. It also follows that if $\langle y_{mf(m)}\rangle$ is a cross-sequence in $\langle y_{mn}\rangle$, then $y_{mf(m)} \in V_k \subset V$, $m \geqslant k$. Consequently $\mathcal{L}_2$-$\lim y_{mf(m)} = b$. We have proved that $\langle y_{mn}\rangle$ is a π-sequence. The point a is a ϱ-point, because $(a,b) \notin \lambda_{12} Z_1$. This is a contradiction with Definition 11 and Theorem 3. Hence $(Y, \mathcal{L}_2, \lambda_2) \in F_1$.

□

Proposition 8. *Let $(X, \mathcal{L}_1, \lambda_1)$, $(Y, \mathcal{L}_2, \lambda_2)$ be Fréchet spaces. Let $(Y, \mathcal{L}_2, \lambda_2) \in F_1$. The topological product $(X \times Y, w)$ is not a Fréchet space iff there is a ϱ-point in X or in Y, or there are strongly coupled σ-points $x \in X$, $y \in Y$.*

P r o o f. The assertion follows straightforwardly from Theorem 3. □

The next three propositions below follow instantly from Theorem 3 and Propositions 6, 7, 8.

Proposition 9. *The topological product $(X \times Y, w)$ of a Hausdorff Fréchet space $(X, \mathcal{L}_1, \lambda_1)$ and a locally compact Hausdorff Fréchet space $(Y, \mathcal{L}_2, \lambda_2)$ is not Fréchet iff there is a ϱ-point in X, or there are strongly coupled σ-points $x \in X$, $y \in Y$.*

Proposition 10. *The topological product of two compact Hausdorff Fréchet spaces $(X, \mathcal{L}_1, \lambda_1)$ and $(Y, \mathcal{L}_2, \lambda_2)$ is not Fréchet iff there are strongly coupled σ-points $x \in X$ and $y \in Y$.*

Ch. T. Kendrick ([KC75]) gave a necessary and sufficient condition for the topological product of a Fréchet space and a first countable Fréchet space not to be Fréchet. We obtain the same result by means of the double convergence.

Proposition (Kendrick). *The topological product $(X \times Y, w)$ of a Fréchet space $(X, \mathcal{L}_1, \lambda_1)$ and a first countable non-isolated Fréchet space $(Y, \mathcal{L}_2, \lambda_2)$ is not Fréchet iff there is a ϱ-point $x \in X$.*

E. Michael ([MI72]) posed a question whether the topological product $X \times Y$ of two compact Hausdorff Fréchet spaces X and Y is Fréchet. Under the assumption that $2^{\aleph_0} = \aleph_1$, T. K. Boehme and M. Rosenfeld answered this question by constructing two compact Hausdorff Fréchet spaces X_e and Y_0 whose topological product is not Fréchet ([BR74]). According to Proposition 10, there are strongly coupled $\varrho'\sigma$-points $x \in X_e$ and $y \in Y_0$. P. Simon ([SI80]) improved the result of Boehme and Rosenfeld and constructed compact Hausdorff Fréchet spaces X and Y such that their topological product is not Fréchet without any additional set-theoretical axioms. This proves the following

Proposition 11. *There exists a $\varrho'\sigma$-point.*

Next we show that there are Fréchet spaces whose convergence closure product is Fréchet, but the topological product is not.

Definition 12. Define the class F_2 as follows: A Fréchet space $(Y, \mathcal{L}_2, \lambda_2)$ belongs to F_2 iff there exists a Fréchet space $(X, \mathcal{L}_1, \lambda_1)$ such that the convergence closure product $(X \times Y, \mathcal{L}_{12}, \lambda_{12})$ is a Fréchet space, but the topological product $(X \times Y, w)$ is not Fréchet.

Proposition 12. *The class F_2 is nonempty.*

Proof. Let $(X, \mathcal{L}_1, \lambda_1)$ and $(Y, \mathcal{L}_2, \lambda_2)$ be compact Hausdorff Fréchet spaces. According to the proof of Proposition 6, there is a point $a = \lim x_{mn}$ in X and a point $b = \lim y_{mn}$ in Y which are coupled by sequences $\langle x_{mn}\rangle$, $\langle y_{mn}\rangle$. Since X and Y are compact, neccessarily $X_\varrho = \emptyset = Y_\varrho$. Consequently, the points a, b are $\varrho'\sigma$-points, $\langle x_{mn}\rangle$ is a σ-sequence of the second kind and $\langle y_{mn}\rangle$ is a σ-sequence of the first kind. Let X' consist of points a and x_{mn}, $m, n \in \mathbb{N}$, and let Y' consist of points b, y_{mn}, $m, n \in \mathbb{N}$. Let $(X', \mathcal{L}'_1, \lambda'_1)$ and $(Y', \mathcal{L}'_2, \lambda'_2)$ be the corresponding subspaces. Then $X'_\varrho = \emptyset = Y'_\varrho$, by Proposition (Boehme, Rosenfeld). We can suppose that y_{mn} are isolated points (compact Hausdorff Fréchet spaces with this property exists, see for example [BR74]). It follows that $\langle y_{mn}\rangle$ is a σ-sequence of the third kind in Y'. Consequently, in view of Theorem 3, the convergence closure product $(X' \times Y', \mathcal{L}'_{12}, \lambda'_{12})$ is a Fréchet space. On the other hand, $(a, b) \in wD$ and $(a, b) \notin \lambda_{12}D$, where D is a set of all points (x_{mn}, y_{mn}), $m, n \in \mathbb{N}$. Therefore the topological product $(X' \times Y', w')$ is not Fréchet. Hence both spaces $(X', \mathcal{L}'_1, \lambda'_1)$ and $(Y', \mathcal{L}'_2, \lambda'_2)$ belong to the class F_2. $\square$

Remark. Let F be a class of nondiscrete Fréchet spaces. Let a space X belong to F. Then it belongs either to F_1 or to F_2. Hence $F = F_1 \cup F_2$. Let X and Y be Fréchet spaces of F. If at least one of them belongs to F_1 the question what are the necessary and sufficient conditions such that their topological product is Fréchet has been answered in Proposition 8. If both the spaces belong to F_2 the question remains open.

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