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ON THE NON-VANISHING OF LOCAL COHOMOLOGY MODULES

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Abstract. It is shown that for any Artinian modules M, $\dim M^\vee$ is the greatest integer i such that $\mathrm{H}^i_{\mathfrak{m}}(M^\vee) \neq 0$.

0. INTRODUCTION

Let R be a Noetherian ring, let $\mathfrak{a}$ be an ideal of R, and let M be an R-module. The local cohomology of M with respect to $\mathfrak{a}$ was introduced by Grothendieck by

$$\mathrm{H}^n_{\mathfrak{a}}(M) = \varinjlim_{\ell} \operatorname{Ext}^n_R(R/\mathfrak{a}^\ell, M).$$

As is well-known, these local cohomology modules can also be computed in the case that $M = R$ by using a Čech resolution of $\operatorname{Spec} R - V(\mathfrak{a})$.

There has been a great deal of work on the question, when $\mathrm{H}^n_{\mathfrak{m}}$ are zero. The first and most basic theorem was given by Grothendieck himself who showed that $\mathrm{H}^n_{\mathfrak{a}}(M) = 0$ if $n > \dim R$. He also showed that the d-th local cohomology of a local ring of dimension d at its maximal ideal is never zero. The local cohomology modules were studied further by Sharp in [5], Macdonald and Sharp in [4] and Foxby in [2]. Let R be a local ring with maximal ideal $\mathfrak{m}$ and let M be an R-module. Then the depth M is the smallest integer i such that $\mathrm{H}^i_{\mathfrak{m}}(M) \neq 0$, cf. [2, 7.11], and $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for all $i > \dim M$, cf. [2, 7.8]. In addition, if M is a finite R-module then $\dim M$ is the greatest integer i such that $\mathrm{H}^i_{\mathfrak{m}}(M) \neq 0$, cf. [5], [4] and [2, 8.29].

The aim of this short paper is to prove a generalization of the last result above. Let $M^\vee$ be the Matlis dual of M. We show that for any Artinian R-module M, $\dim M^\vee$ is the greatest integer i such that $\mathrm{H}^i_{\mathfrak{m}}(M^\vee) \neq 0$. We know that, even if M is an Artinian R- module, then $M^\vee$ need not be finite (when R is not complete). Thus the result in this paper is a generalization of the above result.

Throughout this paper the ring R will be commutative and Noetherian and will have non-zero identity elements. We write "finite" for "finitely generated". Also, we shall use $\mathrm{Max}(R)$ to denote the set of all maximal ideals of R. For an R-module M its injective envelope is denoted by $\mathrm{E}(M)$. The phrase "$(R, \mathfrak{m})$ local" will mean that "R is local and has $\mathfrak{m}$ as its unique maximal ideal". We denote the Matlis dual of an R-module M by $M^\vee = \mathrm{Hom}(M, \mathrm{E}(R/\mathfrak{m}))$. Also for $x \in R$ the R-module $\mathrm{Hom}(R/(x), M)$ is denoted by $\underline{M}_x$.

1. Notation

This section contains the notation, including the definitions and results that we use in this paper.

Let M be an R-module, and let $\mathfrak{a}$ be an ideal of R. Let

$$\Gamma_{\mathfrak{a}}(M) = \{x \in M \mid x\mathfrak{a}^t = 0 \text{ for some } t \in \mathbb{N}\}.$$

Then $\Gamma_{\mathfrak{a}}$ is a covariant, additive and R-linear functor from the category of R-modules and R-homomorphisms to itself; it is called the section functor with support in $V(\mathfrak{a})$.

For $i \in \mathbb{N}_0$, the i-th right derived functor of $\Gamma_{\mathfrak{a}}$ is denoted by $\mathrm{H}^i_{\mathfrak{a}}$ and will be referred to as the i-th local cohomology functor with respect to the ideal $\mathfrak{a}$.

Now we recall some notions from [6] and [7]. For any $\mathfrak{m} \in \mathrm{Max}\, R$, let $\mathrm{D}_{\mathfrak{m}}(-) = \mathrm{Hom}(-, \mathrm{E}(R/\mathfrak{m}))$. We say a module is *cocyclic* if it is a submodule of $\mathrm{D}_{\mathfrak{m}}(R)$ for some $\mathfrak{m} \in \mathrm{Max}\, R$. We also say $\mathfrak{p} \in \mathrm{Spec}\, R$ is a *coassociated prime ideal* of M if there exists a cocyclic homomorphic image L of M such that $\mathrm{Ann}(L) = \mathfrak{p}$. The set of all coassociated prime ideals of M is denoted by $\mathrm{Coass}(M)$.

In addition, we say $\mathfrak{p} \in \mathrm{Spec}\, R$ is in the *cosupport* of M if there exists a cocyclic homomorphic image L of M such that $\mathfrak{p} \supseteq \mathrm{Ann}(L)$. The cosupport of M is denoted by $\mathrm{Cosupp}(M)$. The set of all $x \in R$ such that $xM \neq M$ is denoted by $\mathrm{w}(M)$ and this is a dual notion of the set $\mathrm{z}(M)$ of zero divisors, cf. [6, 1.12].

The *magnitude* of M is the supremum of $\dim R/\mathfrak{p}$ where $\mathfrak{p}$ runs over the set $\mathrm{Cosupp}\, M$ and we denote it $\mathrm{mag}\, M$. If $M = 0$ we write $\mathrm{mag}\, M = -\infty$. Note that $\mathrm{mag}\, M$ is the supremum of $\dim R/\mathfrak{p}$ where $\mathfrak{p}$ runs over the set $\mathrm{Coass}\, M$, cf. [6, 2.6]. In addition, if M is an Artinian R-module, then $\mathrm{mag}\, M = \dim R/\mathrm{Ann}(M)$, cf. [6, 2.3].

We say that a sequence of elements $x_1, \ldots, x_n$ of R is an *M-coregular sequence* if

(i) $x_i \notin \mathrm{w}(\mathrm{Hom}(A/(x_i, \ldots, x_{i-1}), M)$ for $i = 1, \ldots, n$,
(ii) $\mathrm{Hom}(A/(x_1, \ldots, x_n), M) \neq 0$.

For any R-module M and any ideal $\mathfrak{a}$ of R the $\mathfrak{a}$-*width* of M is defined by

$$\operatorname{cograde}(\mathfrak{a}, M) = \inf\{\ell \geqslant 0 \mid \operatorname{Tor}_{\ell}(R/\mathfrak{a}, M) \neq 0\}.$$

If $\operatorname{Tor}_\ell(R/\mathfrak{a}, M) = 0$ for all ℓ, then $\operatorname{cograde}(\mathfrak{a}, M) = \infty$. Note that $\operatorname{cograde}(\mathfrak{a}, M)$ may be infinite even when $\mathfrak{a}$ is a proper ideal and $M \neq 0$ (for example, let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a} = \mathfrak{m}$, and $M = \mathrm{E}(R/\mathfrak{p})$ where $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathfrak{p} \neq \mathfrak{m}$). If $(R, \mathfrak{m})$ is a local ring, the $\mathfrak{m}$-width of M is called the *width* of M.

1.1 Definition. Let M be an R-module and let $\mathfrak{p} \in \operatorname{Supp} M$. Define the M-height of $\mathfrak{p}$, $\operatorname{ht}(\mathfrak{p}, M)$, to be the supremum of lengths of chains of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r = \mathfrak{p}$ in the support of M which terminate with $\mathfrak{p}$.

If $\mathfrak{a}$ is an ideal of R we define $\operatorname{ht}(\mathfrak{a}, M)$, the M-height of $\mathfrak{a}$, by

$$\operatorname{ht}(\mathfrak{a}, M) = \inf\{\operatorname{ht}(\mathfrak{p}, M) \mid \mathfrak{p} \in \operatorname{Supp} M \cap V(\mathfrak{a})\}.$$

Remark. Let $(R, \mathfrak{m})$ be local and let M be an R-module. Then we have $\operatorname{ht}(\mathfrak{m}, M) = \dim M$, and also $\operatorname{ht}(\mathfrak{a}, R) = \operatorname{ht} a$ for any ideal $\mathfrak{a}$ of R.

1.2 Definition. Let M be an R-module and $\mathfrak{p} \in \operatorname{Cosupp} M$. Define the M-coheight of $\mathfrak{p}$, $\operatorname{coht}(\mathfrak{p}, M)$, to be the supremum of lengths of chains of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r = \mathfrak{p}$ in the cosupport of M which terminate with $\mathfrak{p}$.

If $\mathfrak{a}$ is an ideal of R we define $\operatorname{coht}(\mathfrak{a}, M)$, the M-coheight of $\mathfrak{a}$, by $\operatorname{coht}(\mathfrak{a}, M) = \inf\{\operatorname{coht}(\mathfrak{p}, M) \mid \mathfrak{p} \in \operatorname{Cosupp} M \cap V(\mathfrak{a})\}$.

Remark. Let $(R, \mathfrak{m})$ be local and let M be an R-module. It is easy to see that $\operatorname{coht}(\mathfrak{m}, M) = \operatorname{mag} M$ and since $\operatorname{Coass} \mathrm{E}(R/\mathfrak{m}) = \operatorname{Ass} R$ by [6, 1.17], we have $\operatorname{Cosupp} \mathrm{E}(R/\mathfrak{m}) = \operatorname{Supp} R$ by [6, 2.6], and hence for any ideal $\mathfrak{a}$ of R, $\operatorname{coht}(\mathfrak{a}, \mathrm{E}(R/\mathfrak{m})) = \operatorname{ht} a$.

1.4 Lemma. *Let M be an R-module and let $\mathfrak{a}$ be an ideal of R. Then by* [7, 1.9] *and* [7, 3.7] *we have*

(a) $\operatorname{cograde}(\mathfrak{a}, M) = \operatorname{grade}(\mathfrak{a}, \mathrm{D}_{\mathfrak{m}}(M))$ *for some* $\mathfrak{m} \in \operatorname{Max} R$.
(b) $\operatorname{coht}(\mathfrak{a}, M) = \operatorname{ht}(\mathfrak{a}, \mathrm{D}_{\mathfrak{m}}(M))$ *for some* $\mathfrak{m} \in \operatorname{Max} R$.

1.5 Theorem. *Let M be an R-module and let $\mathfrak{p} \in \operatorname{Supp} M$. Then*

(a) *For any ideal $\mathfrak{a}$, $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ if $i < \operatorname{grade}(\mathfrak{a}, M)$.*
(b) *Let $\mathfrak{p} \in \operatorname{Max} R$. Then $(\mathrm{H}^i_{\mathfrak{p}}(M))_{\mathfrak{p}} \neq 0$ if $i = \operatorname{grade}(\mathfrak{p}, M)$.*
(c) *For any ideal $\mathfrak{a}$, $(\mathrm{H}^i_{\mathfrak{a}}(M))_{\mathfrak{p}} = 0$ if $i > \operatorname{ht}(\mathfrak{p}, M)$.*
(d) *If M is a finite R-module, then $(\mathrm{H}^i_{\mathfrak{p}}(M))_{\mathfrak{p}} \neq 0$ if $i = \operatorname{ht}(\mathfrak{p}, M)$.*

Proof. (a) See [2, 7.10].

(b) Set $\operatorname{grade}(\mathfrak{p}, M) = t$. Since $\operatorname{Supp}\operatorname{Ext}^t(R/\mathfrak{p}, M) \subseteq \operatorname{Supp} R/\mathfrak{p} \cap \operatorname{Supp} M = \{\mathfrak{p}\}$ we have $t = \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p})$. Now use [2, 7.11].

(c) It is easy to see that $\operatorname{ht}(\mathfrak{p}, M) = \dim_{R_\mathfrak{p}}(M_\mathfrak{p})$. Now use [2, 7.8].

(d) Use (iii) and [2, 8.29]. □

1.6 Corollary. *Let M be an R-module and let $\mathfrak{p} \in \operatorname{Cosupp} M$. Then*

(a) *For any ideal $\mathfrak{a}$, $\mathrm{H}^i_\mathfrak{a}(\mathrm{D}_\mathfrak{m}(M)) = 0$ for some $\mathfrak{m} \in \operatorname{Max} R$, if $i < \operatorname{cograde}(\mathfrak{a}, M)$.*

(b) *Let $\mathfrak{p} \in \operatorname{Max} R$. Then $(\mathrm{H}^i_\mathfrak{p}(\mathrm{D}_\mathfrak{m}(M)))_\mathfrak{p} \neq 0$ for some $\mathfrak{m} \in \operatorname{Max} R$, if $i = \operatorname{cograde}(\mathfrak{p}, M)$. (Note that $\mathfrak{m} = \mathfrak{p}$.)*

(c) *For any ideal $\mathfrak{a}$, $(\mathrm{H}^i_\mathfrak{a}(\mathrm{D}_\mathfrak{m}(M)))_\mathfrak{p} = 0$ for some $\mathfrak{m} \in \operatorname{Max} R$, if $i > \operatorname{coht}(\mathfrak{p}, M)$.*

Proof. Use (1.4) and (1.5). □

1.7 Remark. In Theorem (1.5)(d) we use the finite condition for the module M. Now it is natural to ask "what is the dual of this result?". In the next section we bring the dual of this part for local rings.

2. Main result

In this section we follow the dual of the proof of [4, 2.2]. First we need a lemma.

2.1 Lemma. *Let $(R, \mathfrak{m})$ be local and let M be a non-zero Artinian R-module of magnitude s. Then the set*

$$\Sigma = \{N \mid N \text{ is a submodule of } M \text{ and } \operatorname{mag} M/N < s\}$$

has a minimal element with respect to inclusion, say N. Then

(a) $\operatorname{mag} N = s$,

(b) *N has no non-zero submodule L such that $\operatorname{mag} N/L < s$,*

(c) *$\operatorname{Coass} N = \{\mathfrak{p} \in \operatorname{Coass} M \mid \dim R/\mathfrak{p} = s\}$, and*

(d) $\mathrm{H}^s_\mathfrak{m}(N^\vee) \cong \mathrm{H}^s_\mathfrak{m}(M^\vee)$.

Proof. Since M is an Artinian R-module, the set Σ has a minimal member, say N.

(a) It follows from the natural exact sequence

$$0 \to N \to M \to M/N \to 0$$

that $\operatorname{Cosupp} N \subseteq \operatorname{Cosupp} M$ and any $\mathfrak{p} \in \operatorname{Coass} M$ having $\dim R/\mathfrak{p} = s$ must belong to $\operatorname{Cosupp} N$ since it can not belong to $\operatorname{Cosupp} M/N$. Hence $\operatorname{mag} N = s$ and

$$\{\mathfrak{p} \in \operatorname{Coass} M \mid \dim R/\mathfrak{p} = s\} \subseteq \operatorname{Coass} N.$$

(b) Suppose that L is a submodule of M such that $L \subseteq N \subseteq M$ and $\operatorname{mag} N/L < s$. The natural exact sequence

$$0 \to N/L \to M/L \to M/N \to 0$$

shows that $\operatorname{mag} M/L < s$; hence $L \in \Sigma$ and therefore $L = N$.

(c) Let $\mathfrak{p} \in \operatorname{Coass} N$. There exists a cocyclic homomorphic image N/L of N such that $\operatorname{Ann} N/L = \mathfrak{p}$. By (b) $\operatorname{mag} N/L = s$, so that $\dim R/\mathfrak{p} = s$, hence $\operatorname{Coass} N \subseteq \{\mathfrak{p} \in \operatorname{Coass} M \mid \dim R/\mathfrak{p} = s\}$. Since the reverse inclusion was established in our proof of (a) above, we have completed the proof of (c).

(d) Since $\operatorname{mag} M/N < s$, it follows from (1.6)(c) that

$$\mathrm{H}^s_{\mathfrak{m}}((M/N)^\vee) = \mathrm{H}^{s+1}_{\mathfrak{m}}((M/N)^\vee) = 0.$$

The claim therefore follows from the long exact sequence of local cohomology modules (with respect to $\mathfrak{m}$) which results from the exact sequence

$$0 \to (M/N)^\vee \to M^\vee \to N^\vee \to 0.$$

□

Now we are ready to prove the main result.

2.2 Theorem. *Let $(R, \mathfrak{m})$ be a local ring and let M be a non-zero Artinian R-module with $\operatorname{mag} M = s$. Then $\mathrm{H}^s_{\mathfrak{m}}(M^\vee) \neq 0$ and*

$$\operatorname{Coass} \mathrm{H}^s_{\mathfrak{m}}(M^\vee) = \{\mathfrak{p} \in \operatorname{Coass} M \mid \dim R/\mathfrak{p} = s\}.$$

Proof. We use induction on s. When $s = 0$, we have $\operatorname{Coass} M = \{\mathfrak{m}\}$. Thus M and hence $M^\vee$ is annihilated by some power of $\mathfrak{m}$, cf. [6, 2.12]. Hence $\mathrm{H}^0_{\mathfrak{m}}(M^\vee) \cong \Gamma_{\mathfrak{m}}(M^\vee) = M^\vee \neq 0$. By [6. 1.7],

$$\begin{aligned}\operatorname{Coass} \mathrm{H}^0_{\mathfrak{m}}(M^\vee) = \operatorname{Coass} M^\vee = \operatorname{Ass} M^{\vee\vee} = \operatorname{Ass} M \\ = \{\mathfrak{p} \in \operatorname{Coass} M \mid \dim R/\mathfrak{p} = 0\}\end{aligned}$$

Thus the result has been proved in this case.

Assume, inductively, that $s = t+1 > 0$ and that the result has been proved when $s = t$. By (2.1), we can assume that M has no non-zero homomorphic image of magnitude less than $t+1$.

We shall make this assumption for the remainder of the inductive step.

Since $t+1 > 0$, we have $\mathfrak{m} \notin \operatorname{Coass} M$, and so there exists $a \in \mathfrak{m}$ which does not belong to $\mathrm{w}(M)$. We suppose that $\mathrm{H}_{\mathfrak{m}}^{t+1}(M^{\vee}) = 0$, and look for a contradiction. When $t = 0$, we have

$$1 \leqslant \operatorname{width} M \leqslant \operatorname{mag} M = 1.$$

So the width $M = 1$ and we have a contradiction, cf. (1.6).

Thus we can assume $t > 0$. We have $\operatorname{mag} \underline{M}_a = t$, and the exact sequence

$$0 \longrightarrow M^{\vee} \xrightarrow{a\cdot} M^{\vee} \longrightarrow (\underline{M}_a)^{\vee} \longrightarrow 0$$

induces a long exact sequence of local cohomology modules

$$\mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee}) \xrightarrow{a\cdot} \mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee}) \longrightarrow \mathrm{H}_{\mathfrak{m}}^{t}((\underline{M}_a)^{\vee}) \longrightarrow 0$$

in view of our supposition that $\mathrm{H}_{\mathfrak{m}}^{t+1}(M^{\vee}) = 0$. Thus for each M-coregular element a belonging to $\mathfrak{m}$ we have

$$\mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee})/a\mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee}) \cong \mathrm{H}_{\mathfrak{m}}^{t}((\underline{M}_a)^{\vee}),$$

and this is non-zero by the induction hypothesis. Thus $\mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee}) \neq 0$.

We now claim that $\mathfrak{m} \in \operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee})$. If this were not the case, we should have, by prime avoidance

$$\mathfrak{m} \nsubseteq \mathrm{w}(\mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee})) \cup \mathrm{w}(M)$$

so that there exists an M-coregular element b that belongs to $\mathfrak{m}$ such that $\mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee}) = b\mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee})$, and this is impossible because $\mathrm{H}_{\mathfrak{m}}^{t}((\underline{M}_b)^{\vee}) \neq 0$.

Thus $\mathfrak{m} \in \operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee})$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the remaining members of $\operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee})$. Again by prime avoidance, there exists $c \in \mathfrak{m} - \left(\bigcup_{i=1}^{r} \mathfrak{p}_i\right) \cup \mathrm{w}(M)$. We again have $\mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee})/c\mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee}) \cong \mathrm{H}_{\mathfrak{m}}^{t}((\underline{M}_c)^{\vee})$, since $c \in \mathfrak{m}$ is an M-coregular element, and by the induction hypothesis, $\mathrm{H}_{\mathfrak{m}}^{t}((\underline{M}_c)^{\vee}) \neq 0$ and

$$\operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{t}((\underline{M}_c)^{\vee}) \subseteq \{\mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} = t\}$$

but, by [6. 1.21],

$$\operatorname{Coass}(\mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee})/c\mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee})) \subseteq \{\mathfrak{p} \in \operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{t}(M^{\vee}) \mid c \in \mathfrak{p}\}$$

and $\mathfrak{m}$ is the only member of the latter set. Since $t > 0$, we have a contradiction.

Thus we have proved that $\mathrm{H}_{\mathfrak{m}}^{t+1}(M^\vee) \neq 0$. To complete the inductive step, it remains for us to prove that $\operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{t+1}(M^\vee) = \operatorname{Coass} M$.

We know that there is an M-coregular element in $\mathfrak{m}$. Moreover, for every M-coregular element $a \in \mathfrak{m}$ we have $\operatorname{mag} \underline{M}_a = t$, so that $\mathrm{H}_{\mathfrak{m}}^{t+1}((\underline{M}_a)^\vee) = 0$, and the long exact sequence of local cohomology modules induced by the exact sequence

$$0 \to M^\vee \xrightarrow{a\cdot} M^\vee \longrightarrow (\underline{M}_a)^\vee \longrightarrow 0$$

yields that $\mathrm{H}_{\mathfrak{m}}^{t+1}(M^\vee) = a\mathrm{H}_{\mathfrak{m}}^{t+1}(M^\vee)$. It therefore follows that $\mathfrak{m} - \mathrm{w}(M) \subseteq \mathfrak{m} - \mathrm{w}(\mathrm{H}_{\mathfrak{m}}^{t+1}(M^\vee))$.

Let $\mathfrak{q} \in \operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{t+1}(M^\vee)$. It follows from the above inclusion relation and the prime avoidance that $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Coass} M$. Since $\mathrm{H}_{\mathfrak{m}}^{t+1}(-)$ is an additive, R-linear functor, it follows that

$$\operatorname{Ann}(M^\vee) \subseteq \operatorname{Ann}(\mathrm{H}_{\mathfrak{m}}^{t+1}(M^\vee)) \subseteq \mathfrak{q} \subseteq \mathfrak{p}.$$

Hence $\mathfrak{q} \supseteq \mathfrak{p}'$ for some $\mathfrak{p}' \in \operatorname{Coass} M$. But each $\mathfrak{p}'' \in \operatorname{Coass} M$ has $\dim R/\mathfrak{p}'' = t+1$, and so $\mathfrak{p}' = \mathfrak{q} = \mathfrak{p}$. Hence

$$\operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{t+1}(M^\vee) \subseteq \operatorname{Coass} M.$$

Now let $\mathfrak{p} \in \operatorname{Coass} M$, so that $\dim R/\mathfrak{p} = t+1$. Thus there exists a $\mathfrak{p}$-secondary submodule Q of M, cf. [3, 5.2] and [6, 1.14]. Note that Q can not have any non-zero homomorphic image of magnitude less than $t+1$ (or else it would have an coassociated prime other than $\mathfrak{p}$). Now if we use Q rather than M in the above, we have $\operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{t+1}(Q^\vee) \subseteq \operatorname{Coass} Q = \{\mathfrak{p}\}$ and $\mathrm{H}_{\mathfrak{m}}^{t+1}(Q^\vee) \neq 0$. Thus $\operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{t+1}(Q^\vee) = \{\mathfrak{p}\}$. However, the exact sequence

$$0 \to (M/Q)^\vee \to M^\vee \to Q^\vee \to 0$$

induces an epimorphism $\mathrm{H}_{\mathfrak{m}}^{t+1}(M^\vee) \to \mathrm{H}_{\mathfrak{m}}^{t+1}(Q^\vee)$ since $\operatorname{mag}(M/Q)^\vee < t+2$. It now follows that

$$\operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{t+1}(Q^\vee) \subseteq \operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{t+1}(M^\vee).$$

Hence $\operatorname{Coass} M \subseteq \operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{t+1}(M^\vee)$. □

2.3 Corollary. *Let $(R, \mathfrak{m})$ be a local ring and let M be a finite R-module with $\dim M = s$. Then*

$$\operatorname{Coass} \mathrm{H}_{\mathfrak{m}}^{s}(\widehat{M}) = \{\mathfrak{p} \in \operatorname{Ass} M \mid \dim R/\mathfrak{p} = s\}.$$

(Note that $\widehat{M}$ is the completion of M in the $\mathfrak{m}$-adic topology.)

Proof. We know that $M^\vee$ is an Artinian R-module, $\operatorname{mag} M^\vee = s$ and $\operatorname{Coass} M^\vee = \operatorname{Ass} M$. Also, we have $\widehat{M} = M^{\vee\vee}$. Now the assertion follows from (2.2). □

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