Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 48 (1998), No. 1, 77–83

Persistent URL: http://dml.cz/dmlcz/127400

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ON OPERATORS WITH THE SAME LOCAL SPECTRA

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(Received June 1, 1995)

Abstract. Let $B(X)$ be the algebra of all bounded linear operators in a complex Banach space X. We consider operators $T_1, T_2 \in B(X)$ satisfying the relation $\sigma_{T_1}(x) = \sigma_{T_2}(x)$ for any vector $x \in X$, where $\sigma_T(x)$ denotes the local spectrum of $T \in B(X)$ at the point $x \in X$. We say then that T_1 and T_2 have the same local spectra. We prove that then, under some conditions, $T_1 - T_2$ is a quasinilpotent operator, that is $\|(T_1 - T_2)^n\|^{1/n} \to 0$ as $n \to \infty$. Without these conditions, we describe the operators with the same local spectra only in some particular cases.

MSC 2000: 47A10

Keywords: Banach space, spectrum, local spectrum

1. Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. For any $T \in B(X)$, denote by $\sigma(T)$ the spectrum of T, and by $\sigma_T(x)$ the local spectrum of T at a point $x \in X$. It is known (see [4] or [5]) that

(a) $\sigma_T(x) \subseteq \sigma(T) \qquad (x \in X);$

(b) $\sigma(T) = \bigcup\{\sigma_T(x)\colon\ x \in X\};$

(c) $\sigma_T(x)$ is a compact set in C, for any $x \in X$;

(d) $\sigma_T(x) = \emptyset$ if and only if $x = 0$.

An operator $N \in B(X)$ is called quasinilpotent if $\|N^n\|^{1/n} \to 0$ as $n \to \infty$, i.e. if $\sigma(N) = 0$. N is obviously quasinilpotent if and only if $\sigma_N(x) = \{0\}$ for any vector $x \in X \setminus \{0\}$.

For arbitrary operators $T_1, T_2 \in B(X)$, we can consider the relation

$$\sigma_{T_1}(x) = \sigma_{T_2}(x) \quad \text{whenever } x \in X. \tag{1}$$

It is obvious that in the relation considered we can suppose that $x \neq 0$.

If T_1, T_2 satisfy the relation (1) (for any vector $x \in X$), we say that they have the same local spectra (SLS in short).

If T_1 and T_2 have SLS, then obviously $\sigma(T_1) = \sigma(T_2)$. In the general case, the point spectra $\sigma_p(T_1)$ and $\sigma_p(T_2)$ can be different. As an example, take any nilpotent operator N_1 in the complex Hilbert space $X = L^2(0,1)$, and the Volterra integral operator N_2 in the same space (N_2 is a quasinilpotent but not nilpotent operator with empty point spectrum). Then $\sigma(N_1) = \sigma(N_2) = \{0\}$, and immediately $\sigma_{N_1}(x) = \sigma_{N_2}(x) = \{0\}$ for any vector $x \neq 0$. But, on the other hand, $\sigma_p(N_1) = \{0\}$ and $\sigma_p(N_2) = \emptyset$.

2. The next theorem completely describes the relationship between operators with SLS if they are commuting and decomposable in the sense of Foiaş [3].

Theorem 1. *Let $T_1, T_2 \in B(X)$ be commuting decomposable operators. Then they have SLS if and only $T_1 - T_2$ is a quasinilpotent operator.*

P r o o f. Assume that $T_1 - T_2$ is a quasinilpotent operator. Then, by a result of [1] (see also [2, Ch. 4]), we have

$$\sigma_{T_1}(x) = \sigma_{T_2+N}(x) = \sigma_{T_2}(x)$$

for any vector $x \in X$.

Next assume that T_1 and T_2 have SLS. Then with notation from [3], we have for any closed set $F \subseteq C$:

$$X_{T_1}(F \cap \sigma(T_1)) = \{x\colon\ \sigma_{T_1}(x) \subseteq F\},$$
$$X_{T_2}(F \cap \sigma(T_2)) = \{x\colon\ \sigma_{T_2}(x) \subseteq F\}.$$

By relation (1), it is obvious that

$$X_{T_1}(F \cap \sigma(T_1)) = X_{T_2}(F \cap \sigma(T_2))$$

for any closed set $F \subseteq C$, so that all conditions from Theorem 3.2 from [3] are satisfied. Hence $T_1 - T_2$ is a quasinilpotent operator, Q.E.D. □

The above theorem completely describes commuting operators with SLS in the class of all normal operators in a Hilbert space, or in the class of all compact operators in a Banach space. Indeed, as is well-known, all these operators are decomposable.

Corollary 1. *Let T_1, T_2 be commuting operators in a finite-dimensional space X. Then they have SLS if and only if $T_1 - T_2$ is a nilpotent operator.*

P r o o f. As is known, any operator in a finite-dimensional space is decomposable, and the assertion follows immediately by Theorem 1. As is also known, nilpotent and quasinilpotent operators in a finite-dimensional space always coincide. □

The next examples show that operators with SLS, in the general case, need not be commuting nor decomposable.

Example 1. Take $X = C^2$, and let $\{e_1, e_2\}$ be the standard basis in C^2. Define

$$N_1(x) = N_1(\xi_1, \xi_2) = (\xi_1 + \xi_2, -\xi_1 - \xi_2),$$
$$N_2(x) = N_2(\xi_1, \xi_2) = (\xi_1 - \xi_2, \xi_1 - \xi_2).$$

Then $N_1^2 = N_2^2 = 0$, thus N_1 and N_2 are nilpotent, and therefore they have SLS. Neverthless, they are obviously noncommuting.

Example 2. Let T_1 be the right shift operator in the space ℓ^2, defined by

$$T_1(\xi_1, \xi_2, \dots) = (0, \xi_1, \xi_2, \dots)$$

for any $x = (\xi_1, \xi_2, \dots) \in \ell^2$.

Take $T_2 = T_1^2$, i.e.

$$T_2(\xi_1, \xi_2, \dots) = (0, 0, \xi_1, \xi_2, \dots).$$

It is known (see [4], [5] or [6]) that

$$\sigma_{T_1}(x) = \sigma(T_1) = \{\lambda\colon\ |\lambda| \leqslant 1\}$$

for any vector $x \in \ell^2 \setminus \{0\}$.

It can be also verified that

$$\sigma_{T_2}(x) = \sigma(T_2) = \{\lambda\colon\ |\lambda| \leqslant 1\}$$

for any vector $x \in \ell^2 \setminus \{0\}$.

Hence, the condition (1) is obviously fulfilled, and T_1 and $T_2 = T_1^2$ obviously commute. On the other hand, if $D = T_1 - T_2 = T_1 - T_1^2$, then

$$\|D^n e_1\|^2 = \|(1 - T_1)^n e_{n+1}\|^2 = \left\| \sum_{k=0}^{n} \binom{n}{k} (-1)^k e_{n+k+1} \right\|^2$$
$$= \sum_{k=0}^{n} \binom{n}{k}^2 \geqslant \sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

so that $\|D^n e_1\|^{1/n} \geqslant \sqrt{2}$ for all $n \geqslant 0$. Hence, D is not a quasinilpotent operator. Together with Theorem 1 this shows that not both operators T_1 and T_2 are decomposable. Moreover, since the analytic functions of decomposable operators are also decomposable, we have that both operators T_1 and T_2 are nondecomposable.

We also note that the operator T_1 from this example has appeared several times in literature as a very useful example for different aims (see [4], [5], [6]).

Describing the general operators which have SLS remains an open question in this paper. We have succeeded only in some particular classes of operators. In the next section we shall analyze general (not necessarily commuting) operators in a finite-dimensional space X which have SLS.

3. Let T be an arbitrary linear operator in a finite-dimensional space X. Denote $\sigma(T) = \sigma_p(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$, where $\lambda_i \neq \lambda_j$ for $i \neq j$, and let $\mathscr{L}_T(\lambda_1), \mathscr{L}_T(\lambda_2), \ldots, \mathscr{L}_T(\lambda_r)$ be the corresponding root spaces of T. Then we have

$$X = \mathscr{L}_T(\lambda_1) \dotplus \mathscr{L}_T(\lambda_2) \dotplus \ldots \dotplus \mathscr{L}_T(\lambda_r). \tag{2}$$

Denote by $E_T(\lambda_i)$ $(i = 1, 2, \ldots, r)$ the projection from X to $\mathscr{L}_T(\lambda_i)$ according to the decomposition (2). Since X is finite-dimensional, all these projections $E_T(\lambda_i)$ $(i = 1, 2, \ldots, r)$ are bounded.

Using the definition of the local spectrum, it is not difficult to see the following.

Proposition 1. *Let T be an arbitrary operator in a finite-dimensional space X and assume that $x \in X \setminus \{0\}$. Then a complex value $\lambda \in \sigma_T(x)$ if and only if $\lambda \in \sigma(T)$ and $E_T(\lambda)x \neq 0$.*

In particular, $\sigma_T(x) = \{\lambda_i\}$ if and only if $x \in \mathscr{L}_T(\lambda_i) \setminus \{0\}$.

Proposition 2. *Let X be a finite-dimensional space. Then operators T_1, T_2 have SLS if and only if they have the same spectra and the same corresponding root spaces.*

Remark. As an immediate consequence, we get that then T_1 and T_2 also have the same algebraic multiplicities of their eigenvalues.

Proof. Let T_1 and T_2 have SLS. Then we obviously have $\sigma(T_1) = \sigma(T_2)$. Denote their common spectrum by $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ where $\lambda_i \neq \lambda_j$ for $i \neq j$, and let $\mathscr{L}_{T_1}(\lambda_i), \mathscr{L}_{T_2}(\lambda_i)$ $(i = 1, 2, \ldots, r)$ be the corresponding root spaces of T_1 and T_2, respectively. If $x \in \mathscr{L}_{T_1}(\lambda_i) \setminus \{0\}$ then $\sigma_{T_1}(x) = \{\lambda_i\} = \sigma_{T_2}(x)$, whence $x \in \mathscr{L}_{T_2}(\lambda_i)$ by Proposition 1. Hence $\mathscr{L}_{T_1}(\lambda_i) \subseteq \mathscr{L}_{T_2}(\lambda_i)$. Similarly $\mathscr{L}_{T_2}(\lambda_i) \subseteq \mathscr{L}_{T_1}(\lambda i)$, and hence $\mathscr{L}_{T_1}(\lambda_i) = \mathscr{L}_{T_2}(\lambda_i)$ for any $i = 1, 2, \ldots, r$.

Conversely, assume that T_1 and T_2 have the same spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ and the same corresponding root spaces. Take any $x \in X \setminus \{0\}$. Then $x = x_1 + x_2 + \ldots + x_r$, where $x_i \in \mathscr{L}_{T_1}(\lambda_i) = \mathscr{L}_{T_2}(\lambda_i)$ for any $i = 1, 2, \ldots, r$. Denoting $S = \{i \in \{1, 2, \ldots, r\}\colon x_i \neq 0\}$ and $F = \{\lambda_i\colon i \in S\}$, we have that $\sigma_{T_1}(x) = \sigma_{T_2}(x) = F$ by Proposition 1.

This completes the proof. □

Using the Jordan form of operators in a finite-dimensional space, one can easily obtain

Corollary 1. *Let X be a finite-dimensional space. Then operators T_1, T_2 have SLS if and only if T_1 is similar to an operator S_1 and T_2 is similar to an operator S_2 such that $S_1 - S_2 = N_1 - N_2$, where N_1, N_2 are nilpotent operators commuting respectively with T_1, T_2.*

4. Next, consider two similar operators $T, S \in B(X)$ in a Banach space X; thus $S = KTK^{-1}$, where $K, K^{-1} \in B(X)$. We know that T and S always have the same spectra. But the following example shows that similar operators need not have the same local spectra.

Example 3. Let $X = C^2$ with the standard basis $\{e_1, e_2\}$. Define operators T and S by

$$T(\xi_1, \xi_2) = (\xi_1, 0), \quad S(\xi_1, \xi_2) = (0, \xi_2).$$

Operators T and S are similar with respect to the operator $K(\xi_1, \xi_2) = (\xi_2, \xi_1)$. Next we have that $\sigma(T) = \sigma(S) = \{0; 1\}$, but $\sigma_T(e_1) = \{1\}$, $\sigma_S(e_1) = \{0\}$. Hence T and S have not the same local spectra.

The next example shows that some similar operators can have the same local spectra.

Example 4. Let X be an arbitrary Banach space, and assume that $T = \lambda I + N$, where $\lambda \in C$ and N is a quasinilpotent operator. If $S = KTK^{-1}$, where $K, K^{-1} \in B(X)$, we find that $\sigma(S) = \sigma(T) = \{\lambda\}$. For any $x \in X \setminus \{0\}$, we easily get $\sigma_T(x) = \sigma_S(x) = \{\lambda\}$, so that T and S have SLS.

We also see that such operators T have, in a sense, extremely large local spectra, for $\sigma_T(x) = \sigma(T)$ for every $x \in X \setminus \{0\}$.

These examples motivate us to introduce the following definition.

Let $\mathscr{M}(X)$ be the class of all operators $T \in B(X)$ in a Banach space X such that T and every operator S similar to T have SLS.

Let $\mathscr{S}(x)$ be the class of all operators $T \in B(X)$ such that

$$\sigma_T(x) = \sigma(T) \tag{3}$$

for every vector $x \in X \setminus \{0\}$.

Obviously, both classes $\mathscr{M}(X)$ and $\mathscr{S}(X)$ contain all operators of the form $\lambda I+N$, where $\lambda \in C$ and N is an arbitrary quasinilpotent operator.

Proposition 3. *Classes $\mathscr{M}(X)$ and $\mathscr{S}(X)$ coincide.*

Proof. Assume that $T \in \mathscr{S}(X)$. By the definition of the local spectrum it is not difficult to see that

$$\sigma_{KTK^{-1}}(x) = \sigma_T(K^{-1}x)$$

whenever $K, K^{-1} \in B(X)$. Hence, obviously,

$$\sigma_{KTK^{-1}}(x) = \sigma(T) = \sigma_T(x)$$

for every vector $x \in X \setminus \{0\}$. Therefore $T \in \mathscr{M}(X)$.

Conversely, assume that $T \in \mathscr{M}(X)$. Then we have

$$\sigma_T(K^{-1}x) = \sigma_T(x) \tag{4}$$

whenever $x \in X$ and $K, K^{-1} \in B(X)$.

If $x, y \neq 0$ are arbitrary but fixed vectors in X, we are now proving that there is an invertible operator $K \in B(X)$ such that $K^{-1}x = y$, i.e. $Ky = x$. If $y = x$, take $K = I$. If $y = \alpha x$ ($\alpha \neq 0$), take $K = \alpha^{-1} I$. If x, y are linearly independent, denote by $E = \operatorname{sp}\{x, y\}$ the linear span over x and y. Then E is closed, and as is well-known, there is a closed subspace F in X such that $X = E \dotplus F$. Define $Kx = y$, $Ky = x$, $Ku = u$ ($u \in F$), and further by linearity. Then $K = K^{-1}$ is bounded on closed invariant subspaces E and F, and the projection from X onto E along F is also bounded. Hence K is bounded on the entire space X.

Now consider relation (4).

By the previous remark we get

$$\sigma_T(x) = \sigma_T(y) \qquad (x, y \neq 0). \tag{5}$$

Since $\sigma(T) = \bigcup\{\sigma_T(x) \colon x \in X\}$, we immediately have relation (3) for every $x \in X \setminus \{0\}$. Therefore $T \in \mathscr{S}(X)$. □

The class of operators $\mathscr{S}(X)$ seems to be important and interesting. However, we shall not develop the questions concerning this class in this paper. We only note that the operators T_1, T_2 from Example 2 both belong to this class.

The class $\mathscr{S}(X)$ can be completely described at least in a finite-dimensional space.

Proposition 4. *If X is a finite-dimensional space, then*

$$\mathscr{S}(X) = \bigl\{\lambda I + N \colon \lambda \in C,\ N \text{ is a nilpotent operator}\bigr\}.$$

P r o o f. Example 4 proves a half of this assertion.

Next, assume that $T \in \mathscr{S}(X)$. By Proposition 1, it is easy to see that then T has exactly one point in its spectrum, thus $T = \lambda I + N$ for some $\lambda \in C$ and a nilpotent operator N.

This completes the proof. □

Acknowledgment. The author is very grateful to the referee for several valuable remarks concerning this paper.

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