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REACTION-DIFFUSION SYSTEMS: STABILIZING EFFECT OF CONDITIONS DESCRIBED BY QUASIVARIATIONAL INEQUALITIES

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Abstract. Reaction-diffusion systems are studied under the assumptions guaranteeing diffusion driven instability and arising of spatial patterns. A stabilizing influence of unilateral conditions given by quasivariational inequalities to this effect is described.

Keywords: reaction-diffusion systems, unilateral conditions, bifurcation, quasivariational inequalities, spatial patterns

MSC 1991: 35B32, 35K57, 35J85

INTRODUCTION

Let us consider a bounded domain in $\mathbb{R}^k$ with a lipschitzian boundary Γ and real differentiable functions f, g on $\mathbb{R}^2$. We will study reaction-diffusion systems

$$\text{(RD)} \qquad \begin{aligned} u_t &= d_1\Delta u + f(u,v), \\ v_t &= d_2\Delta v + g(u,v) \quad \text{on } [0,+\infty)\times\Omega \end{aligned}$$

with positive parameters d_1, d_2. Let $\Gamma_{\mathcal{D}}$, $\Gamma_{\mathcal{N}}$, $\Gamma_{\mathcal{U}}$ be pairwise disjoint subsets of Γ, $\operatorname{meas}(\Gamma \setminus \Gamma_{\mathcal{D}} \cup \Gamma_{\mathcal{N}} \cup \Gamma_{\mathcal{U}}) = 0$, $\operatorname{meas}\Gamma_{\mathcal{D}} > 0$. Suppose that $\bar{u}$, $\bar{v}$ are nonnegative constants, $f(\bar{u},\bar{v}) = g(\bar{u},\bar{v}) = 0$. Particularly, $\bar{u}$, $\bar{v}$ is a constant solution of (RD) with boundary conditions

$$\text{(CC)} \qquad u = \bar{u},\ v = \bar{v} \text{ on } \Gamma_{\mathcal{D}},\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_{\mathcal{N}} \cup \Gamma_{\mathcal{U}}.$$

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Simultaneously, we will consider unilateral conditions of the type

$$
\text{(UC)}\qquad \begin{cases} u = \bar{u}, \quad v = \bar{v} \quad \text{on} \quad \Gamma_{\mathcal{D}}, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 \quad \text{on} \quad \Gamma_{\mathcal{N}}, \\ \dfrac{\partial v}{\partial n} = 0, \quad u(x) \geqslant \bar{u} - \displaystyle\int_{\Gamma_{\mathcal{U}}} \Phi(x,y) \frac{\partial u}{\partial n}(y)\, \mathrm{d}\Gamma(y), \quad \dfrac{\partial u}{\partial n} \geqslant 0, \\ \qquad \left(u(x) - \bar{u} + \displaystyle\int_{\Gamma_{\mathcal{U}}} \Phi(x,y) \frac{\partial u}{\partial n}(y)\, \mathrm{d}\Gamma(y) \right) \dfrac{\partial u(x)}{\partial n} = 0 \quad \text{on} \quad \Gamma_{\mathcal{U}} \end{cases}
$$

where $\Phi \in C^2(\Gamma)$, $\Phi \geqslant 0$ (see Remark 1.1 for details). We will always suppose that

$$
\text{(SIGN)}\qquad b_{11} > 0,\ b_{12} < 0,\ b_{21} > 0,\ b_{22} < 0,\ b_{11} + b_{22} < 0,\ \det b_{ij} > 0
$$

where $b_{11} = \frac{\partial f}{\partial u}(\bar{u}, \bar{v})$, $b_{12} = \frac{\partial f}{\partial v}(\bar{u}, \bar{v})$, $b_{21} = \frac{\partial g}{\partial u}(\bar{u}, \bar{v})$, $b_{22} = \frac{\partial f}{\partial v}(\bar{u}, \bar{v})$. This condition corresponds to a class of activator-inhibitor (or prey-predator) systems. Our aim is to show a certain stabilizing influence of unilateral conditions of the type (UC) to the following effect (diffusion driven instability) known for the classical problem (RD), (CC) under the assumption (SIGN): The constant solution $[\bar{u}, \bar{v}]$ is stable as a solution of the corresponding problem without any diffusion but as a solution of (RD), (CC) it is stable only for some $[d_1, d_2] \in \mathbb{R}^2_+$ (domain of stability) and unstable for the other $[d_1, d_2] \in \mathbb{R}^2_+$ (domain of instability). See Fig. 1, Proposition 1.1 and references therein. Spatially nonconstant stationary solutions (spatial patterns) bifurcate from $\bar{u}$, $\bar{v}$ at the border between the domain of stability and instability while bifurcation is excluded in the domain of stability. We will prove that bifurcation of stationary solutions of the unilateral problem (RD), (UC) is excluded not only in the domain of stability (corresponding to (RD), (CC)) but also in a part of the domain of instability (see Theorem 1.1). This means that if a unilateral condition is prescribed for the activator (or prey) then the bifurcation can occur in a certain sense later than in the classical case (see Interpretation). This can be understood as a stabilizing effect of unilateral conditions given for the activator.

In the sequel, we will suppose without loss of generality $\bar{u} = \bar{v} = 0$. Then the stationary problem corresponding to (RD), (UC) can be formulated in the weak sense as a quasivariational inequality

$$
\text{(I)}\qquad \begin{aligned} & u, v \in \mathbf{H},\ u \in K_u, \\ & \langle d_1 u - b_{11} A u - b_{12} A v + N_1(u,v), \varphi - u \rangle \geqslant 0 \text{ for all } \varphi \in K_u, \\ & d_2 u - b_{21} A u - b_{22} A v + N_2(u,v) = 0 \end{aligned}
$$

where

(A) $\quad A\colon \mathbb{V} \to \mathbb{V}$ is a completely continuous symmetric positive operator in a suitable Hilbert space $\mathbb{V}$ (see Weak Formulation 1.1), $\mathbb{H}$ is a suitable subspace of $\mathbb{V}$,

(N) $N_j\colon \mathbb{V}\times\mathbb{V}\to\mathbb{V}$ ($j=1,2$) are completely continuous operators satisfying

$$\lim_{\|U\|\to 0}\frac{\|N_j(U)\|}{\|U\|}=0,$$

and K_u, $u\in\mathbb{H}$ is a system of closed convex sets in $\mathbb{V}$ satisfying certain assumptions (see Weak Formulation 1.2). The main result will be formulated for such abstract problems. The inequality (I) will be compared with the corresponding system of equations

$$\text{(E)}\qquad \begin{aligned} &d_1u-b_{11}Au-b_{12}Av+N_1(u,v)=0,\\ &d_2v-b_{21}Au-b_{22}Av+N_2(u,v)=0\end{aligned}$$

which can represent the weak formulation of the stationary problem corresponding to (RD), (CC) (see Weak Formulation 1.1). In fact, the corresponding linearized systems

$$(\mathrm{I_L})\qquad \begin{aligned} &u,v\in\mathbf{H}, u\in K_u,\\ &\langle d_1u-b_{11}Au-b_{12}Av,\varphi-u\rangle\geqslant 0 \text{ for all } \varphi\in K_u,\\ &d_2v-b_{21}Au-b_{22}Av=0\end{aligned}$$

and

$$(\mathrm{E_L})\qquad \begin{aligned} &d_1u-b_{11}Au-b_{12}Av=0,\\ &d_2v-b_{21}Au-b_{22}Av=0\end{aligned}$$

will play an essential role. Of course, the problem $(\mathrm{I_L})$ is nonlinear again.

Notice that a result of the type mentioned was briefly explained in [5] for a special case of unilateral conditions described by variational inequalities and generalized in [8] for conditions described by inclusions. The conditions (UC) with $\Phi=0$ are included in [5], [8] but the general case ($\Phi\neq 0$) is contained neither in [5] nor in [8]. (On the other hand, conditions given by inclusions cannot be described by quasivariational inequalities in general.) The opposite, destabilizing effect of unilateral conditions given by quasivariational inequalities for the function v describing the inhibitor (predator) was described in [6]. There it is proved that if such a unilateral condition is prescribed for v then stationary spatially nonhomogeneous solutions bifurcate already in the domain of stability of the constant solution of (RD), (CC) where the bifurcation for (RD), (CC) is excluded. In terms of an interpretation, it means that if a unilateral condition is prescribed for the inhibitor then the bifurcation arises in a certain sense sooner than in the case of classical boundary conditions (see Weak Formulation 1.2, Motivation of Unilateral Conditions and Interpretation). An analogous result for the case of variational inequalities was proved in [1], [13] and generalized to inclusions in [7]. A certain destabilizing effect of unilateral conditions given by variational inequalities for the inhibitor in terms of loss of stability of the trivial solution of the linearized unilateral problem was proved in [2], [3], [13].

Analogously as in the case of bifurcation problems for inequalities, one of the basic dificulties in the study of bifurcations of (I) is that the "linearization" ($\mathrm{I_L}$) of the problem (I) is a strongly nonlinear problem.

1. General remarks

For simplicity, it will be always supposed that $\bar{u} = \bar{v} = 0$.

Notation.

$\mathbb{R}_+$—the set of all positive reals, $\mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}_+$,
$q > 2$—a given real number such that $W^1_2(\Omega) \subset L_q(\Omega)$, i.e. $q < \frac{2k}{k-2}$ or q arbitrary if $k \geqslant 3$ or $k = 2$, respectively,
$\mathbb{V} = \{\varphi \in H^1(\Omega);\ \varphi = 0 \text{ on } \Gamma_{\mathcal{D}} \text{ in the sense of traces}\}$ ($H^1(\Omega)$ is the usual Sobolev space),
$H^{q^*}_\Delta(\Omega) = \{\varphi \in H^1(\Omega);\ \Delta\varphi \in L_{q^*}(\Omega)\}$ where $\frac{1}{q} + \frac{1}{q^*} = 1$,
$\mathbb{H} = H^{q^*}_\Delta(\Omega) \cap \mathbb{V}$,
$\langle \varphi, \psi \rangle = \int_\Omega \nabla\varphi \nabla\psi \,\mathrm{d}x$, $\|\varphi\|^2 = \langle \varphi, \varphi, \rangle$—the inner product and the norm on $\mathbb{V}$ (which is equivalent to the usual norm on $\mathbb{V}$ under the assumption $\operatorname{meas}\Gamma_{\mathcal{D}} > 0$),
$|\!|\!|\varphi|\!|\!| = \left(\int_\Omega (|\nabla\varphi|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_\Omega |\Delta\varphi|^{q^*} \,\mathrm{d}x\right)^{\frac{1}{q^*}}$—the norm on $\mathbb{H}$,
A, N_j—operators introduced in Weak Formulation 1.1 (see also Lemma 1.1),
$U = [u, v]$—elements of $\mathbb{V} \times \mathbb{V}$, $AU = [Au, Av]$ for $U = [u, v] \in \mathbb{V} \times \mathbb{V}$,
$\langle U, W \rangle = \langle u, w \rangle + \langle v, z \rangle$, $\|U\|^2 = \|u\|^2 + \|v\|^2$ for $U = [u, v]$, $W = [w, z] \in \mathbb{V} \times \mathbb{V}$,
$|\!|\!|U|\!|\!|^2 = |\!|\!|u|\!|\!|^2 + |\!|\!|v|\!|\!|^2$ for $U = [u, v] \in \mathbb{H} \times \mathbb{H}$,
$H^{\frac{1}{2}}(\Gamma)$, $H^{-\frac{1}{2}}(\Gamma)$—the space of traces of functions from $H^1(\Omega)$ and its dual space (see [9]),
$\Phi(x, y)$—a given smooth nonnegative function on $\Gamma \times \Gamma$—Remark 1.1,
K_u ($u \in \mathbf{H}$)—a system of closed convex sets introduced in Weak Formulation 1.2,
$\rightarrow$, $\rightharpoonup$ —strong convergence, weak convergence,

κ_j, e_j ($j = 1, 2, \ldots$)—the characteristic values and characteristic vectors of the operator A, i.e. eigenvalues and eigenvectors of $-\Delta u = \lambda u$ with (CC) (see Weak Formulation 1.1),

$C_E^j = \{d = [d_1, d_2] \in \mathbb{R}_+^2;\ d_2 = \frac{b_{12}b_{21}/\kappa_j^2}{d_1 - b_{11}/\kappa_j} + \frac{b_{22}}{\kappa_j}\}$, $j = 1, 2, 3, \ldots$—hyperbolas from Fig. 1,

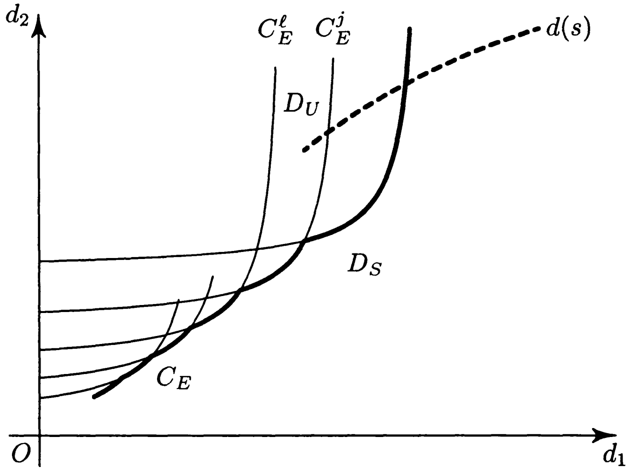

Fig. 1

C_E—the envelope of the hyperbolas C_E^j, $j = 1, 2, 3, \ldots$ (see Fig. 1),

D_S—domain of stability—the set of all $d \in \mathbb{R}_+^2$ lying to the right from C_E (see Fig. 1),

D_U—domain of instability—the set of all $d \in \mathbb{R}_+^2$ lying to the left from C_E (see Fig. 1),

$C_r^R = \{d = [d_1, d_2] \in C_E;\ r \leqslant d_2 \leqslant R\}$,

$C_r^R(\varepsilon) = \{d = [d_1, d_2] \in \overline{D_U};\ r \leqslant d_2 \leqslant R,\ \operatorname{dist}(d, C_E) < \varepsilon\}$,

$E_B(d) = \{U \in \mathbb{V} \times \mathbb{V};\ (\mathrm{E_L}) \text{ is fulfilled}\}$, $E_S(\lambda) = \{u \in \mathbb{V};\ (2.5) \text{ is fulfilled}\}$,

critical point of ($\mathrm{E_L}$)—a parameter $d \in \mathbb{R}_+^2$ for which ($\mathrm{E_L}$) has a nontrivial solution,

critical point of ($\mathrm{I_L}$)—a parameter $d \in \mathbb{R}_+^2$ for which ($\mathrm{I_L}$) has a nontrivial solution,

bifurcation point of (E) or (I)—a parameter $d^0 \in \mathbb{R}_+^2$ such that in any neighbourhood of $[d^0, 0, 0]$ in $\mathbb{R}_+^2 \times \mathbb{V} \times \mathbb{V}$ there exists $[d, U] = [d, u, v]$, $\|U\| \neq 0$ satisfying (E) or (I), respectively,

bifurcation point of (2.7) or (2.8)—a parameter $s_0 \in \mathbb{R}$ such that in any neighbourhood of $[s_0, 0, 0]$ in $\mathbb{R} \times \mathbb{V} \times \mathbb{V}$ there exists $[s, U] = [s, u, v]$, $\|U\| \neq 0$ satisfying (2.7) or (2.8), respectively.

Of course, any bifurcation point of (E) is simultaneously a critical point of ($\mathrm{E_L}$). Similarly for (I), ($\mathrm{I_L}$) (see Lemma 1.2).

Weak Formulation 1.1. Set

$$n_1(u,v) = b_{11}u + b_{12}v - f(u,v),\ n_2(u,v) = b_{21}u + b_{22}v - g(u,v)$$

and suppose that

$$|n_j(u,v)| \leqslant c(1 + |u|^{q-1} + |v|^{q-1}),\ j = 1,2 \tag{1.1}$$

with $q > 2$ or $2 < q < \frac{2k}{k-2}$ in the case $k = 2$ or $k > 2$, respectively. Then the Nemyckij operator $u, v \to n_j(u,v)$ of L_q into L_{q^*} is continuous. Define operators $A\colon \mathbb{V} \to \mathbb{V}$, $N_j\colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ $(j = 1,2)$ by

$$\begin{aligned}\langle Au, \varphi\rangle &= \int_\Omega u\varphi \,\mathrm{d}x \quad \text{for all } u, \varphi \in \mathbb{V},\\ \langle N_j(U), \varphi\rangle &= \int_\Omega n_j(u,v)\varphi \,\mathrm{d}x \quad \text{for all } U = [u,v] \in \mathbb{V} \times \mathbb{V},\ \varphi \in \mathbb{V},\ j = 1,2.\end{aligned}$$

Then (A) and (N) follow from the compactness of the imbedding $\mathbb{V} \subset L_q$ and the continuity of the Nemyckij operator of $L_q(\Omega)$ into $L_{q*}(\Omega)$, $\frac{1}{q} + \frac{1}{q^*} = 1$ (see Lemma A1 and Consequence A1 in Appendix for details). The system of operator equations (E) is the weak formulation of the stationary problem

$$\begin{aligned}d_1\Delta u + f(u,v) &= 0,\\ d_2\Delta v + g(u,v) &= 0\end{aligned} \tag{$\mathrm{RD_S}$}$$

with the classical boundary conditions (CC). Further,

$$\begin{aligned}d_1 u - b_{11}Au - b_{12}Av + \lambda Au &= 0,\\ d_2 v - b_{21}Au - b_{22}Av + \lambda Av &= 0\end{aligned} \tag{EP}$$

is the weak formulation of the eigenvalue problem

$$\begin{aligned}d_1\Delta u + b_{11}u + b_{12}v &= \lambda u,\\ d_2\Delta v + b_{21}u + b_{22}v &= \lambda v\end{aligned} \tag{$\mathrm{RD_\lambda}$}$$

with (CC) describing the stability of the trivial solution of (RD), (CC) (see e.g. [14]).

Remark 1.1. For any $u \in H^{q^*}_\Delta(\Omega)$ we can define the normal derivative $\frac{\partial u}{\partial n}$ at Γ as a functional from $H^{-\frac{1}{2}}(\Gamma)$ such that the Green formula holds and the mapping $u \to \frac{\partial u}{\partial n}$ of $H^{q^*}_\Delta(\Omega)$ into $H^{-\frac{1}{2}}(\Gamma)$ is continuous (see Lemma A2 in Appendix for details). Consider a nonnegative function Φ on $\Gamma \times \Gamma$ which can be extended onto $\bar{\Omega} \times \bar{\Omega}$ such that

$$(\Phi) \qquad \Phi \in C^2(\bar{\Omega} \times \bar{\Omega}),\ \Phi(x,y) = 0 \text{ for all } x,y \in \Gamma_{\mathcal{D}} \times \Gamma.$$

Particularly, $\Phi(x,\cdot) \in W_2^{\frac{1}{2}}(\Gamma)$ for any $x \in \Gamma$ and the boundary conditions (UC) for $u,v \in \mathbb{H}$ can be understood in the sense of functionals $\frac{\partial u}{\partial n}, \frac{\partial v}{\partial n} \in H^{-\frac{1}{2}}(\Gamma)$. Particularly, integrals are understood as the duality between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$, $\frac{\partial u}{\partial n} \geqslant 0$ on $\Gamma_{\mathcal{U}}$ means $\int_\Gamma \frac{\partial u}{\partial n}\varphi \, \mathrm{d}\Gamma \geqslant 0$ for all $\varphi \in H^{\frac{1}{2}}(\Gamma)$ satisfying $\varphi \geqslant 0$ on $\Gamma_{\mathcal{U}}$, $\varphi = 0$ on $\Gamma_{\mathcal{D}} \cup \Gamma_{\mathcal{N}}$, the last condition in (UC) (with $\bar{u} = \bar{v} = 0$) means

$$\int_{\Gamma_{\mathcal{U}}} \left[u(x) + \int_{\Gamma_{\mathcal{U}}} \Phi(x,y)\frac{\partial u}{\partial n}(y)\,\mathrm{d}\Gamma(y)\right] \frac{\partial u}{\partial n}(x)\,\mathrm{d}\Gamma(x) = 0.$$

(Note that clearly $\int_{\Gamma_{\mathcal{U}}} \Phi(\cdot,y)\frac{\partial u}{\partial n}(y)\,\mathrm{d}\Gamma(y) \in H^{\frac{1}{2}}(\Gamma)$ under our assumption.)

Weak Formulation 1.2. For any $u \in \mathbb{H}$, define a closed convex set K_u in $\mathbb{V}$ by

$$K_u = \left\{\varphi \in \mathbb{V};\ \varphi(x) \geqslant -\int_{\Gamma_{\mathcal{U}}} \Phi(x,y)\frac{\partial u}{\partial n}(y)\,\mathrm{d}\Gamma(y) \text{ on } \Gamma_{\mathcal{U}}\right\}$$

where Φ is from Remark 1.1 and the integral is understood as the duality between $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$ (see Remark 1.1). It is natural to introduce a weak solution of the problem ($\mathrm{RD_S}$), (UC) (with $\bar{u} = \bar{v} = 0$) as a solution of the quasivariational inequality (I). It is easy to see that such a solution u,v satisfies ($\mathrm{RD_S}$) a.e. on Ω and the boundary conditions (UC) are fulfilled in the sense of Remark 1.1. Particularly, $\frac{\partial u}{\partial n} \geqslant 0$ on $\Gamma_{\mathcal{U}}$.

By using Weak Formulation 1.2 and Remark 1.1 (see also Lemma A2 in Appendix) it is easy to see that the sets K_u satisfy the following conditions:

(0) if u,v satisfy (I) with some $[d_1,d_2] \in \mathbb{R}_2^+$ then $0 \in K_u$,

(HK) if $u \in \mathbb{H}$, $t > 0$ then $u \in K_u$ if and only if $tu \in K_{tu}$,

(CK) if $u_n \in \mathbb{H}$, $\varphi_n \in K_{u_n}$, $u_n \rightharpoonup u$ in $\mathbb{H}$, $\varphi_n \rightharpoonup \varphi$ in $\mathbb{V}$ then $\varphi \in K_u$,

(AK) if $u_n, u \in \mathbb{H}$, $u_n \rightharpoonup u$ in $\mathbb{H}$, $\varphi \in K_u$,
then there exist $\varphi_n \in K_{u_n}$, $\varphi_n \to \varphi$ in $\mathbb{V}$.

(We can choose $\varphi_n = \varphi + f_n - f$ where

$$f_n(x) = \int_{\Gamma_{\mathcal{U}}} \Phi(x,y)\frac{\partial u_n}{\partial n}(y)\,\mathrm{d}\Gamma(y),\ f(x) = \int_{\Gamma_{\mathcal{U}}} \Phi(x,y)\frac{\partial u}{\partial n}(y)\,\mathrm{d}\Gamma(y),$$

$f_n \to f$ in $\mathbb{V}$ under the assumption (Φ)).

Proposition 1.1. *Let* (SIGN) *hold and let* $\operatorname{meas}\Gamma_{\mathcal{D}} > 0$. *Then* $\bigcup_{j=1}^{\infty} C_E^j$ *is the set of all critical points of* ($\mathrm{E_L}$). *Particularly, there are no bifurcation points of* (E) *in* D_S. *If* $d \in D_S$ *then all eigenvalues of* (EP) *(i.e. of* (RD_λ), (CC)*) have negative real parts. If* $d \in D_U$ *then there exists at least one positive eigenvalue.*

If $d \in C_E^s$ *for* $s = j, \dots, j+k-1$ *(i.e. either* k *is the multiplicity of the eigenvalue* κ_j, $C_E^j = \dots = C_E^{j+k-1}$, *or* d *is in the intersection of two different hyperbolas* C_E^j, C_E^ℓ *and* k *is the sum of the multiplicities of* κ_j, κ_ℓ*) then* $E_B(d) = \operatorname{Lin}\{[\frac{d_2\kappa_s - b_{22}}{b_{21}} e_s, e_s]\}_{s=j}^{j+k-1}$.

Particularly, the trivial solution of (RD), (CC) *is stable and unstable for* $d \in D_S$ *and* $d \in D_U$, *respectively.*

Proof. For the special case $k = 1$ see e.g. [10], for the general case see [3]. See also Weak Formulation 1.1. □

Observation 1.1. For any $u \in \mathbb{H}$, denote by P_u the projection onto the closed convex set K_u in $\mathbb{V}$, i.e.

$$P_u z \in K_u, \ \|P_u z - z\| = \min_{\varphi \in K_u} \|\varphi - z\| \text{ for all } u \in \mathbb{H}, z \in \mathbb{V}.$$

It is well-known and easy to see that for any $v \in \mathbb{H}$ and $z \in \mathbb{V}$, $P_v z$ is the only element from K_v satisfying

$$\langle P_v z - z, y - P_v z \rangle \geqslant 0 \quad \text{for all} \quad y \in K_v$$

(see e.g. [15]). Further, $P_{tv} tz = t P_v z$ for all $v \in \mathbb{H}$, $z \in \mathbb{V}$, $t > 0$.

Observation 1.2. According to Observation 1.1, the problem (I) and ($\mathrm{I_L}$) is equivalent to

$$\begin{aligned} &d_1 u - P_{d_1 u}\big(b_{11} A u + b_{12} A v - N_1(u,v)\big) = 0,\\ &d_2 v - b_{21} A u - b_{22} A v + N_2(u,v) = 0 \end{aligned}$$

and

$$\begin{aligned} &d_1 u - P_{d_1 v}\big(b_{11} A u + b_{12} A v\big) = 0,\\ &d_2 v - b_{21} A u - b_{22} A v = 0, \end{aligned}$$

respectively.

Lemma 1.1. *The operators* A *and* N_j *are completely continuous as the mappings* $\mathbb{H} \to \mathbb{H}$ *and* $\mathbb{H} \times \mathbb{H} \to \mathbb{H}$, *respectively. If* $U_n = [u_n, v_n] \to 0$, $W_n = [w_n, z_n] = \frac{U_n}{\|\!|U_n|\!\|} \rightharpoonup$

$W = [w, z]$ *in* $\mathbb{H}$, $d_n = [d_1^n, d_2^n] \to d = [d_1, d_2]$ *then*

$$(1.2) \qquad \begin{aligned} P_{w_n}\Big[(d_1^n)^{-1}\Big(b_{11}Aw_n + b_{12}Az_n - \frac{N_1(u_n, v_n)}{\|U_n\|}\Big)\Big] &\to P_w[d_1^{-1}(b_{11}Aw + b_{12}Az)], \\ b_{21}Aw_n + b_{22}Az_n - \frac{N_2(u_n, v_n)}{\|U_n\|} &\to b_{21}Aw + b_{22}Az \text{ in } \mathbb{H}. \end{aligned}$$

Proof. We will prove only the first convergence in (1.2), the proof of the other assertions is simpler. Set

$$y_n = P_{w_n}\Big[(d_1^n)^{-1}\Big(b_{11}Aw_n + b_{12}Az_n - \frac{N_1(u_n, v_n)}{\|U_n\|}\Big)\Big].$$

It follows from Observation 1.1 that

$$\Big\langle y_n - (d_1^n)^{-1}\Big(b_{11}Aw_n + b_{12}Az_n - \frac{N_1(u_n, v_n)}{\|U_n\|}\Big), \psi - y_n\Big\rangle \geqslant 0 \text{ for all } \psi \in K_{w_n}.$$

Choosing $\psi = y_n \pm \varphi$ with an arbitrary $\varphi \in \mathcal{D}(\Omega)$ we obtain

$$\int_\Omega \nabla y_n \nabla\varphi - (d_n^1)^{-1}\Big(b_{11}w_n + b_{12}z_n - \frac{n_1(u_n, v_n)}{\|U_n\|)}\Big)\varphi \, dx = 0 \text{ for all } \varphi \in \mathcal{D}(\Omega),$$

i.e.

$$(1.3) \qquad \Delta y_n = (d_1^n)^{-1}\Big(b_{11}w_n + b_{12}z_n - \frac{n_1(u_n, v_n)}{\|U_n\|}\Big)$$

in the sense of distributions and, as a result of (1.1), $\Delta y_n \in L_{q^*}(\Omega)$ (see Weak Formulation 1.1). Hence, $y_n \in \mathbb{H}$. Let $\psi \in K_w$ be fixed. It follows from (AK) that there exist $\psi_n \in K_{w_n}$ such that $\psi_n \to \psi$ in $\mathbb{V}$. Put

$$x_n = (d_1^n)^{-1}\Big(b_{11}Aw_n + b_{12}Az_n - \frac{N_1(u_n, v_n)}{\|U_n\|}\Big).$$

It follows from (A), (N) that $x_n \to x = d_1^{-1}(b_{11}Aw + b_{12}Az)$ in $\mathbb{V}$. We get

$$\|x_n - y_n\| = \|x_n - P_{w_n}x_n\| \leqslant \|x_n - \psi_n\|$$

and therefore $y_n = (y_n - x_n) + x_n$ is bounded in $\mathbb{V}$. Thus we can suppose $y_n \rightharpoonup y$ in $\mathbb{V}$ and we obtain

$$\|x - y\| \leqslant \liminf \|x_n - y_n\| \leqslant \limsup \|x_n - y_n\| \leqslant \|x - \psi\|.$$

We have $y_n = P_{w_n} x_n \in K_{w_n}$, (CK) implies $y \in K_w$ and therefore the last inequality holding for any $\psi \in K_w$ means $y = P_w x$. Setting $\psi = y$, the last inequality gives $x_n - y_n \to x - y$, i.e. $y_n \to y$ in $\mathbb{V}$. Analogously as (1.3) above we obtain

$$\Delta y = -d_1^{-1}(b_{11}w + b_{12}z) \in L_{q^*}(\Omega). \tag{1.4}$$

It follows from (1.1) that $\frac{n_1(u_n,v_n)}{\|U_n\|} \to 0$ in $L_{q*}(\Omega)$ (see Lemma A1 in Appendix for detailes). Hence, (1.3), (1.4) gives $\Delta y_n \to \Delta y$ in $L_{q^*}(\Omega)$ and finally $y_n \to y$ in $\mathbb{H}$. □

Remark 1.2. If d_n, U_n satisfy (I), $d_n \to d, \|U_n\| \to 0$ then also $|||U_n||| \to 0$. In other words, if d is a bifurcation point in the sense of our definition (i.e. with respect to the norm $\|\cdot\|$) then it is simultaneously a bifurcation point with respect to the norm $|||\cdot|||$. Otherwise we would have $d_n = [d_1^n, d_2^n], U_n = [u_n, v_n]$ satisfying (I), $d_n \to d$, $\|U_n\| \to 0$, $|||U_n||| \geqslant \varepsilon > 0$. Setting $W_n = [w_n, z_n] = \frac{U_n}{\|U_n\|}$ we would have $|||W_n||| \to \infty$. Dividing (I) by $\|U_n\|$, we would obtain analogously as in the proof of Lemma 1.1 that

$$\Delta w_n = (d_1^n)^{-1}\Big(b_{11}w_n + b_{12}z_n - \frac{n_1(u_n,v_n)}{\|U_n\|}\Big)$$
$$\Delta z_n = (d_2^n)^{-1}\Big(b_{21}w_n + b_{22}z_n - \frac{n_2(u_n,v_n)}{\|U_n\|}\Big)$$

in Ω. But the right hand side should be bounded in $L_{q^*}(\Omega)$ (see Lemma A1 in Appendix for details), i.e. $\Delta w_n, \Delta z_n$ should be bounded in $L_{q^*}(\Omega)$, i.e. W_n bounded in $\mathbb{H}$, which is a contradiction.

Lemma 1.2. *Let* (SIGN) *be fulfilled. Then any bifurcation point d^0 of* (I) *is simultaneously a critical point of* ($\mathrm{I_L}$).

Proof. Let $d^0 = [d_1^0, d_2^0]$ be a bifurcation point of (I). According to Remark 1.2 and Observation 1.2, there exist $d^n = [d_1^n, d_2^n]$, $U_n = [u_n, v_n]$ such that $d^n \to d^0$, $|||U_n||| > 0$, $|||U_n||| \to 0$,

$$u_n, v_n \in \mathbb{H}, \quad u_n \in K_{u_n},$$
$$d_1^n u_n - P_{d_1^n u_n}\big(b_{11}Au_n + b_{12}Av_n - N_1(u_n,v_n)\big) = 0,$$
$$d_2^n v_n - b_{21}Au_n - b_{22}Av_n + N_2(u_n,v_n) = 0.$$

We can suppose $\frac{U_n}{|||U_n|||} \rightharpoonup W = [w,z]$ in $\mathbb{H}$. Dividing the last equations by $d_1^n|||U_n|||$, $d_2^n|||U_n|||$, using Lemma 1.1 and (HK) we obtain $\frac{u_n}{|||U_n|||} \to w$, $\frac{v_n}{|||U_n|||} \to z$ in $\mathbb{H}$,

$$w, z \in \mathbb{H}, \quad w \in K_w,$$
$$d_1^0 w - P_{d_1^0 w}(b_{11}Aw + b_{12}Az) = 0,$$
$$d_2^0 z - b_{21}Aw - b_{22}Az = 0.$$

Hence, d^0 is a critical point by Observation 1.2.

□

2. Main results

We will always have in mind operators, spaces and convex sets from Weak Formulations 1.1 and 1.2. Hence, solutions of the quasivariational inequality (I) or ($\mathrm{I_L}$) are weak stationary solutions of (RD), (UC) or of the corresponding linearized system with (UC). In fact, however, we could formulate our results for general Hilbert spaces $\mathbb{H} \subset \mathbb{V}$, operators A, N and systems of closed convex sets $K_u, u \in \mathbb{H}$ such that the conditions (A), (N), (0), (HK), (CK), (AK) are fulfilled and that the assertion of Lemma 1.1 holds.

If $d \in C_E$ is such that there is $U = [u, v] \in E_B(d)$ satisfying $u \in K_u$, $\|U\| \neq 0$ then d is simultaneously a critical point of ($\mathrm{I_L}$) and it can be also a bifurcation point of (I). We are interested in cases when this situation is excluded and therefore we shall deal with critical points d of ($\mathrm{E_L}$) satisfying the condition

$$\text{if } U = [u, v] \in E_B(d),\ \|U\| \neq 0 \text{ then } u \notin K_u. \tag{2.1}$$

(Note that clearly $U \in \mathbb{H} \times \mathbb{H}$ for any $U \in E_B(d)$.)

Observation 2.1. Consider the situation from Weak Formulations 1.1 and 1.2. If u changes its sign on $\Gamma_{\mathcal{U}}$ for any $[u, v] \in E_B(d)$ then the condition (2.1) is fulfilled. Indeed, if $[u, v] \in E_B(d)$ then $\frac{\partial u}{\partial n} = 0$ on $\Gamma_{\mathcal{U}}$ and the assertion follows.

Observation 2.2. If (2.1) holds for some $d^0 \in C_E^j$ then it holds for all $d \in C_E^j \setminus \bigcup_{\ell \in I} C_E^\ell$ where I is the set of all indices of all hyperbolas which do not coincide with C_E^j, i.e. it holds for all $d \in C_E^j$ possibly with the exception of the intersection points with the other hyperbolas (see Proposition 1.1 and Fig. 1). Indeed, Proposition 1.1 gives $E_B(d) = \mathrm{Lin}\left\{\left[\frac{d_2\kappa_s - b_{22}}{b_{21}} e_s, e_s\right]\right\}_{s=j}^{j+k-1}$ for all $d \in C_E^j \setminus \bigcup_{\ell \in I} C_E^\ell$ where k is the multiplicity of κ_j and therefore (2.1) for such d means $u \notin K_u$ for any $u = \sum_{s=j}^{j+k-1} c_s e_s$, $c_s \in \mathbb{R}$, and our assertion follows. For $\tilde{d} \in C_E^j \cap C_E^\ell$, $C_E^\ell \neq C_E^j$, $j < \ell$ we have $E_B(d) = \mathrm{Lin}\{[\frac{d_2\kappa_s - b_{22}}{b_{21}} e_s, e_s]\}_{s=j}^{j+k-1}$ where k is the sum of the multiplicities of κ_j, κ_ℓ by Proposition 1.1. Similarly as above, it follows that (2.1) is fulfilled for $d = \tilde{d}$ if and only if it holds for all $d \in C_E^j \cup C_E^\ell$. Analogously for the case $j > \ell$.

In general, it follows from Proposition 1.1 and the assumption (HK) that it is sufficient to know eigenvectors of the operator A (i.e. of $-\Delta$ with (CC) in case of the operator from Weak Formulation 1.1) for verifying the condition (2.1).

Theorem 2.1. *Suppose that* (SIGN) *holds and* $\operatorname{meas}\Gamma_D > 0$. *Then there is no critical point of* $(\mathrm{I_L})$ *in* D_S. *If* (2.1) *is fulfilled for all* $d \in C_r^R$ *with some* $0 < r \leqslant R$ *then there exists* $\varepsilon > 0$ *such that there is no critical point of* $(\mathrm{I_L})$ *in* $C_r^R(\varepsilon)$. *Particularly, there is no bifurcation point of* (I) *in* $C_r^R(\varepsilon) \cup D_S$.

Remark 2.1. Recall that if S is a linear completely continuous symmetric operator in a real Hilbert space $\mathbb{V}$ and if

$$\lambda_m = \max_{\|\varphi\|=1} \langle S\varphi, \varphi\rangle > 0$$

then λ_m is the greatest eigenvalue of S and $u \neq 0$ is a corresponding eigenvector if and only if $\frac{\langle Su,u\rangle}{\|u\|^2} = \lambda_m$.

Proof of Theorem 2.1. First, let $d = [d_1, d_2] \in C_E$ be fixed and consider the problem

$$\begin{aligned} &u, v \in \mathbb{H},\ u \in K_u,\\ &\langle \lambda u - b_{11}Au - b_{12}Av, \varphi - u\rangle \geqslant 0 \text{ for all } \varphi \in K_u,\\ &d_2 v - b_{21}Au - b_{22}Av = 0 \end{aligned} \tag{2.2}$$

with the only parameter λ. Under the conditions (A), (SIGN), the existence of the inverse $\left(I - \frac{b_{22}}{d_2}A\right)^{-1}$ to $\left(I - \frac{b_{22}}{d_2}A\right)$ in $\mathbb{V}$ is ensured and the last equation in (2.2) is equivalent to

$$v = \left(I - \frac{b_{22}}{d_2}A\right)^{-1} \frac{b_{21}}{d_2} Au. \tag{2.3}$$

Substituting into the inequality in (2.2) we obtain

$$\langle \lambda u - Su, \varphi - u\rangle \geqslant 0 \text{ for all } \varphi \in K_u \tag{2.4}$$

where

$$Su = b_{11}Au + \frac{b_{12}b_{21}}{d_2} A\left(I - \frac{b_{22}}{d_2}A\right)^{-1} Au$$

is a linear completely continuous symmetric operator in $\mathbb{V}$. Analogously, the system of equations

$$\begin{aligned} \lambda u - b_{11}Au - b_{12}Av &= 0,\\ d_2 v - b_{21}Au - b_{22}Av &= 0 \end{aligned}$$

is equivalent to

$$\lambda u - Su = 0 \tag{2.5}$$

with (2.3).

Let $[\lambda, d_2]$ be a critical point of $(\mathrm{I_L})$, i.e. there exists a nontrivial $u \in \mathbb{H}$ satisfying (2.4), $u \in K_u$. We can suppose $\|u\| = 1$ in view of the condition (HK) and set $\varphi = 0$ in (2.4) in view of the condition (0). Hence, $\lambda \leqslant \langle Su, u\rangle$. It follows from Proposition 1.1 and the above considerations that $\lambda_m = d_1$ is the greatest eigenvalue of the operator S and this together with Remark 2.1 gives $\lambda \leqslant d_1$. The point $[d_1, d_2] \in C_E$ was arbitrary and it follows that there is no critical point of $(\mathrm{I_L})$ in $\mathrm{D_S}$.

Further, let us show that

$$\text{if } d = [d_1, d_2] \in C_r^R \text{ and } [\lambda, d_2] \text{ is a critical point of } (\mathrm{I_L}) \text{ then } \lambda < d_1. \tag{2.6}$$

We already know that $\lambda \leqslant d_1$ under the assumptions of (2.6). Suppose that $\lambda = d_1$, i.e. $d = [d_1, d_2] \in C_r^R$ is a critical point of $(\mathrm{I_L})$. Then there is $U = [u, v]$, $\|U\| \neq 0$ satisfying $(\mathrm{I_L})$, that means also (2.3), (2.4) hold with $\lambda = d_1$. Particularly, $u \in K_u$ and the assumption (2.1) implies $[u, v] \notin E_B(d)$, that means $u \notin E_S(d_1)$ according to the above considerations. Remark 2.1 implies $\langle Su, u\rangle < d_1$ (because $\langle Su, u\rangle$ attains its maximum only at eigenvectors of S). Simultaneously, setting $\varphi = 0$ in (2.4) we obtain

$$d_1 \langle u, u\rangle \leqslant \langle Su, u\rangle\,,$$

which is a contradiction. Hence, (2.6) is proved.

Suppose that the assertion of Theorem 2.1 is not true. It follows from (2.6) that then there exist $d^n = [d_1^n, d_2^n] \in D_U$ and $U_n = [u_n, v_n] \in \mathbb{H}$ such that $d_2^n \in [r, R]$, $d^n \to d^0 \in C_r^R$, $\|\!|U_n|\!\| = 1$, $U_n \rightharpoonup U = [u, v]$ in $\mathbb{H}$,

$$\begin{aligned} &u_n - P_{u_n}[(d_1^n)^{-1}(b_{11}Au_n - b_{12}Av_n)] = 0,\\ &d_2^n v_n - b_{12}Au_n - b_{22}Av_n = 0. \end{aligned}$$

(see also Observation 1.2). Lemma 1.1 (for $N = 0$) implies that $U_n \to U$ in $\mathbb{H}$, $\|\!|U|\!\| = 1$. The limiting process and Observation 1.2 give $(\mathrm{I_L})$ with $d = d^0$. That means $d^0 \in C_r^R$ is a critical point of $(\mathrm{I_L})$, which contradicts (2.6). The last assertion of Theorem 2.1 follows from Lemma 1.2. $\square$

From the point of view of an interpretation, it is natural to consider a curve d: $\mathbb{R} \to \mathbb{R}_+^2$ and bifurcation problems

$$\begin{aligned} &d_1(s)u - b_{11}Au - b_{12}Av + N_1(u, v) = 0,\\ &d_2(s)v - b_{21}Au - b_{22}Av + N_2(u, v) = 0 \end{aligned} \tag{2.7}$$

and

$$(2.8)\qquad \begin{aligned} &u, v \in \mathbb{H},\ u \in K_u,\\ &\langle d_1(s)u - b_{11}Au - b_{12}Av + N_1(u,v), \varphi - u\rangle \geqslant 0 \text{ for all } \varphi \in K_u,\\ &d_2(s)v - b_{21}Au - b_{22}Av + N_2(u,v) = 0 \end{aligned}$$

with a single parameter $s \in \mathbb{R}$. (See also Interpretation.)

Consequence 2.1. *Suppose that* (SIGN) *holds and* meas $\Gamma_{\mathcal{D}} > 0$. *Consider a curve $d\colon \mathbb{R} \to \mathbb{R}^2_+$ such that $d(s) \in D_S$ for $s < s_0$, $d(s) \in D_U$ for $s > s_0$, $d(s_0) = d^0 \in C_E$. Then there is no bifurcation point of the inequality* (2.8) *in* $(-\infty, s_0)$. *If d^0 satisfies* (2.1) *then there exists $\varepsilon > 0$ such that there is no bifurcation point of* (2.8) *in* $(-\infty, s_0 + \varepsilon)$.

P r o o f. The first assertion follows directly from Theorem 2.1. Let d^0 satisfy (2.1) and suppose by way of contradiction that there exist bifurcation points s_n of (2.8), $s_n \geqslant s_0$, $s_n \to s_0$. Then $[d_1(s_n), d_2(s_n)]$ are critical points of $(\mathrm{I_L})$ by Lemma 1.2 and the same considerations as in the last part of the proof of Theorem 2.1 imply that d^0 is a critical point of $(\mathrm{I_L})$, which contradicts Theorem 2.1. □

Motivation of Unilateral Conditions. The unilateral condition in (UC) can describe for instance a semipermeable membrane allowing the flux through the boundary only in one direction, or some other kind of regulation by a certain source. Consider that a reaction described by our system takes place in a domain Ω which is embedded in a reservoir with fixed concentrations $\bar{u}$, $\bar{v}$ of the activator and inhibitor. None of these substances can cross the part $\Gamma_{\mathcal{N}}$ of the boundary, both substances can flow through $\Gamma_{\mathcal{D}}$ in both directions. The part $\Gamma_{\mathcal{U}}$ represents a semipermeable membrane allowing the flux of the activator only inwards Ω and no flux of the inhibitor through the boundary. The case $\Phi \equiv 0$ corresponds to the situation when the concentration of u (activator or prey) outside Ω is precisely $\bar{u}$ and the natural flux into the domain balances the concentration in Ω near $\Gamma_{\mathcal{U}}$ in case of its decrease in Ω. An increase of the concentration of the activator in Ω is not influenced by the concentration in the reservoir because the flux outwards Ω is not allowed. The case $\Phi \not\equiv 0$ corresponds to an analogous situation but with the concentration in the reservoir depending on the amount of the material just flowing into Ω (cf. [11], where existence results for some other problems with boundary conditions of this type are given). If $\Phi(x,y) = \Phi(y)$ then the concentration in the whole reservoir remains homogeneous and depends only on the flux throug the whole $\Gamma_{\mathcal{U}}$. (This corresponds to an "infinite diffusion" of the activator in the reservoir.) But we can describe also the case when the flux at a given point $x \in \Gamma_{\mathcal{U}}$ influences the concentration only in a neigbourhood of x. We can choose Φ such that $\Phi(x,y) > 0$ for y only from a neighbourhood of x and $\Phi(x,y) = 0$ elsewhere, or such that $\Phi(x,y)$ is small if y is far from x in some sense.

Interpretation. The changing of the diffusion parameters along a given curve $d\colon \mathbb{R}_+ \to \mathbb{R}_+^2$ going from the domain of stability D_S to the domain of instability D_U can correspond to a development of the system described by our equations. The simplest example is the curve $d_1(s) = d_1^0 s^{-1}$, $d_2(s) = d_2^0 s^{-1}$ which can correspond to a growth of the domain Ω, s being proportional to the size of Ω. (This can be shown by a simple substitution.) Particularly, if $d^0 \in C_E$, $d(s) \in D_S$ for $s < s_0 = 1$ then the constant solution * of the problem

$$u_t = d_1(s)\Delta u + f(u,v),$$
$$v_t = d_2(s)\Delta v + g(u,v)$$

with (CC) is stable and no bifurcation can occur as far as $s < s_0$ (see Proposition 1.1), i.e. for domains of size less than a certain critical size. The stability of the constant solution is lost when the critical size of the domain is reached and simultaneously stationary spatially nonhomogeneous solutions (spatial patterns) bifurcate from the constant solution under certain additional assumptions (e.g. odd multiplicity of the corresponding eigenvalue—see e.g. [10]).

The sense of Theorem 2.1 (or Consequence 2.1) is that if the activator (prey) is regulated by the unilateral condition under consideration then (under certain assumptions) spatial patterns can arise only later from the point of view of the development of the system (e.g. from the point of view of the growth of the domain) than in the case of the corresponding classical problem (RD), (CC). On the other hand, it is proved in [6] that if the inhibitor (predator) is regulated by a unilateral condition then spatial patterns arise already in the domain of stability, i.e. sooner from the point of view of the development, e.g. already for smaller domains Ω than for (RD), (CC).

Appendix

Lemma A1. *If* (1.1) *holds and* $U_n = [u_n, v_n] \to 0$ *in* $\mathbb{V}$ *then* $\frac{n_j(u_n,v_n)}{\|U_n\|} \to 0$ *in* L_{q^*}. *Particularly, if* $U_n = [u_n, v_n] \in \mathbb{H}$, $U_n \to 0$ *in* $\mathbb{H}$ *then* $\frac{n_j(u_n,v_n)}{|\!|\!|U_n|\!|\!|} \to 0$ *in* L_{q^*}.

Consequence A1. *Under the assumption* (1.1) *the condition* (N) *is fulfilled.*

* which is supposed to be zero in our mathematical considerations

P r o o f. Let $\varepsilon > 0$ be arbitrary, let c_1 be the constant of the imbedding $W_2^1(\Omega) \subset L_q(\Omega)$. It follows from the definition of n_j (Weak Formulation 1.1) that there exists $\delta > 0$ such that

$$\text{(A1)} \qquad \text{if } |s|^{q-1} + |t|^{q-1} < \delta \text{ then } \frac{n_j(s,t)}{|s|+|t|} \leqslant \frac{\varepsilon}{c_1}.$$

The continuity of the imbedding $W_2^1 \subset L_q$ implies that $\frac{u_n}{\|U_n\|}, \frac{v_n}{\|U_n\|}$ are bounded in $L_q(\Omega)$ and it follows that there is $\xi > 0$ such that

$$\text{(A2)} \qquad \begin{aligned} &\text{if } N \subset \Omega,\ \operatorname{meas} N < \xi \\ &\text{then } c^{q^*}(\delta^{-1}+1)^{q^*} \int_N \Big(\frac{|u_n(x)|^{q-1} + |v_n(x)|^{q-1}}{\|U_n\|}\Big)^{q^*} \,\mathrm{d}x < \varepsilon \end{aligned}$$

where c is from the assumption (1.1). The Jegorov theorem ensures the existence of $N \subset \Omega$ such that

$$\text{(A3)} \qquad \operatorname{meas} N < \xi,\ U_n \to 0 \text{ uniformly on } \Omega \setminus N$$

and therefore there is n_0 such that

$$\text{(A4)} \qquad |u_n(x)|^{q-1} + |v_n(x)|^{q-1} < \delta \text{ for all } x \in \Omega \setminus N,\ n \geqslant n_0.$$

Introduce the sets

$$N_\delta^n = \{x \in \Omega;\ |u_n(x)|^{q-1} + |v_n(x)|^{q-1} \geqslant \delta\}.$$

Writing $\frac{n_j(u_n,v_n)}{\|U_n\|} = \frac{n_j(u_n,v_n)}{|u_n(x)|+|v_n(x)|} \frac{|u_n(x)|+|v_n(x)|}{\|U_n\|}$ for x such that $|u_n(x)| + |v_n(x)| > 0$ we obtain by (A1)

$$\text{(A5)} \quad \Big(\int_{\Omega\setminus N_\delta^n} \Big(\frac{n_j(u_n,v_n)}{\|U_n\|}\Big)^{q^*} \mathrm{d}x\Big)^{\frac{1}{q^*}} \leqslant \frac{\varepsilon}{c_1}\Big(\int_\Omega \Big(\frac{|u_n(x)| + |v_n(x)|}{\|U_n\|}\Big)^{q^*} \mathrm{d}x\Big)^{\frac{1}{q^*}} < \varepsilon.$$

We have $N_\delta^n \subset N$ for all $n \geqslant n_0$ by (A4). Hence, we obtain by using (1.1), (A2) that

$$\text{(A6)} \quad \begin{aligned} \int_{N_\delta^n} \Big(\frac{n_j(u_n,v_n)}{\|U_n\|}\Big)^{q^*} \mathrm{d}x &\leqslant \int_{N_\delta^n} \Big(\frac{c(1 + |u_n(x)|^{q-1} + |v_n(x)|^{q-1})}{\|U_n\|}\Big)^{q^*} \mathrm{d}x \\ &\leqslant \int_{N_\delta^n} \Big(\frac{c(\delta^{-1}+1)(|u_n(x)|^{q-1} + |v_n(x)|^{q-1})}{\|U_n\|}\Big)^{q^*} \mathrm{d}x < \varepsilon. \end{aligned}$$

But $\varepsilon > 0$ was arbitrarily small and therefore our assertion follows from (A5), (A6). $\square$

Further, let T_0 be the unique linear continuous mapping of $H^1(\Omega)$ onto the space of traces $H^{\frac{1}{2}}(\Gamma)$ of functions from $H^1(\Omega)$ such that $T_0 u$ is the restriction of u to Γ for $u \in C(\bar{\Omega}) \cap H^1(\Omega)$ (see e.g. [9]).

Lemma A2. *There is a uniquely defined continuous mapping $T_n\colon H^{q^*}_\Delta(\Omega) \to H^{-\frac{1}{2}}(\Gamma)$ such that $T_n u = \frac{\partial u}{\partial n}$ if $u \in C^1(\bar{\Omega})$ and*

$$\text{(A7)}\quad \int_\Omega \Delta u \varphi \,\mathrm{d}x = \int_\Gamma T_n u T_0 \varphi \,\mathrm{d}\Gamma - \int_\Omega \nabla u \nabla \varphi \,\mathrm{d}x \text{ for all } u \in H^{q^*}_\Delta(\Omega),\ \varphi \in H^1(\Omega)$$

where the integral over Γ is understood as the value $\langle T_n u, T_0\varphi\rangle_{\frac{1}{2}}$ of the functional $T_n u \in H^{-\frac{1}{2}}(\Gamma)$ at $T_0\varphi \in H^{\frac{1}{2}}(\Gamma)$. We can write $\frac{\partial u}{\partial n}$ instead of $T_n u$.

Proof. There exists a linear continuous mapping $\varphi \to \tilde{\varphi}$ of $H^{\frac{1}{2}}(\Gamma)$ into $H^1(\Omega)$ such that $T_0\tilde{\varphi} = \varphi$ for all $\varphi \in H^{\frac{1}{2}}(\Gamma)$ (see [9]). Hence, for any $u \in H^{q^*}_\Delta(\Omega)$, we can define $T_n u$ by

$$\int_\Gamma T_n u \varphi \,\mathrm{d}\Gamma = \langle T_n u, \varphi\rangle_{\frac{1}{2}} = \int_\Omega \nabla u \nabla \tilde{\varphi} \,\mathrm{d}x + \int_\Omega \Delta u \tilde{\varphi} \,\mathrm{d}x$$
$$\text{for all } u \in H^{q^*}_\Delta(\Omega),\ \varphi \in H^{\frac{1}{2}}(\Gamma).$$

It follows from the continuity of the imbedding $H^1(\Omega) \subset L_{q^*}(\Omega)$ that

$$\begin{aligned}\|T_n u\|_{H^{-\frac{1}{2}}} &= \sup_{\|\varphi\|_{H^{\frac{1}{2}}} \leqslant 1} \langle T_n u, \varphi\rangle_{\frac{1}{2}} \leqslant C \sup_{\|\tilde{\varphi}\|_{H^1} \leqslant 1} \left(\int_\Omega \nabla u \nabla \tilde{\varphi} \,\mathrm{d}x + \int_\Omega \Delta u \tilde{\varphi} \,\mathrm{d}x \right) \\ &\leqslant C \sup_{\|\tilde{\varphi}\|_{H^1} \leqslant 1} \left[\left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla \tilde{\varphi}|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left(\int_\Omega |\Delta u|^{q^*} \,\mathrm{d}x \right)^{\frac{1}{q^*}} \left(\int_\Omega |\tilde{\varphi}|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \right] \leqslant C\|u\|_{H^{q^*}_\Delta}.\end{aligned}$$

This means that the linear mapping $T_n\colon H^{q^*}_\Delta(\Omega) \to H^{-\frac{1}{2}}(\Gamma)$ is continuous. □

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