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ON NONCONVEX FUNCTIONAL EVOLUTION INCLUSIONS INVOLVING m-DISSIPATIVE OPERATORS

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1. Introduction

In a recent paper Avgerinos-Papageorgiou [2], proved an existence result for a class of evolution inclusions driven by m-dissipative operators and with a nonconvex set-valued perturbation. In this paper we extend the work of Avgerinos-Papageorgiou [2] in several directions. First, we consider functional-evolution inclusions; i.e. the system under consideration has a memory feature. Second, the multivalued perturbation consists of the extreme points of the original convex-valued orientor field. We emphasize that this "extreme points multifunction", in general is not closed valued and/or lower semicontinuous. So the general theoretical framework of [2] fails. Third, we prove that these "extremal" trajectories are in fact dense in the topology of uniform convergence, in the solution set of the original evolution inclusion, obtaining this way a new strong relaxation theorem. We remark, that in the context of control systems, this density result produces new nonlinear, infinite dimensional "bang-bang" principles. In addition our work here extends those of Cellina-Marchi [8], who studied maximal monotone differential inclusions in $\mathbb{R}^n$ and of Attouch-Damlamian [1], who considered evolution inclusions in a Hilbert space, monitored by subdifferential operators and with a convex set-valued perturbation. A comprehensive introduction to the subject of functional-differential inclusions and their application to optimal control problems, can be found in the recent book of Kisielewicz [11].

2. Mathematical preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper, we will be using the following notations:

$$P_{f(c)}(X) = \{A \subset X \colon A \quad \text{nonempty, closed and (convex)}\},$$
$$P_{(w)k(c)}(X) = \{A \subset X \colon A \quad \text{nonempty, (weakly-)compact, (convex)}\}.$$

A multifunction $F \colon \Omega \to P_f(X)$ is said to be measurable, if for all $x \in X$, the function $\omega \to d(x, F(\omega)) = \inf\{\|x - z\| \colon z \in F(\omega)\}$ is Borel measurable. Now let $\mu(\cdot)$ be a finite measure defined on (Ω, Σ). We define S_F^p $(1 \leqslant p \leqslant +\infty)$ to be the set of all $L^p(\Omega, X)$-selectors of $F(\cdot)$; i.e. $S_F^p = \{f \in L^p(\Omega, X) \colon f(\omega) \in F(\omega)\ \mu-\text{a.e.}\}$. This set may be empty. It is nonempty if and only if the function $\omega \to \inf\{\|z\| \colon z \in F(\omega)\}$ belongs to $L^p(\Omega, \mathbb{R}^+)$. Recall that a subset K of $L^p(\Omega, X)$ is *decomposable* if for every triple $(f, g, A) \in K \times K \times \Sigma$, we have $f\chi_A + g\chi_{A^c} \in K$, where χ_A denotes the characteristic function of the set A. Clearly S_F^p is decomposable.

On $P_f(X)$ we can define a generalized metric, known in the literature as the "Hausdorff metric", by setting, for $A, B \in P_f(X)$,

$$h(A, B) = \max\left\{\sup\{d(a, B) \colon a \in A\},\ \sup\{d(b, A) \colon b \in B\}\right\}$$

(recall that $d(a, B) = \inf\{\|a - b\| \colon b \in B\}$; similary for $d(b, A)$). The metric space $(P_f(X), h)$ is complete. A multifunction $F \colon X \to P_f(X)$ is said to be *Hausdorff continuous* (H-continuous) if it is continuous from X into $(P_f(X), h)$.

Let Y, Z be Hausdorff topological spaces. A multifunction $G \colon Y \to 2^Z \setminus \{\emptyset\}$ is said to be lower semicontinuous (denoted as l.s.c.), if for all $U \subset Z$ open $F^-(U) = \{y \in Y \colon F(y) \cap U \neq \emptyset\}$ is open in Y.

Let $A \colon D(A) \subset X \to 2^X$ be a set-valued operator with domain $D(A)$. We say that A is *accretive*, if for every $x_1, x_2 \in D(A)$, for every $y_i \in A(x_i)$, $i = 1, 2$, and for every $\lambda > 0$, we have $\|x_1 - x_2\| \leqslant \|x_1 - x_2 + \lambda(y_1 - y_2)\|$. Another equivalent definition, can be given using the duality map of X, which is the set-valued function $J \colon X \to 2^{X^*}$ defined as $J(x) = \{x^* \in X^* \colon (x^*, x) = \|x\|^2 = \|x^*\|^2\}$. Clearly the values of $J(\cdot)$ are nonempty, closed, convex, bounded subsets of X^*; moreover we recall that if X^* is strictly convex, the duality map $J(\cdot)$ is single-valued and w^*-demicontinuous, and furthermore if X^* is locally uniformly convex, then $J(\cdot)$ is single-valued and continuous. Using $J(\cdot)$ we can define the upper semi-inner product on X (denoted by $(\cdot, \cdot)_+$) as follows:

$$(x, y)_+ = \sup\{(x^*, y) \colon x^* \in J(x)\}$$

for all $x, y \in X$. So $A(\cdot)$ is accretive if and only if for every $x_1, x_2 \in D(A)$, for every $y_i \in A(x_i)$, $i = 1, 2$, it follows $(x_1 - x_2, y_1 - y_2)_+ \geqslant 0$. We say that $A(\cdot)$ is *m-accretive*, if it is accretive and for each $\lambda > 0$, $I + \lambda A$ is surjective, where I is the identity operator on X. A is said to be *m-dissipative* if $-A$ is m-accretive. It is well known that an m-dissipative operator A generates, on $\overline{D(A)}$, a semigroup $\{S(t)\}_{t\geqslant 0}$ of non expansive mappings, via the Crandall-Liggett formula

$$S(t)x = \lim_{n\to+\infty} \left(I - \frac{t}{n}A\right)^{-n} x, \quad t \geqslant 0,\ x \in \overline{D(A)} \quad \text{(see [4])}.$$

The semigroup is said to be *compact* if, for each $t > 0$, $S(t)$ is a compact operator.

Finally if $T = [0, b]$, by $L^1_w(T, X)$ we will denote the space of all equivalence classes of Bochner integrable functions $x\colon T \to X$ with the (weak) norm

$$\|x\|_w = \sup\left\{\left\|\int_t^{t'} x(s)\,\mathrm{d}s\right\| : 0 \leqslant t \leqslant t' \leqslant b\right\}.$$

The setting of our problem is the following: let $T = [0, b]$, $T_0 = [-r, 0]$ $(r > 0)$, $\hat{T} = [-r, b]$ and let X be a separable reflexive Banach space, with uniformly convex dual. We consider the following multivalued Cauchy problem:

$$(1)\qquad \begin{cases} \dot{x}(t) \in Ax(t) + F(t, x_t), \\ x(v) = w(v), \ v \in T_0. \end{cases}$$

Here $x_t(\cdot) \in C(T_0, X)$ is the function defined by $x_t(v) = x(t+v)$. So $x_t(\cdot)$ describes the past evolution of the state, from time $t - r$ until the present time t. Also $A\colon D(A) \subset X \to 2^X$ is an m-dissipative operator.

In conjunction with (1), we also consider the following Cauchy problem:

$$(2)\qquad \begin{cases} \dot{x}(t) \in Ax(t) + \operatorname{ext}\ F(t, x_t), \\ x(v) = w(v), \ v \in T_0. \end{cases}$$

Here $\operatorname{ext} F(t, y)$ stands for the extreme points of the orientor field $F(t, y)$. By an *integral solution* of (1) (resp. of (2)), we mean a function $x \in C(\hat{T}, X)$ such that there exists $f \in L^1(T, X)$ with $f(t) \in F(t, x_t)$ (resp. $f(t) \in \operatorname{ext} F(t, x_t)$) a.e. in T and $x(\cdot)$ is an integral solution in the sense of Benilan [6] of the Cauchy problem

$$\begin{cases} \dot{x}(t) \in Ax(t) + f(t), \\ x(0) = w(0); \end{cases}$$

that is for each $[z, y] \in GrA$ and $0 \leqslant s \leqslant t \leqslant b$, we have

$$\frac{1}{2}\|x(t) - z\|^2 \leqslant \frac{1}{2}\|x(s) - z\|^2 + \int_s^t (f(\tau) + y, x(\tau) - z)_+ \,\mathrm{d}\tau.$$

Recall that if $\dim X < \infty$ or more generally if X is a Hilbert space, $f \in L^2(T, X)$ and $A = \partial\varphi$, where φ is a proper, lower semicontinuous, convex $\overline{\mathbb{R}}$-valued function on X, then integral solutions coincide with strong solutions (see [4]).

3. Extremal trajectories

In this section, we estabilish the existence of integral solutions for problem (2). For this we will need the following hypotheses on the data:

$H(A)$: $A\colon D(A) \subset X \to 2^X$ is a multivalued m-dissipative operator which generates a compact semigroup on $\overline{D(A)}$.

$H(F)$: $F\colon T \times C(T_0, X) \to P_{wkc}(X)$ is multifunction such that

j) $\forall x \in C(T_0, X)$, $t \to F(t, x)$ is measurable;

jj) for a.e. $t \in T$, $x \to F(t, x)$ is H-continuous;

jjj) $\exists a, c \in L^p(T, \mathbb{R}^+)$, $1 < p < \infty$:

$$\|F(t, x)\| = \sup\{\|z\|\colon z \in F(t, x)\} \leqslant a(t) + c(t)\|x\|_\infty,$$

a.e. in T, $\forall x \in C(T_0, X)$.

H_0: $w \in C(T_0, X)$ and $w(0) \in \overline{D(A)}$.

Remark 1. Hypotheses $H(F)$ j) and jj) and Theorem 3.3 of [12] imply that $(t, x) \to F(t, x)$ is jointly measurable.

First we prove a lemma that we will need in the sequel

Lemma. *If $(f_n)_n \subset L^p(T, X)$, $1 < p < \infty$, $\sup\{\|f_n\|_p\colon n \in N\} < \infty$ and $f_n \to 0$ in ${L^1}_w(T, X)$ then $f_n \to 0$ weakly in $L^p(T, X)$.*

Proof. From Theorem 1, p. 98, of [9], we know that $L^p(T, X)^* = L^q(T, X^*)$, with $\frac{1}{p} + \frac{1}{q} = 1$. Let $((\cdot, \cdot))$ denote the duality brackets for the pair $(L^p(T, X)$, $L^q(T, X^*))$. Since, by hypothesis, $(f_n)_n$ is bounded in $L^p(T, X)$ and the space of X^*-valued simple functions on T is dense in $L^q(T, X^*)$, we only need to show that $((f_n, s)) \to 0$ as $n \to \infty$, for each simple function $s\colon T \to X^*$ of the form

$$s(t) = v_k, \quad t \in (t_{k-1}, t_k), \quad v_k \in X^*, \quad k = 1, \ldots, N,$$
$$\text{with} \quad 0 = t_0 < t_1 < \ldots < t_N = b.$$

We have:

$$|((f_n, s))| = \left|\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (f_n(\tau), v_k)\, \mathrm{d}\tau\right| \leqslant \sum_{k=1}^{N} \left\| \int_{t_{k-1}}^{t_k} f_n(\tau)\, \mathrm{d}\tau \right\| \|v_k\|$$
$$\leqslant \|f_n\|_w \sum_{k=1}^{N} \|v_k\| \to 0 \quad \text{as} \quad n \to \infty,$$

which was to be proved. □

Now we are ready for the existence theorem concerning Cauchy problem (2).

Theorem 1. *If hypotheses $H(A)$, $H(F)$ and H_0 hold, then problem* (2) *admits an integral solution.*

Proof. We start by deriving an a priori bound for the solutions of the problem (1) (hence of (2) too). So let $x(\cdot) \in C(\hat{T}, X)$ be such a solution and let $y \in C(T, X)$ the unique integral solution of

$$\begin{cases} \dot{x}(t) \in Ax(t), \\ x(0) = w(0) \in \overline{D(A)} \end{cases}$$

(cf. [6]). Then from Benilan's inequality [6], we have

$$\|x(t) - y(t)\| \leqslant \int_0^t \|f(s)\| \, \mathrm{d}s$$

where $f \in L^p(T, X)$, $f(t) \in F(t, x_t)$ a.e. in T. So we have

$$\|x(t)\| \leqslant \|y\|_\infty + \int_0^t (a(s) + c(s)\|x_s\|_\infty) \, \mathrm{d}s,$$

hence

$$\|x_t\|_\infty \leqslant \|y\|_\infty + \|a\|_1 + \int_0^t c(s)\|x_s\|_\infty \, \mathrm{d}s, \ \forall \, t \in T.$$

Here $\|x_t\|_\infty$ is the ess sup of $x_t(\cdot)$ over the interval $[t - r, t]$, while $\|y\|_\infty$ is the ess sup of $y(\cdot)$ over $T = [0, b]$.

Invoking Gronwall's inequality, we deduce that there exists $M_1 > 0$ such that, for all $t \in \hat{T}$ and all solutions $x(\cdot)$ of the problem (1), we have $\|x(t)\| \leqslant M_1$. Hence without any loss of generality, put $\gamma(t) = a(t) + c(t)M_1$, $\gamma \in L^p(T, \mathbb{R}^+)$, we may assume that

$$\|F(t, x)\| = \sup\{\|z\| \colon z \in F(t, x)\} \leqslant \gamma(t), \quad \text{a.e. in } \ T, \ \forall \, x \in C(T_0, X).$$

Otherwise in what follows we replace $F(t, x)$ by $F(t, p_{M_1}(x))$ with $p_{M_1}(\cdot)$ being the M_1-radial retraction. Note that by virtue of Lipschitzness of $p_{M_1}(\cdot)$, $F(t, p_{M_1}(x))$ has the same measurability and continuity properties as $F(\cdot, \cdot)$ and moreover $|F(t, p_{M_1}(x))| \leqslant \gamma(t)$ a.e.

Set

$$V = \{h \in L^p(T, X) \colon \|h(t)\| \leqslant \gamma(t) \quad \text{a.e. in} \quad T\}$$

and let $\eta\colon L^p(T,X) \to C(T,X)$ be the map which assigns to each $h \in L^p(T,X)$, the unique integral solution of the Cauchy problem

$$\begin{cases} \dot{x}(t) \in Ax(t) + h(t), \\ x(0) = w(0) \in \overline{D(A)}. \end{cases}$$

The fact that the above Cauchy problem has an integral solution which is actually unique is due to [6] (see also [4]).

Let $\hat{\eta}\colon L^p(T,X) \to C(\hat{T},X)$ be defined by, for each $h \in L^p(T,X)$,

$$\hat{\eta}(h)(t) = \begin{cases} \eta(h)(t), & \forall t \in T, \\ w(t), & \forall t \in T_0. \end{cases}$$

Since V is bounded and, by hypothesis $H(A)$, the operator $A(\cdot)$ generates a compact semigroup, from Theorem 1 of [3], we have that $\hat{\eta}(V)$ is relatively compact in $C(\hat{T},X)$, hence $\eta(V)$ is relatively compact in $C(T,X)$. Set $K = \overline{\text{conv}}\hat{\eta}(V)$, from Mazur's Theorem we have that K is a compact and convex subset of $C(\hat{T},X)$. In what follows K is endowed with the $C(\hat{T},X)$-topology.

Define $R\colon K \to P_{wkc}(L^p(T,X))$ by

$$R(x) = \{h \in L^p(T,X)\colon h(t) \in F(t,x_t) \text{ a.e. in } T\}.$$

From Theorem 1.1 of [14], we know that there exists a continuous function $r\colon K \to L^1{}_w(T,X)$ such that $r(x) \in \text{ext}\, R(x)$, $\forall x \in K$.

Since for every $x \in K$,

$$\text{ext}\, R(x) = \{h \in L^p(T,X)\colon h(t) \in \text{ext}F(t,x_t) \text{ a.e. in } T\}$$

(cf. [5]), it follows that $r(x)(t) \in \text{ext}\, F(t,x_t)$, a.e. in T.

Let $\hat{\xi} = \hat{\eta} \circ r\colon K \to K$. Recalling that $J(\cdot)$ is a continuous single valued map and using Theorem 1 of [3], we have that $\hat{\eta}(\cdot)$ is sequentially continuous from $L^p(T,X)$ with the weak topology into $C(\hat{T},X)$. Combining this with the Lemma, we get that $\hat{\xi}(\cdot)$ is continuous. Then, by Schauder's fixed point Theorem, we have that there exists $x \in K$ such that $x = \hat{\xi}(x)$. So $x \in C(\hat{T},X)$ is the desired integral solution of the problem (2). □

Let $S(w) \subset C(\hat{T}, X)$ be the solution set of the Cauchy problem (1) and $S_e(w) \subset C(\hat{T}, X)$ the solution set of the problem (2). We saw that under the hypotheses of theorem 1, $\emptyset \neq S_e(w) \subset S(w)$.

In this section, by strengthening our hypothesis on the orientor field, we show that $S_e(w)$ is dense in $S(w)$ for the $C(\hat{T}, X)$-topology.

The stronger hypothesis on F that we will need, is the following:

$H(F)_1$: $F: T \times C(T_0, X) \to P_{wkc}(X)$ is a multifunction such that

j) $\forall x \in C(T_0, X)$, $t \to F(t,x)$ is measurable;

jj)' $\exists k \in L^1(T, \mathbb{R}^+)$: $h(F(t,x), F(t,x')) \leqslant k(t)\|x - x'\|_\infty$,

$$\text{a.e. in } T, \ \forall, x, x' \in C(T_0, X);$$

jjj) $\exists a, c \in L^p(T, \mathbb{R}^+)$, $1 < p < \infty$:

$$\|F(t,x)\| = \sup\{\|z\| : z \in F(t,x)\} \leqslant a(t) + c(t)\|x\|_\infty,$$

a.e. in T, $\forall x \in C(T, X)$.

Theorem 2. *If hypotheses $H(A)$, $H(F)_1$ and H_0 hold then $S_e(w)$ is dense in $S(w)$ for the $C(\hat{T}, X)$-topology.*

P r o o f. Fixed $x \in S(w)$, let $f \in L^p(T, X)$: $f(t) \in F(t, x_t)$, a.e. in T, such that $x(\cdot)$ is the integral solution of the Cauchy problem

$$\begin{cases} \dot{x}(t) \in Ax(t) + f(t), \\ x(0) = w(0) \in \overline{D(A)} \end{cases}$$

on T and $x(v) = w(v)$ for all $v \in T_0$. Let K be the compact subset of $C(\hat{T}, X)$ as in the proof of Theorem 1. Given $z \in K$ and $\varepsilon > 0$, let $\Gamma_\varepsilon : T \to 2^X \setminus \{\emptyset\}$ be defined by

$$\Gamma_\varepsilon(t) = \left\{ u \in X : \|f(t) - u\| < \frac{\varepsilon}{2M_1 b} + d(f(t), F(t, z_t)), u \in F(t, z_t) \right\}$$

with M_1 being the a priori bound for the elements of $S(w)$ obtained in the beginning of the proof of Theorem 1. We have

$$Gr\Gamma_\varepsilon = \left\{ (t, u) \in GrF(\cdot, z.) : \|f(t) - u\| < \frac{\varepsilon}{2M_1 b} + d(f(t), F(t, z_t)) \right\}.$$

From hypotheses $H(F)_1$ j) and jj)' and Theorem 3.3 of [12] we have that the function $t \to F(t, z_t)$ is measurable and so $GrF(\cdot, z.) \in B(T) \times B(C(T_0, X))$ with $B(T)$

(resp. $B(C(T_0, X))$) being the Borel σ-field of T (resp. of $C(T_0, X)$). Apply Aumann's selection Theorem (cf. [11], theorem 3.11, p. 47) to get a measurable map $h_\varepsilon : T \to X$ such that $h_\varepsilon(t) \in \Gamma_\varepsilon(t)$ a.e. in T. Then let $\Sigma_\varepsilon : K \to 2^{L^p}(T, X)$ be defined by

$$\Sigma_\varepsilon(z) = \Big\{h \in L^p(T, X) \colon h(t) \in F(t, z_t) \text{ and } \|f(t) - h(t)\| < \frac{\varepsilon}{2M_1 b} + d(f(t), F(t, z_t)) \text{ a.e. in } T\Big\}.$$

We have just proved that, for all $z \in K$ and all $\varepsilon > 0$, $\Sigma_\varepsilon(z) \neq \emptyset$ and clearly $\Sigma_\varepsilon(\cdot)$ has decomposable values. Furthermore, from Proposition 4 of [7], we know that $z \to \Sigma_\varepsilon(z)$ is l.s.c., hence $z \to \overline{\Sigma_\varepsilon(z)}$ is l.s.c. and it has nonempty, closed and decomposable values. Apply Theorem 3 of [7] to get a continuous map $u_\varepsilon :$ $K \to L^1(T, X)$ such that $u_\varepsilon(z) \in \overline{\Sigma_\varepsilon(z)}$, $\forall z \in K$. We have:

$$\begin{aligned} \|f(t) - u_\varepsilon(z)(t)\| &\leqslant \frac{\varepsilon}{2M_1 b} + d(f(t), F(t, z_t)) \\ &\leqslant \frac{\varepsilon}{2M_1 b} + k(t)\|x_t - z_t\|_\infty, \quad \text{a.e. in } T. \end{aligned}$$

Use Theorem 1.1 of [14] to get a continuous map $v_\varepsilon : K \to L^1_w(T, X)$ with the properties:

$$v_\varepsilon(z) \in \{h \in L^p(T, X) \colon h(t) \in \operatorname{ext} F(t, z_t), \quad \text{a.e. in } T\}$$

and

$$\|u_\varepsilon(z) - v_\varepsilon(z)\|_w < \varepsilon, \quad \forall z \in K.$$

Let $\varepsilon = 1/n$, $u_{1/n} = u_n$ and $v_{1/n} = v_n$. Since $\hat{\eta}(v_n)$ maps K into K, from Schauder's fixed point Theorem we have that there exists $x_n \in K$ such that $x_n = \hat{\eta}(v_n)(x_n)$, therefore $x_n \in S_e(w)$. Since K is a compact subset of $C(\hat{T}, X)$, by passing to a subsequence if necessary, we may assume that $x_n \to z$ in $C(\hat{T}, X)$. From inequality (2,4), p. 124 of [4], we have that (recall that the duality map $J(\cdot)$ is single valued):

$$\begin{aligned} (4.1) \quad \|x(t) - x_n(t)\|^2 &\leqslant 2\int_0^t (J(x(s) - x_n(s)), f(s) - v_n(x_n)(s))\, \mathrm{d}s \\ &\leqslant 2\int_0^t (J(x(s) - x_n(s)), f(s) - u_n(x_n)(s))\, \mathrm{d}s \\ &\quad + 2\int_0^t (J(x(s) - x_n(s)), u_n(x_n)(s) - v_n(x_n)(s))\, \mathrm{d}s, \\ &\quad \forall t \in T, \ \forall n \in \mathbb{N}. \end{aligned}$$

Since X^* is uniformly convex, from Proposition 32.22, p. 860 of [15], we have that $J(x(\cdot) - x_n(\cdot)) \to J(x(\cdot) - x(\cdot))$ in $C(T, X^*)$ as $n \to \infty$, while from the lemma in the section 3, we have $u_n(x_n) - v_n(x_n) \to 0$ weakly in $L^p(T, X)$. So we obtain

$$\lim_{n\to+\infty} \int_0^t (J(x(s) - x_n(s)), u_n(x_n)(s) - v_n(x_n)(s))\, \mathrm{d}s = 0. \tag{4.2}$$

On the other hand

$$\begin{aligned}
&\int_0^t (J(x(s) - x_n(s)), f(s) - u_n(x_n)(s))\, \mathrm{d}s \\
&\leqslant \int_0^t \|J(x(s) - x_n(s))\|\ \|f(s) - u_n(x_n)(s)\|\, \mathrm{d}s \\
&\leqslant \int_0^t \Big(\frac{1}{2nM_1 b} + k(s)\|x_s - (x_n)_s\|_\infty\Big)\|x(s) - x_n(s)\|\, \mathrm{d}s \\
&\leqslant \frac{1}{n} + \int_0^t k(s)\|x_s - (s_n)_s\|_\infty^2\, \mathrm{d}s \to \int_0^t k(s)\|x_s - z_s\|_\infty^2\, \mathrm{d}s, \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence, by (4.1) and (4.2), it follows

$$\|x_t - z_t\|_\infty^2 \leqslant 2\int_0^t k(s)\|x_s - z_s\|_\infty^2\, \mathrm{d}s,\ \forall t \in T.$$

Applying Gronwall's inequality we get $x = z$. Since $x_n \in S_e(w)$ and $x_n \to x$ in $C(\hat{T}, X)$, we have that $S(w)$ is included in the clousure of $S_e(w)$ in $C(\hat{T}, X)$. It remains to show that $S(w)$ is closed in $C(\hat{T}, X)$.

So let $x_n \in S(w)$ and assume that $x_n \to x$ in $C(\hat{T}, X)$. Then on T we have that there exists $f_n \in V$ such that $f_n(t) \in F(t, (x_n)_t)$ a.e. in $T, x_n = \eta(f_n)$ and $x_n(v) = w(v)$ on T. By passing to a subsequence, if it is necessary, we may assume that $f_n \to f$ weakly in $L^p(T, X)$. Put $G, G_n \colon T \to P_{wkc}(X)$ the multifunctions defined by $G(t) = F(t, x_t)$, $G_n(t) = F(t, (x_n)_t)$, $\forall t \in T$, $\forall n \in N$, for every $g \in L^p(T, X)^* = L^q(T, X^*)$, we have

$$((f_n, g)) \leqslant \sigma(g, S^p_{G_n}),$$

where σ is the support function of S_{G_n}, defined by

$$\sigma(g, S^p_{G_n}) = \sup\Big\{((g, h)) \colon h \in S^p_{G_n}\Big\}.$$

But $\sigma(g, S^p_{G_n}) = \int_0^b \sigma(g(t), G_n(t))\, \mathrm{d}t$ (cf. the proof of Theorem 3.1 of [13]). □

Passing to the limit as $n \to \infty$ and using hypothesis $H(F)_1$ jj)′ we have

$$\begin{aligned}
((f, g)) &\leqslant \limsup_{n\to+\infty} \sigma(g, S^p_{G_n}) \\
&\leqslant \int_0^b \limsup_{n\to+\infty} \sigma(g(t), G_n(t))\, \mathrm{d}t = \int_0^b \sigma(g(t), G(t))\, \mathrm{d}t = \sigma(g, S^p_G).
\end{aligned}$$

Since $g \in L^q(T, X^*)$ was arbitrary, we deduce that $f \in S^p_G$. Also, as in the proof of Theorem 1, we have $\eta(f_n) \to \eta(f)$ in $C(T, X)$, hence $x_n = \hat{\eta}(f_n) \to \hat{\eta}(f) = x$ in $C(\hat{T}, X)$ with $f(t) \in F(t, x_t)$ a.e. in T. Then $x \in S(w)$ and so $S(w)$ is closed in $C(\hat{T}, X)$.

5. Distributed parameter control systems with delay

In this section we illustrate the applicability of our abstract results, through an example of a nonlinear parabolic distributed parameter control system with delay. Let $T = [0, b]$ and Z be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ. Here $t \in T$ is the time variable and $z \in Z$ the space variable. We consider the problem

$$(3)\qquad \begin{cases} \dfrac{\partial x}{\partial t} + \sum\limits_{\alpha \leqslant m} (-1)^\alpha D^\alpha A_\alpha(z, \eta(x(t,z))) = f(t, z, x(t-r, z))u(t, z), \\ \qquad \text{a.e. on } T \times Z, \\ D^\beta x|_{T\times\Gamma} = 0, \ |\beta| \leqslant m - 1, x(v, z) = w(v, z) \\ \qquad \text{a.e. on } Z, \text{ for all } v \in T_0 = [-r, 0], \\ |u(t, z)| \leqslant \gamma. \end{cases}$$

Here $f\colon T \times Z \times \mathbb{R} \to \mathbb{R}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \sum\limits_{k=1}^{n} \alpha_k$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ where as always, $D_k = \frac{\partial}{\partial z_k}$, $k = 1, \ldots, n$, $\eta(x) = (D^\alpha x\colon |\alpha| \leqslant m)$ and $A_\alpha\colon Z \times \mathbb{R}^d \to \mathbb{R}$, with $d = \frac{(n+m)!}{n!m!}$. The hypotheses on the data are the following:

$\underline{H}(A)_1$: $A_\alpha\colon Z \times \mathbb{R}^d \to \mathbb{R}$ are functions such that

1) $\forall \eta \in \mathbb{R}^d$, $z \to A_\alpha(z, \eta)$ is measurable;
2) $\forall z \in Z$, $\eta \to A_\alpha(z, \eta)$ is continuous;
3) there exist $p \geqslant 2$, $a \in L^q(Z, \mathbb{R}^+)$ and $c \in L^\infty(Z, \mathbb{R}^+)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ such that $|A_\alpha(z, \eta)| \leqslant a(z) + c(z)\|\eta\|^{p-1}$, a.e. on Z and $\forall \eta \in \mathbb{R}^d$;
4) $\exists d^* > 0$ such that $\sum_{\alpha \leqslant m}(A_\alpha(z, \eta) - A_\alpha(z, \eta'))(\eta_\alpha - \eta'_\alpha) \geqslant d^* \sum\limits_{|\gamma|=m} |\eta_\gamma - \eta'_\gamma|^p$, a.e. on Z and $\forall \eta \in \mathbb{R}^d$.
5) $\exists r > 0$ such that $\sum_{|\alpha| \leqslant m} A_\alpha(z, \eta)\eta_\alpha \geqslant r \sum_{|\gamma|=m} |\eta_\gamma|^p$, a.e. on Z and $\forall \eta \in \mathbb{R}^d$.

$\underline{H}(f)$: $f\colon T \times Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

1) $\forall x \in \mathbb{R}$, $(t, z) \to f(t, z, x)$ is measurable;
2) $\exists k\colon T \times Z \to \mathbb{R}^+$ such that if $\hat{k}$ is the function defined by $\hat{k}(t) = k(t, \cdot)$ then $\hat{k} \in L^2([0, T], L^\infty(Z, \mathbb{R}^+))$ and $|f(t, z, x) - f(t, z, y)| \leqslant k(t, z)|z - y|$, a.e. in $T \times Z$ and $\forall x, y \in \mathbb{R}$;
3) there exist $a_1 \in L^2(T \times Z, \mathbb{R}^+)$ and $c_1 \in L^\infty(T \times Z, \mathbb{R}^+)$: $|f(t, z, x)| \leqslant a_1(t, z) + c_1(t, z)|x|$, a.e. in $T \times Z$, $\forall x \in \mathbb{R}$.

$\underline{H}_0$: $w(\cdot,\cdot) \in C(T_0, L^2(Z, \mathbb{R}))$.

By an *admissible state control-pair* we mean two functions $x \in C(\hat{T}, L^2(Z, \mathbb{R}))$ and $u \in L^\infty(T \times Z, \mathbb{R})$ satisfying the problem (3).

We have the following bang-bang principle for control system (3).

Theorem 3. *If hypotheses $H(A)_1$, $H(f)$ and H_0 hold and if $[x, u]$ is an admissible state control pair then for every $\varepsilon > 0$ there exists another admissible state control pair $[y, v]$ such that*

$$\lambda\{(t,z) \in T \times Z : |v(t,z| \neq \gamma\} = 0 \quad \textit{and} \quad \sup\left\{ \int_Z |x(t,z) - y(t,z)|^2 \, \mathrm{d}z : t \in T \right\} < \varepsilon$$

(here $\lambda(\cdot)$ stands for the Lebesgue product measure on $T \times Z$).

Proof. Let $X = W_0^{m,p}(Z)$, $H = L^2(Z, \mathbb{R})$ and $X^* = W^{-m,q}(Z)$ $(\frac{1}{p} + \frac{1}{q} = 1)$. From the Sobolev embedding theorem we know that $X \hookrightarrow H \hookrightarrow X^*$, with all embeddings being compact; i.e. (X, H, X^*) is an evolution triple with compact embeddings (cf. [15], p. 416). Consider the Dirichelet form $\widetilde{\alpha}: X \times X \to \mathbb{R}$ defined by

$$\widetilde{\alpha}(x,y) = \sum_{|\alpha| \leqslant m} \int_Z A_\alpha(z, \eta(x(z))) D^\alpha y(z) \, \mathrm{d}z$$

for all $x, y \in X$. Let $\hat{A}_\alpha : X \to L^q(Z, \mathbb{R})$ be the function defined by

$$\hat{A}_\alpha(x)(\cdot) = A_\alpha(\cdot, \eta(x(\cdot))), \ \forall x \in X.$$

Because of hypothesis $\underline{H}(A)_1$ and Krasnoselskii's Theorem (cf. [15], p. 561), we have that $\hat{A}_\alpha$ is continuous and in addition, because of $\underline{H}(A)_1$ (3),

$$\exists \hat{a}, \hat{c} > 0 : \|\hat{A}_\alpha(x)\|_q \leqslant \hat{a} + \hat{c}\|x\|_x^{p-1}, \ \forall x \in X.$$

Hence applying Holder's inequality, we get

$$|\widetilde{\alpha}(x,y)| \leqslant \sum_{|\alpha| \leqslant m} \|\hat{A}_\alpha(x)\|_q \, \|D^\alpha y\|_p \leqslant (\hat{a} + \hat{c}\|x\|_X^{p-1})\|y\|_X, \ \forall x, y \in X.$$

Thus there exists a nonlinear operator $A: X \to X^*$ satisfying

$$\langle A(x), y \rangle = \widetilde{\alpha}(x,y),$$

for all $x, y \in X$ and with $\langle \cdot, \cdot \rangle$ denoting the duality brackets for the pair (X, X^*). Observe that

$$\|A(x)\|_{X^*} \leqslant \hat{a} + \hat{c}\|x\|_x^{p-1}, \ \forall x \in X.$$

Moreover if $x_n \to x$ in X, from the continuity of $\hat{A}_\alpha$ and using once more Holder's inequality, we have

$$|\tilde{\alpha}(x_n, y) - \tilde{\alpha}(x, y)| \leqslant \sum_{|\alpha| \leqslant m} \|\hat{A}_\alpha(x_n) - \hat{A}_\alpha(x)\|_q \|y\|_X$$

which implies that

$$\|A(x_n) - A(x)\|_{X^*} \leqslant \sum_{|\alpha| \leqslant m} \|\hat{A}_\alpha(x_n) - \hat{A}_\alpha(x)\|_q \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore $A(\cdot)$ is continuous.

Recall that $|x| = \left(\int_Z \sum_{|\gamma|=m} |D^\gamma x(z)|^p \, \mathrm{d}z \right)^{1/p}$ is an equivalent norm on $W_0^{m,p}(Z)$. So from hypothesis $\underline{H}(A)_1$ (4), we get that there exists $\hat{d} > 0$ such that

$$\hat{d}\|x - y\|_x^p \leqslant \langle A(x) - A(y), x - y \rangle, \ \forall x, y \in X. \tag{5.1}$$

Next let $A_H \colon D(A_H) \subset H \to H$ be defined by $A_H(x) = A(x)$ for all $x \in D(A_H) = \{x \in X \colon A(x) \in H\}$. We know that A is the energetic extension of A_H and from hypothesis $\underline{H}(A)_1$ (5), we obtain that it is also coercive (i.e. $\lim\limits_{\|u\| + \infty} \frac{\langle Au, u \rangle}{\|u\|} = \infty$); therefore A_H is maximal monotone (cf. [4], p. 140). Recall that on a Hilbert space, maximal monotonicity is equivalent to m-accretivity (cf. [15], p. 821). So let $\{S(t) \colon t \in T\}$ bet the semigroup of nonlinear contractions generated by $-A_H$. From Theorem 30.A, p. 771 of [15], we know that $S(t)\overline{D(A_H)} = S(t)H \subset D(A_H)$ (smoothing effect on initial data; see also [4], p. 144). Moreover from [4], p. 144, we have that there exists $\tilde{c} > 0$ such that

$$\left\| t \frac{\mathrm{d}}{\mathrm{d}t} S(t)x \right\|_H = \|tA_H S(t)x\|_H \leqslant \tilde{c}\|x\|_H, \quad \forall t, x \in T \times H. \tag{5.2}$$

Because of (5.1) and the fact that $W_0^{m,p}(Z)$ embeds compactly in $L^2(Z, \mathbb{R})$, we obtain that $(I + A)^{-1}$ is compact. Now note that $S(t)x = (I + A)^{-1}(I + A^\circ)S(t)x$, for all $x \in H$. Use (5.2) together with the compactness of the resolvent $(I + A)^{-1}$ to conclude that $\{S(t) \colon t \in]0, b]\}$ is a compact semigroup.

Next put $\hat{f} \colon T \times C(T_0, H) \to H$ the function defined by

$$\hat{f}(t, y)(z) = f(t, z, y(-r)(z)), \quad \text{for all} \quad t \in T, y \in C(T_0, H) \quad \text{and} \quad z \in Z,$$

and $U = \{u \in L^\infty(Z, \mathbb{R}) \colon \|u\|_\infty \leqslant \gamma\}$, let $F \colon T \times C(T_0, H) \to P_{wkc}(H)$ be defined by

$$F(t, y) = \hat{f}(t, y)U, \quad \text{for all} \quad (t, y) \in T \times C(T_0, H).$$

Observe that, for every $h \in H$, we have

$$(\hat{f}(t,y),h)_H = \int_Z f(t,z,y(-r)(z))h(z)\,\mathrm{d}z,$$

so, by Fubini's Theorem, the function $t \to \hat{f}(t,y)$ is weakly measurable. Then, taking into account the separability of $H = L^2(Z,\mathbb{R})$, by Pettis measurability Theorem we get that $t \to \hat{f}(t,y)$ is measurable from T into H, therefore (cf. [9], p. 42) it follows that $t \to F(t,y)$ is measurable.

Moreover, form $\underline{H}(f)(2)$, we have that there exists $\widetilde{k} \in L^1(T,\mathbb{R}^+)$ such that

$$h(F(t,x),F(t,y) \leqslant \widetilde{k}(t)\|x-y\|_\infty, \quad \text{a.e. in} \quad T, \forall x,y \in H,$$

and, from $\underline{H}(f)(3)$, there exist $\hat{a}_1 \in L^2(T,\mathbb{R}^+)$ and $\hat{c}_1 \in L^\infty(T,\mathbb{R}^+)$ with the property

$$\|F(t,x)\| \leqslant \hat{a}_1(t) + \hat{c}_1(t)\|x\|_\infty, \quad \text{a.e. in} \quad T, x \in C(T_0,H).$$

Since $\operatorname{ext} F(t,x) = \operatorname{ext} \hat{f}(t,x)U \subset \hat{f}(t,x) \operatorname{ext} U$ and (cf. [10], p. 79)

$$\operatorname{ext} U = \Big\{v \in L^\infty(Z,\mathbb{R})\colon \lambda_0\{z \in Z\colon |v(z)| \neq \gamma\} = 0\Big\},$$

where λ_0 is the Lebesgue measure on Z, we can rewrite the problem (3) in the following equivalent deparametrized problem

$$\begin{cases} \dot{x}(t) \in -A_H x(t) + F(t,x_t), \\ x(v) = \hat{w}(v), \ v \in T, \end{cases}$$

with $\hat{w}(v) = w(v,\cdot) \in L^2(Z,\mathbb{R}) = H = \overline{D(A_H)}$. Then theorem 2 will give us the desidered "bang-bang" admissible pair $[y,v]$ such that

$$\lambda\{(t,z) \in T\times Z\colon |v(t,z)| \neq \gamma\} = 0 \quad \text{and} \quad \sup\Big\{\int_Z |x(t,z)-y(t,z)|^2\,\mathrm{d}z\colon t \in T\Big\} < \varepsilon.$$

□

Now suppose that we are also given a continuous cost functional $V\colon C(\hat{T}, L^2(Z,\mathbb{R})) \to \mathbb{R}$, which has to be minimized over the set $S(\hat{w})$ of trajectories of (3). In other words, if $m = \inf\{V(x)\colon x \in S(\hat{w})\}$, our problem is the following

(P) is there exists a trajectory $\bar{x} \in S(\hat{w})$ such that $V(\bar{x}) = m$?

Using theorems 2 and 3 and recalling that $S(\hat{w})$ is a compact subset of $C(\hat{T}, L^2(Z,\mathbb{R}))$ we get the following

Theorem 4. *If hypotheses $H(A)_1$, $H(f)$ and H_0 hold, then (P) has s solution and for every $\varepsilon > 0$ there exists $y \in C(\hat{T}, L^2(Z,\mathbb{R}))$ a trajectory generated by a "bang-bang" control $v \in L^\infty(T \times Z,\mathbb{R}^+)$ (i.e. $\lambda\{(t,z) \in T \times Z\colon |v(t,z)| \neq \gamma\} = 0$) such that $V(y) \leqslant m + \varepsilon$.*

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